Lecture 8: Ultrasound-modulated optical tomography

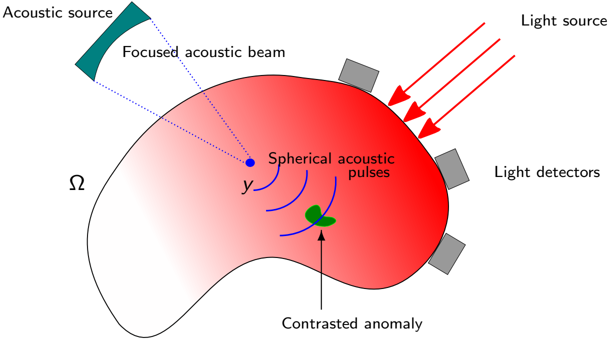
Habib Ammari

Department of Mathematics, ETH Zürich

- Reconstruction algorithm for ultrasound-modulated diffuse optical tomography.
- Diffuse optical imaging: low resolution.
- By mechanically perturbing the medium $\rightarrow$ achieve a significant resolution enhancement.
    - Spherical acoustic wave: propagating inside the medium $\rightarrow$ optical parameter of the medium: perturbed.
    - Cross-correlations of the boundary measurements of the intensity of the light propagating in the perturbed medium and in the unperturbed one $\rightarrow$ two iterative algorithms for reconstructing the optical absorption coefficient:
        - Spherical Radon transform inversion $\rightarrow$ nonlinear system: solved iteratively or by optimal control.

Ultrasound-modulated optical tomography

- Acoustically modulated optical tomography:



- Record the variations of the light intensity on the boundary due to the propagation of the acoustic pulses.

Ultrasound-modulated optical tomography

- g: the light illumination; a: optical absorption coefficient; l: extrapolation length. Fluence Φ (in the unperturbed domain):

$$\begin{cases} -\Delta\Phi + a\Phi = 0 \quad \text{in } \Omega, \\ l\dfrac{\partial\Phi}{\partial\nu} + \Phi = g \quad \text{on } \partial\Omega. \end{cases}$$

- Acoustic pulse propagation: $a \to a_u(x) = a(x + u(x))$.
- Fluence Φ_u (in the displaced domain):

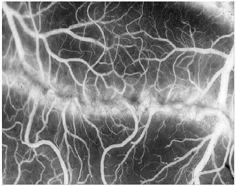
$$\begin{cases} -\Delta\Phi_u + a_u\Phi_u = 0 \quad \text{in } \Omega, \\ l\dfrac{\partial\Phi_u}{\partial\nu} + \Phi_u = g \quad \text{on } \partial\Omega. \end{cases}$$

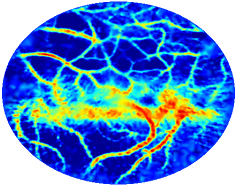
- u: thin spherical shell growing at a constant speed; y: source point; r: radius.
- Cross-correlation formula:

$$M(y,r) := \int_{\partial\Omega} \left(\frac{\partial\Phi}{\partial\nu}\Phi_u - \frac{\partial\Phi_u}{\partial\nu}\Phi \right) = \int_\Omega (a_u - a)\Phi\Phi_u \approx \underbrace{\int_\Omega u \cdot \nabla a \, |\Phi|^2}_{Taylor+Born}.$$

Ultrasound-modulated optical tomography

- Helmholtz decomposition: $\Phi^2 \nabla a = \nabla \psi + \nabla \times A$.
- Spherical Radon transform: $\nabla \psi = -\frac{1}{c} \nabla \mathcal{R}^{-1} \left[\int_0^r \frac{M(y,\rho)}{\rho^{d-2}} d\rho \right]$.
- System of nonlinearly coupled elliptic equations: $\nabla \cdot \Phi^2 \nabla a = \Delta \psi$ and $\Delta \Phi + a\Phi = 0$.
- Fixed point and Optimal control algorithms.
- Reconstruction for a realistic absorption map.
- Proofs of convergence for highly discontinuous absorption maps (bounded variation).





Ultrasound-modulated optical tomography

- Ω: acoustically homogeneous.
- Displacement field: spherical acoustic pulse generated at y.
- $P : \Omega \longrightarrow \Omega$: the displacement. $u = P^{-1} - Id$: small compared to the size of Ω.
- Typical form of u:

$$u_{y,r}^{\eta}(x) = -\eta \frac{r_0}{r} w \left(\frac{|x-y|-r}{\eta} \right) \frac{x-y}{|x-y|}, \quad \forall x \in \mathbb{R}^d.$$

- w: shape of the pulse; $\mathrm{supp}(w) \subset [-1, 1]$ and $\|w\|_\infty = 1$. η: thickness of the wavefront, y: source point; r: radius.
- Thin spherical shell growing at a constant speed.

Ultrasound-modulated optical tomography

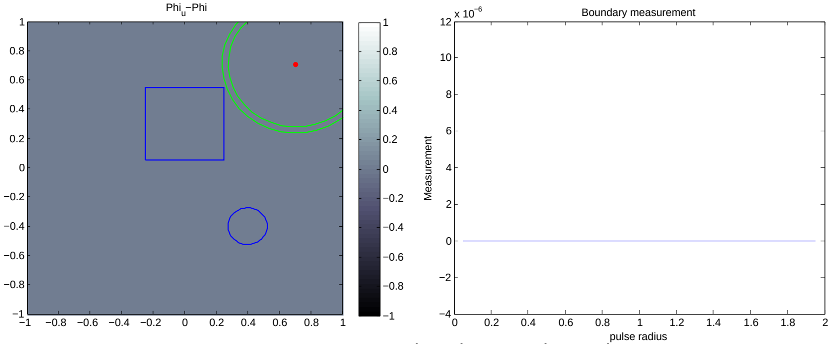
- Pulse propagation: $a \to a_u(x) = a(x + u(x))$.
- Fluence Φ_u:

$$\begin{cases} -\Delta \Phi_u + a_u \Phi_u = 0 & \text{in } \Omega, \\ l \dfrac{\partial \Phi_u}{\partial \nu} + \Phi_u = g & \text{on } \partial\Omega, \end{cases}$$

- Cross-correlation formula:

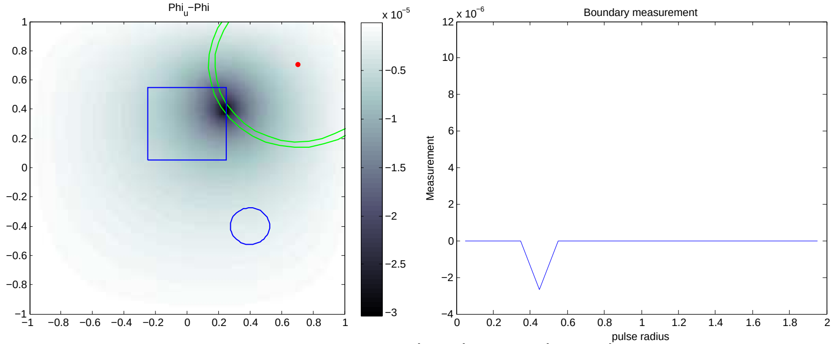
$$M_u := \int_{\partial\Omega} (\partial_\nu \Phi \Phi_u - \partial_\nu \Phi_u \Phi) = \int_\Omega (a_u - a) \Phi \Phi_u$$

Ultrasound-modulated optical tomography



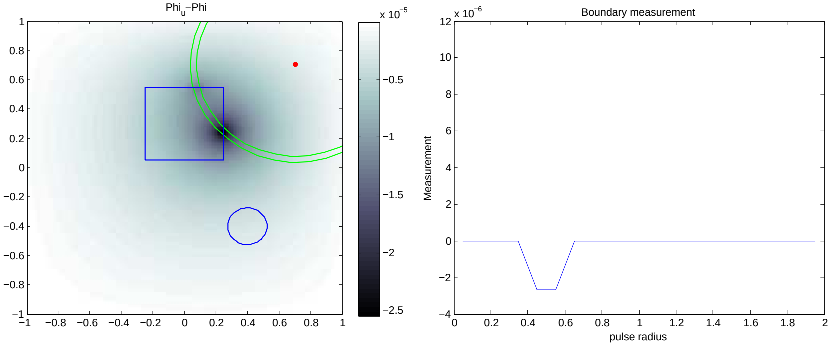
$\Phi_u - \Phi$ (left); M_u (right).

Ultrasound-modulated optical tomography



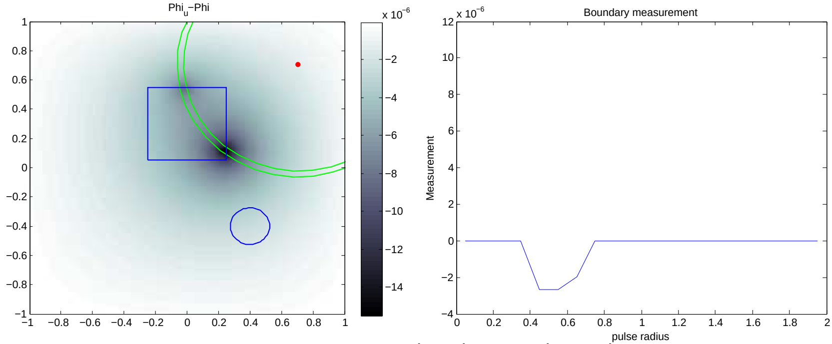
$\Phi_u - \Phi$ (left); M_u (right).

Ultrasound-modulated optical tomography



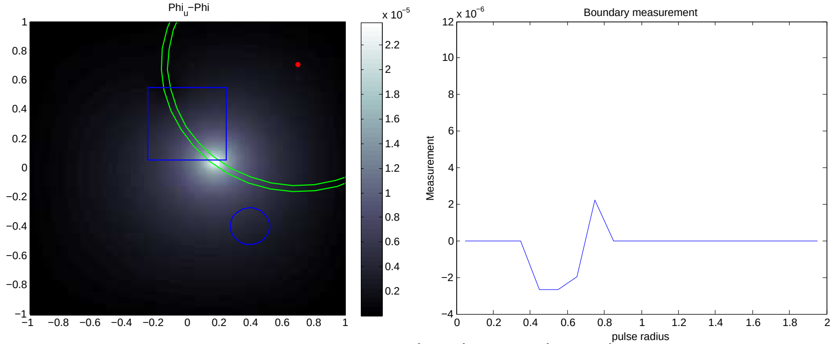
$\Phi_u - \Phi$ (left); M_u (right).

Ultrasound-modulated optical tomography



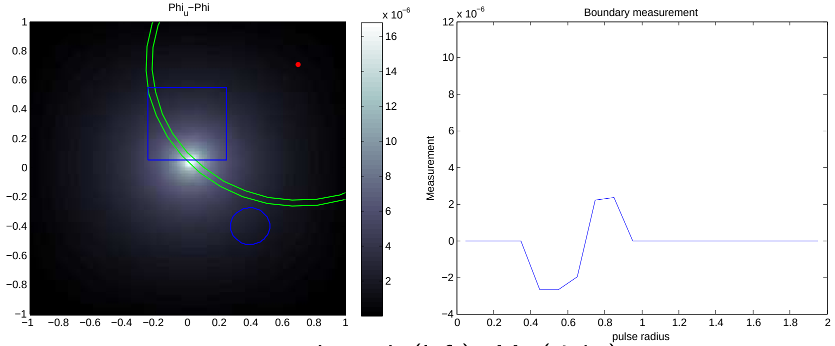
$\Phi_u - \Phi$ (left); M_u (right).

Ultrasound-modulated optical tomography



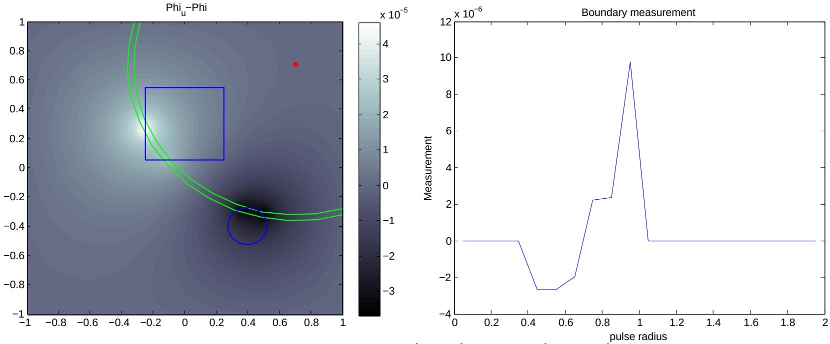
$\Phi_u - \Phi$ (left); M_u (right).

Ultrasound-modulated optical tomography



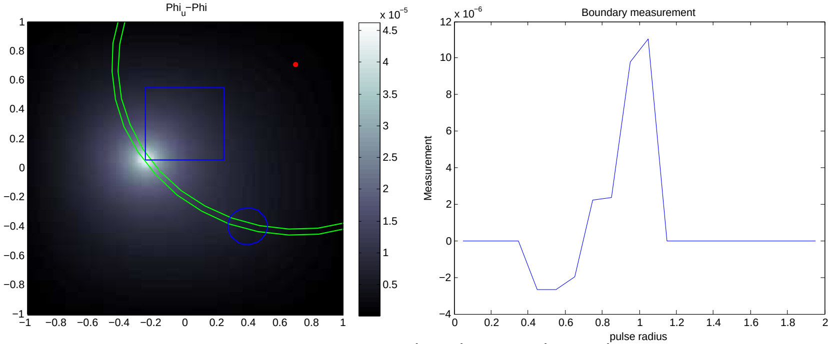
$\Phi_u - \Phi$ (left); M_u (right).

Ultrasound-modulated optical tomography



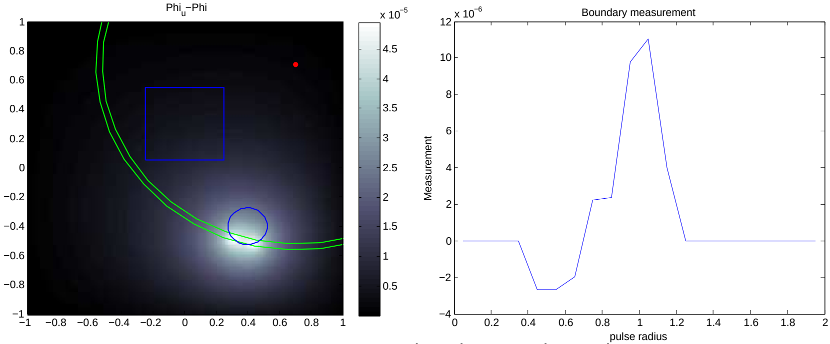
$\Phi_u - \Phi$ (left); M_u (right).

Ultrasound-modulated optical tomography



$\Phi_u - \Phi$ (left); M_u (right).

Ultrasound-modulated optical tomography



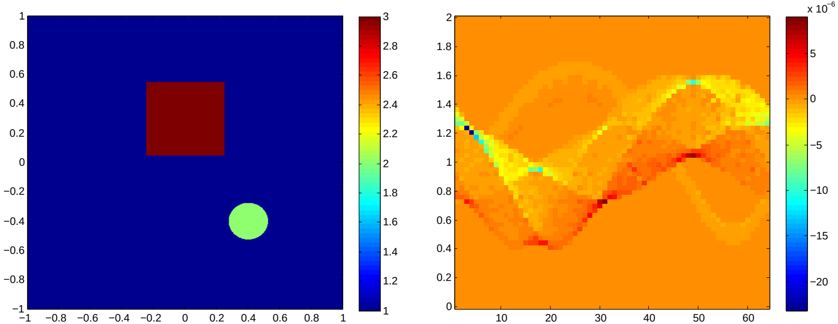
$\Phi_u - \Phi$ (left); M_u (right).

Ultrasound-modulated optical tomography

- Displacement field u: depends on the center y, the radius r and the wavefront thickness η.
- Measurements:

$$M_\eta(y,r) = \frac{1}{\eta^2}\int_\Omega (a_{u^\eta_{y,r}} - a)\Phi\Phi_{u^\eta_{y,r}}$$

Ultrasound-modulated optical tomography



True absorbtion (left) and measurements M_u (right) for 64 pulses centered on the unit circle.

Ultrasound-modulated optical tomography

- Small η:

$$M_\eta(y,r) \approx \frac{1}{\eta^2}\int_\Omega \nabla a . u^\eta_{y,r}\Phi^2 .$$

- Asymptotic behavior:

$$\lim_{\eta\to 0} M_\eta(y,r) = -cr^{d-2}\int_{S^{d-1}} (\Phi^2\nabla a)(y+r\xi).\xi \, d\sigma(\xi) =: M(y,r)$$

  $c > 0$: depends on the shape of u and on d. Expansion uniform in (y,r); Error $= \mathcal{O}(\eta)$.

- Reconstruct a from M.

Ultrasound-modulated optical tomography

- Spherical means Radon transform:

$$\mathcal{R}[f](y,r) = \int_{S^{d-1}} f(y + r\xi) d\sigma(\xi) \quad y \in S, \; r > 0,$$

- Derivative of $\mathcal{R}$:

$$\partial_r(\mathcal{R}[f])(y,r) = \int_{S^{d-1}} \nabla f(y + r\xi) \cdot \xi d\sigma(\xi).$$

Ultrasound-modulated optical tomography

- Helmholtz decomposition of $\Phi^2 \nabla a$:

$$\Phi^2 \nabla a = \nabla \psi + \nabla \times A.$$

- Measurement interpretation:

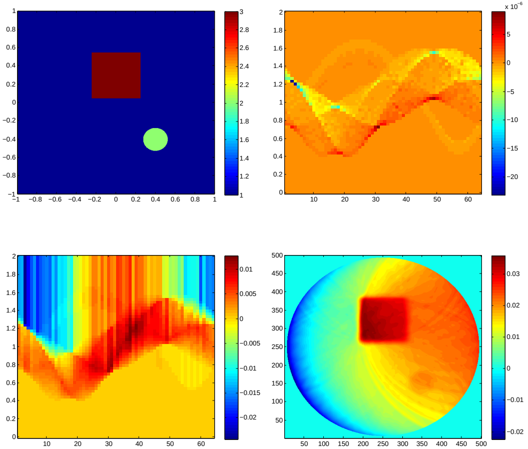
$$\int_{S^{d-1}} (\Phi^2 \nabla a)(y + r\xi).\xi d\sigma(\xi) = \int_{S^{d-1}} \nabla \psi (y + r\xi).\xi d\sigma(\xi).$$

- Relate M to $\partial_r \mathcal{R}[\psi]$.
- Reconstruction formula for ψ:

$$\psi = -\frac{1}{c} \mathcal{R}^{-1} \left[\int_0^r \frac{M(y, \rho)}{\rho^{d-2}} d\rho \right]$$

(up to an additive constant).

Ultrasound-modulated optical tomography



True absorbtion a; M_u; $\mathcal{R}[\psi]$; ψ.

Ultrasound-modulated optical tomography

- Reconstruct a knowing only ψ in the Helmholtz decomposition:

$$\Phi^2 \nabla a = \nabla \psi + \nabla \times A \quad ?$$

- Divergence of the Helmholtz decomposition:

$$\nabla \cdot (\Phi^2 \nabla a) = \Delta \psi.$$

- Assume $a = a_0$ (a known constant on $\Omega \backslash \Omega'$) and $g \geq 0$ on $\partial\Omega$:

$$(E_2): \begin{cases} \nabla \cdot (\Phi^2 \nabla a) = \Delta \psi \text{ in } \Omega', \\ \qquad a = a_0 \text{ on } \partial\Omega'. \end{cases}$$

- Φ: unknown in Ω.

Ultrasound-modulated optical tomography

Coupled elliptic system:

$$(E) : \begin{cases} (E_1) : \begin{cases} -\Delta\Phi + a\Phi = 0 \text{ in } \Omega, \\ l\dfrac{\partial\Phi}{\partial\nu} + \Phi = g \text{ on } \partial\Omega, \end{cases} \\ (E_2) : \begin{cases} \nabla \cdot (\Phi^2 \nabla a) = \Delta\psi \text{ in } \Omega', \\ a = a_0 \text{ on } \partial\Omega', \\ a = a_0 \text{ in } \Omega\backslash\Omega', \end{cases} \end{cases}$$

ψ, $l > 0$, g, and $a_0 > 0$: known.

Ultrasound-modulated optical tomography

- Convergence result for the fixed point scheme provided ψ is small.
- Optimal control and Landweber schemes:
  - $F[a] := \nabla \cdot (\Phi^2[a]\nabla a)$;
  - Optimal control: $\min \|F[a] - \Delta\psi\|$;
  - Landweber sequence:

$$a^{(n+1)} = a^{(n)} - \mu dF[a^{(n)}]^*(F[a^{(n)}] - \Delta\psi),$$

  - $\mu > 0$: relaxation parameter.
  - Convergence results assuming a good initial guess.

Ultrasound-modulated optical tomography

- Fixed point algorithm:
  - Initial guess $a^{(0)} = a_0$.
  - For $n \geq 1$, solve

$$\begin{cases} -\Delta\phi^{(n)} + Ta^{(n-1)}\phi^{(n)} & = & 0 & \text{in } \Omega\,, \\ l\dfrac{\partial\phi^{(n)}}{\partial\nu} + \phi^{(n)} & = & g & \text{on } \partial\Omega\,; \end{cases}$$

    $Ta := \min\{\max\{a, \underline{a}\}, \overline{a}\}\,.$
  - Find $a^{(n)}$ by solving

$$\begin{cases} -\nabla\cdot((\phi^{(n)})^2\nabla a^{(n)}) & = & \Delta\psi & \text{in } \Omega'\,, \\ a^{(n)} & = & a_0 & \text{on } \partial\Omega'\,. \end{cases}$$

    and defining $a^{(n)} = a_0$ in $\Omega \setminus \Omega'$.
  - For $\|\Delta\psi\|_{L^\infty(\Omega)}$ small enough, $a^{(n)} \to a_*$; a_*: true optical absorption coefficient.

Ultrasound-modulated optical tomography

- $\mathcal{Q} = \{a \in L^\infty(\Omega) : \underline{a} < a < \overline{a}\}$;

$$\begin{array}{rcl} F_1 : \mathcal{Q} & \rightarrow & H^1(\Omega) \\ a & \mapsto & F_1[a] = \varphi; \end{array}$$

- φ:

$$\left\{ \begin{array}{rcl} -\Delta\varphi + a\varphi & = & 0 \quad \text{in } \Omega\,, \\ l\frac{\partial\varphi}{\partial\nu} + \varphi & = & g \quad \text{on } \partial\Omega\,. \end{array} \right.$$

Ultrasound-modulated optical tomography

- For all $a \in \mathcal{Q}$, $F_1[a] \in L^\infty(\Omega)$; There exists a positive constant $\Lambda(\underline{a}, \overline{a})$ s.t.
$$\big|F_1[a](x)\big| \leq \Lambda, \quad \forall\, x \in \Omega.$$
- For any $\Omega' \Subset \Omega$, there exists a positive constant $\lambda(\Omega', \underline{a}, \overline{a})$ s.t.
$$\lambda \leq F_1[a](x), \quad \forall\, x \in \Omega'.$$
- F_1: Fréchet differentiable: $dF_1[a](h) = \phi$ for $h \in L^\infty(\Omega)$; ϕ solves
$$\begin{cases} -\Delta\phi + a\phi &= -h\varphi \quad \text{in } \Omega\,, \\ l\frac{\partial\phi}{\partial\nu} + \phi &= 0 \qquad \text{on } \partial\Omega \end{cases}$$
with $\varphi = F_1[a]$.
- $dF_1[a]$: continuously extended to $L^2(\Omega)$
$$\|dF_1[a]\|_{\mathcal{L}(L^2(\Omega), H^1(\Omega))} \leq C\Lambda\,.$$

Ultrasound-modulated optical tomography

- Open set of $L^\infty(\Omega)$:

$$\mathcal{P} = \left\{ \rho \in L^\infty(\Omega) : \frac{\lambda}{2} < \rho < 2\Lambda \text{ in } \Omega' \right\} .$$

- Let

$$\begin{array}{rcl} F_2 : \mathcal{P} & \to & W^{1,2}(\Omega) \\ \phi & \mapsto & F_2[\phi] = a, \end{array}$$

$$\left\{ \begin{array}{r} \nabla \cdot (\Phi^2 \nabla a) = \Delta \psi \text{ in } \Omega', \\ a = a_0 \text{ on } \partial\Omega', \\ a = a_0 \text{ in } \Omega \backslash \Omega'. \end{array} \right.$$

Ultrasound-modulated optical tomography

- F_2: Fréchet differentiable: for $h \in L^\infty(\Omega)$,

$$dF_2[\phi](h) = Q,$$

Q solves

$$\begin{cases} -\nabla \cdot (\phi^2 \nabla Q) & = & \nabla \cdot (2\phi h \nabla a) \quad \text{in } \Omega', \\ Q & = & 0 \quad \text{on } \partial\Omega'. \end{cases}$$

- $dF_2[\phi]$ can be extended continuously to $L^2(\Omega)$ and

$$\|dF_2[\varphi]\|_{\mathcal{L}(L^2(\Omega), W^{1,2}(\Omega))} \leq \frac{2\Lambda}{\lambda^2} c_2(\lambda, \Lambda, M),$$

M: an upper bound of $\|\nabla \cdot (\phi \nabla a)\|_{L^\infty(\Omega')}$.

Ultrasound-modulated optical tomography

- Assume that $\underline{a}$, $\overline{a}$, and M:given.
- If $\|\Delta\psi\|_{L^\infty(\Omega)}$: sufficiently small, then the iteration sequence in the algorithm converges in $L^2(\Omega)$ to a_*.

Ultrasound-modulated optical tomography

- Introduce the map on $\mathcal{Q}$:

$$F[a] = F_2 \circ F_1[a].$$

- $dF[a] : L^\infty(\Omega) \to L^2(\Omega)$

$$dF[a](h) = dF_2[F_1[a]](dF_1[a](h)).$$

- $dF[a]$ can be extended continuously to $L^2(\Omega)$ with

$$\begin{aligned}
&\|dF[a]\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \\
&\leq \|dF_1[a]\|_{\mathcal{L}(L^2(\Omega),W^{1,2}(\Omega))}\|dF_2[a]\|_{\mathcal{L}(L^2(\Omega),W^{1,2}(\Omega))} \\
&\leq C\|\Delta\psi\|_{L^\infty(\Omega)}\,.
\end{aligned}$$

Ultrasound-modulated optical tomography

- Recall from the algorithm that $a^{(0)} = a_0$: the initial guess for the true coefficient a_* and for $n \geq 1$, define

$$a^{(n)} = F[Ta^{(n-1)}] \quad n \geq 1,$$

$Tp = \min\{\max\{p, \underline{a}\}, \overline{a}\}$.

- For all $m, n \geq 1$,

$$\begin{aligned}
&\|F[Ta^{(n)}] - F[Ta^{(m)}]\|_{L^2(\Omega)} \\
&= \left\| \int_0^1 dF[(1-t)Ta^{(n)} + tTa^{(m)}](a^{(m)} - a^{(n)})dt \right\|_{L^2(\Omega)} \\
&\leq C\|\Delta\psi\|_{L^\infty(\Omega)}\|a^{(m)} - a^{(n)}\|_{L^2(\Omega)}\,.
\end{aligned}$$

- If $\|\Delta\psi\|_{L^\infty(\Omega)}$: small enough, then $F \circ T : L^2(\Omega) \to L^2(\Omega)$: contraction map.

Ultrasound-modulated optical tomography

- Admissible set K: closed and convex in $H_0^1(\Omega)$:

$$K := \{a - a_0 \in W_0^{1,4}(\Omega) : \underline{a} \leq a \leq \overline{a} \text{ and } \|\nabla a\|_{L^4(\Omega)} \leq \theta\};$$

  θ: to be determined.

- Internal data map $F : K \to H^{-1}(\Omega)$. For all $a \in K$,

$$F[a](v) = \int_\Omega F_1[a]^2 \nabla a \cdot \nabla v \quad \text{for all } v \in H_0^1(\Omega)\,.$$

- F: Fréchet differentiable in K and

$$dF[a](h, v) = \int_\Omega (2F_1[a]dF_1[a](h)\nabla a + F_1[a]^2 \nabla h) \cdot \nabla v \, dx$$

  for all $a \in K$, $h \in W_0^{1,4}(\Omega) \cap L^\infty(\Omega)$ and $v \in H_0^1(\Omega)$.

Ultrasound-modulated optical tomography

- Assume $0 < \theta < \frac{C_\Omega \lambda^2}{\Lambda^2}$; C_Ω: constant.
- $dF[a]$: well-defined on $H^1_0(\Omega)$ and there exists a positive constant C s.t. for all $h \in H^1_0(\Omega)$,

$$\|dF[a](h)\|_{H^{-1}(\Omega)} \geq C\|h\|_{H^1_0(\Omega)}.$$

$$dF[a](h) : v \in H^1_0(\Omega) \mapsto dF[a](h, v).$$

Ultrasound-modulated optical tomography

- Consider $\Delta\psi \in H^{-1}(\Omega)$; Rewrite

$$\nabla \cdot F_1[a]^2 \nabla a = \Delta\psi \,,$$

in the sense of distributions, as $F[a] = \Delta\psi$.

- Projection T from $H^1_0(\Omega)$ onto K (closed and convex).
- Optimal control algorithm: minimize the discrepancy between $F[a]$ and $\Delta\psi$:

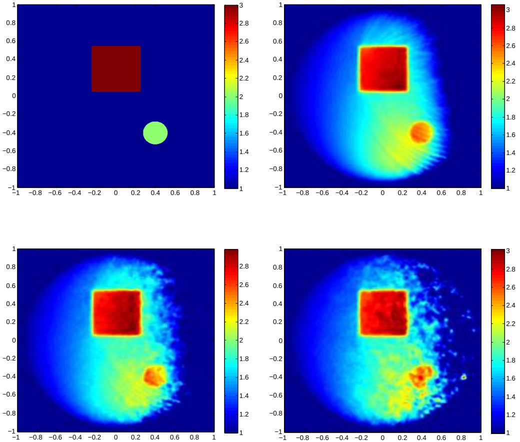
$$\min_{a \in K} J[a] := \frac{1}{2} \| F[a] - \Delta\psi \|^2_{H^{-1}(\Omega)} \,.$$

  - Initial guess $a^{(0)} = a_0$.
  - For $n \geq 1$,

$$a^{(n+1)} = Ta^{(n)} - \eta dF[Ta^{(n)}]^* (F[Ta^{(n)}] - \Delta\psi) \,, \quad \eta : \text{ step size.}$$

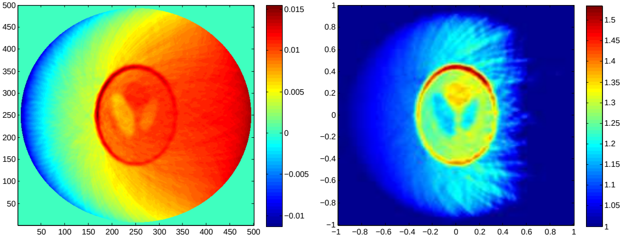
  - For $||a_0 - a_*||_{H^1_0(\Omega)}$ and μ: small enough, $\{a^{(n)}\} \to a_*$ in $H^1_0(\Omega)$; a_* : true optical absorption coefficient.

Ultrasound-modulated optical tomography



Reconstruction of a from noisy measurements : true a; noise level: 0%, 5%, and 10%.

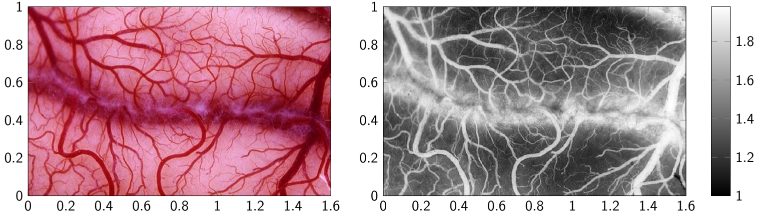
Ultrasound-modulated optical tomography



Reconstruction of the Shepp-Logan phantom for 128 acoustic pulses.

Ultrasound-modulated optical tomography

Realistic biological light absorption map:



Ultrasound-modulated optical tomography

Reconstruction of the absorption map:

- Minimal regularity assumption on a (SBV^∞; change of function):

$$\widetilde{a} := a - a_0 - \frac{\psi}{\phi^2}.$$

- a and ψ: same set of discontinuities.