Lecture 9: Acousto-electric imaging

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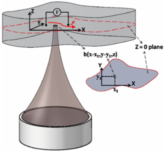
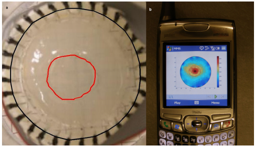
Acousto-electric imaging

- Acousto-electric effect:
  - Acoustic pressure: $p(x,t) = p_0 b(x) a(t)$; p_0: amplitude; b: beam pattern; a: ultrasound waveform.
  - Acousto-electric effect:

$$\Delta\sigma = \eta\sigma p; \quad \eta: \text{ interaction constant.}$$

- Acousto-electric imaging:
  - Change of conductivity induces a change of the boundary voltage measurements.
  - Scan the sample, record the boundary variations, and determine the conductivity distribution.





Acousto-electric imaging

- Acousto-electric imaging: mathematical and numerical framework.
- u the voltage potential induced by a current g in the absence of acoustic perturbations:

$$\begin{cases} \nabla_x \cdot (\sigma(x) \nabla_x u) = 0 \text{ in } \Omega \, , \\ \sigma(x) \dfrac{\partial u}{\partial \nu} = g \text{ on } \partial\Omega \, . \end{cases}$$

- Suppose σ bounded from below and above and known in a neighborhood of the boundary $\partial\Omega$: $\sigma = \sigma_\star$; Set $\Omega' \Subset \Omega$ where σ: unknown.

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- Use of focalized ultrasonic waves with D as a focal spot $\rightarrow$

$$\sigma_\delta(x) \quad = \quad \sigma(x)\Big[1 + \chi(D)(x)\,(\nu(x) - 1)\Big],$$

with $\nu(x) = \eta p(x)$: known.

- u_δ induced by g in the presence of acoustic perturbations localized in the focal spot $D := z + \delta B$:

$$\begin{cases} \nabla_x \cdot (\sigma_\delta(x) \nabla_x u_\delta(x)) = 0 \text{ in } \Omega, \\ \sigma(x) \dfrac{\partial u_\delta}{\partial \nu} = g \text{ on } \partial\Omega. \end{cases}$$

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- Suppose the focal spot D to be a disk and $u \in W^{2,\infty}(D)$. Then,

$$\int_{\partial\Omega} (u_\delta - u) g \, d\sigma = |\nabla u(z)|^2 \int_D \sigma(x) \frac{(\nu(x) - 1)^2}{\nu(x) + 1} dx$$
$$+ O(|D|^{1+\beta}),$$

  - $O(|D|^{1+\beta}) \leq C|D|^{1+\beta} ||\nabla u||_{L^\infty(D)} |\nabla^2 u||_{L^\infty(D)}$ with C: independent of D and u.
  - β: depends only on Ω', ν, $\sup_\Omega \sigma, \min_\Omega \sigma$.

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- Suppose $\sigma \in \mathcal{C}^{0,\alpha}(D)$, $0 \leq \alpha \leq 2\beta \leq 1$. Then

$$\begin{aligned}\mathcal{E}(z) &:= \left(\int_D \frac{(\nu(x)-1)^2}{\nu(x)+1}\,dx\right)^{-1}\int_{\partial\Omega}(u_\delta - u)g\,d\sigma \\ &= \sigma(z)\,|\nabla u(z)|^2 + O(|D|^{\alpha/2}).\end{aligned}$$

- $\mathcal{E}(z)$: electrical energy density; known function from the boundary measurements.

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- Substitute σ by $\mathcal{E}/|\nabla u|^2$.
- Nonlinear PDE (the 0–Laplacian)

$$\left\{ \begin{array}{l} \nabla_x \cdot \left(\dfrac{\mathcal{E}}{|\nabla u|^2} \nabla u \right) = 0 \quad \text{in } \Omega \, , \\ \dfrac{\mathcal{E}}{|\nabla u|^2} \dfrac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega \, . \end{array} \right.$$

- g such that u has no critical point inside Ω.
- Choose two currents g_1 and g_2 s.t. $\nabla u_1 \times \nabla u_2 \neq 0$ for all $x \in \Omega$.

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- Substitution algorithm.
- Polarized measurements:

$$\mathcal{E}_{ij} := \int_{\partial\Omega} (u_\delta^{(j)} - u^{(j)}) g_i \, d\sigma, \quad i,j = 1,2.$$

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$$\begin{cases} \nabla_x \cdot \left(\dfrac{\mathcal{E}_{jj}}{|\nabla u^{(j)}|^2} \nabla u^{(j)} \right) = 0 & \text{in } \Omega \,, \\ \dfrac{\mathcal{E}_{jj}}{|\nabla u^{(j)}|^2} \dfrac{\partial u^{(j)}}{\partial \nu} = g_j & \text{on } \partial\Omega \,. \end{cases}$$

- Proper set of measurements: (g_1, g_2) s.t.

$$|\nabla u^{(j)}| > 0; (\mathcal{E}_{ij})_{i,j} : \text{ invertible in } \Omega'.$$

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- Notion of proper set of measurements:
- Let $g_j \in L^2_0(\partial\Omega)$, $j = 1, 2$ and let $u^{(j)}$ be the voltage potential induced by g_j, in the absence of ultrasonic perturbations, that is,

$$\begin{cases} \nabla \cdot \Big(\sigma(x) \nabla u^{(j)} \Big) = 0 & \text{in } \Omega\,, \\ \sigma \dfrac{\partial u^{(j)}}{\partial \nu} = g_j & \text{on } \partial\Omega\,, \end{cases}$$

with the convention that $\int_{\partial\Omega} u^{(j)} = 0$.
Let

$$\mathcal{E}_{jl}[\sigma](x) := \sigma(x) \nabla u^{(j)}(x) \nabla u^{(l)}(x), \quad j, l = 1, 2\,,$$

and

$$\mathcal{E}[\sigma] := (\mathcal{E}_{jl}[\sigma])_{j,l=1,2}\,.$$

- (g_1, g_2): proper set of measurements if
  - (i) $|\nabla u^{(j)}| > 0$ in Ω';
  - (ii) The matrix $\mathcal{E}[\sigma]$: invertible (or equivalently, $|\det \mathcal{E}[\sigma]| > 0$) for all $x \in \Omega'$.

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- Evidence of the possibility of constructing proper sets of measurements:
- f: smooth function on $\partial\Omega$ s.t. there exist P and Q on $\partial\Omega$ such that $f|_{\Gamma_1}$ and $f|_{\Gamma_2}$: one-to-one, where Γ_1 and Γ_2: two parts of $\partial\Omega$, connecting P and Q.
- For all positive and smooth function σ, $\nabla v \neq 0$ in Ω where v:

$$\left\{ \begin{array}{rcll} \nabla \cdot \sigma \nabla v &=& 0 & \text{in } \Omega\,, \\ v &=& f & \text{on } \partial\Omega\,. \end{array} \right.$$

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- Fix $x_0 \in \Omega$. Let $X \in \Gamma_1$ and $Y \in \Gamma_2$ be s.t.

$$f(X) = f(Y) = v(x_0).$$

- Unique pair due to the one-to-one property of f on Γ_1 and Γ_2.
- v: continuous and v does not attain local extreme value in $\Omega \Rightarrow$ the level set $\{x : v(x) = x_0\}$: curve connecting X and Y.
- Level set divides Ω into two subdomains $\Omega^{\pm}$.
- On Ω^+, $v > v(x_0)$ and on Ω^-, $v < v(x_0)$.
- Hopf's lemma $\Rightarrow \nabla v(x) \neq 0$.
- No critical points.
- Take any $g_j = \partial v_j / \partial \nu, j = 1, 2$, with f_1 and f_2 satisfying: for all $\alpha \in \mathbb{R}$, there exist P and Q in $\partial\Omega$ s.t. $f_1 - \alpha f_2$: one-to-one on each of two curves along $\partial\Omega$ connecting P and Q.
- If $\partial\Omega$ does not contain any line segment, then choose $f_1 = x_1 + M$ and $f_2 = x_2 + M$ for a sufficiently large number M.

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- Linearisation:
  - σ: small perturbation of conductivity profile σ_0: $\sigma = \sigma_0 + \delta\sigma$.
  - $u_0^{(j)}$ and $u^{(j)} = u_0^{(j)} + \delta u^{(j)}$: potentials corresponding to σ_0 and σ with the same Neumann boundary data g_j.
  - $\delta u^{(j)}$ satisfies $\nabla \cdot \left(\sigma \nabla \delta u^{(j)}\right) = -\nabla \cdot (\delta\sigma \nabla u_0^{(j)})$ in Ω with the homogeneous Dirichlet boundary condition.
  - $$\begin{aligned} \mathcal{E}_{jj} &= (\sigma_0 + \delta\sigma)|\nabla(u_0^{(j)} + \delta u^{(j)})|^2 \\ &\approx \sigma_0 |\nabla u_0^{(j)}|^2 + \delta\sigma |\nabla u_0^{(j)}|^2 + 2\sigma_0 \nabla u_0^{(j)} \cdot \nabla \delta u^{(j)} ; \end{aligned}$$
  - Neglecting $\delta\sigma \nabla u_0^{(j)} \cdot \nabla \delta u^{(j)}$ and $\delta\sigma |\nabla \delta u^{(j)}|^2 \Rightarrow$

$$\delta\sigma \approx \frac{\mathcal{E}_{jj}}{\left|\nabla u_0^{(j)}\right|^2} - \sigma_0 - 2\sigma_0 \frac{\nabla \delta u^{(j)} \cdot \nabla u_0^{(j)}}{\left|\nabla u_0^{(j)}\right|^2} .$$

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- Substitution algorithm.
    - Start from an initial guess for the conductivity σ;
    - Solve the corresponding Dirichlet conductivity problem

$$\begin{cases} \nabla \cdot (\sigma \nabla u_0) = 0 & \text{in } \Omega , \\ u_0 = \psi & \text{on } \partial\Omega . \end{cases}$$

    - ψ: Dirichlet data measured as a response to the current $g = g_1$ in absence of elastic deformation;
    - Define the discrepancy between the data and the guessed solution by

$$\epsilon_0 := \frac{\mathcal{E}_{11}}{|\nabla u_0|^2} - \sigma .$$

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- Introduce the corrector, δu, computed as the solution to

$$\left\{ \begin{array}{ll} \nabla \cdot (\sigma \nabla \delta u) = -\nabla \cdot (\varepsilon_0 \nabla u_0) & \text{in } \Omega \, , \\ \delta u = 0 & \text{on } \partial\Omega \, ; \end{array} \right.$$

- Update the conductivity

$$\sigma := \frac{\mathcal{E}_{11} - 2\sigma \nabla \delta u \cdot \nabla u_0}{|\nabla u_0|^2} \, .$$

- Iteratively update the conductivity, alternating directions of currents (with $g = g_2$ and $\mathcal{E}_{11}$ replaced with $\mathcal{E}_{22}$).

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- Optimal control algorithm
- (g_1, g_2): proper set of measurements.
- σ and $\tilde{\sigma}$ be two $\mathcal{C}^1$-conductivities with $\sigma(x_0) = \tilde{\sigma}(x_0)$ for some $x_0 \in \overline{\Omega}$,

$$\| \log \sigma - \log \widetilde{\sigma} \|_{W^{1,\infty}(\Omega)} \leq C \| \mathcal{E}[\sigma] - \mathcal{E}[\tilde{\sigma}] \|_{W^{1,\infty}(\Omega)} .$$

- $\sigma \mapsto \mathcal{E}[\sigma]$: Fréchet differentiable.
- $\Rightarrow$

$$\mathrm{Ker}(d\mathcal{E}[\sigma])|_{H_0^{1,\infty}(\Omega)} = \{0\} ,$$

provided that $\sigma > c_0 > 0$ for some constant c_0.
- For all $h \in W_0^{1,\infty}(\Omega)$,

$$\begin{aligned} \| d\mathcal{E}[\sigma](h) \|_{W_0^{1,\infty}(\Omega)} &= \lim_{t \to 0} \frac{\| \mathcal{E}[\sigma + th] - \mathcal{E}[\sigma] \|_{W_0^{1,\infty}(\Omega)}}{|t|} \\ &\geq \frac{1}{C} \lim_{t \to 0} \frac{\| \log(\sigma + th) - \log \sigma \|_{W_0^{1,\infty}(\Omega)}}{|t|} \\ &= \frac{1}{C\sigma} \| h \|_{W_0^{1,\infty}(\Omega)} . \end{aligned}$$

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- Admissible set of conductivities: open subset of $W^{1,\infty}(\Omega)$

$$A = \{\sigma \in W^{1,2}(\Omega) : c_0 < \sigma < C_0, |\nabla\sigma| < C_1\}\,.$$

- For $j, l = 1, 2$, the map $\sigma \mapsto \mathcal{E}_{jl}[\sigma]$: Fréchet differentiable.
- For $h \in W_0^{1,2}(\Omega)$ s.t. $\sigma + h \in A$,

$$d\mathcal{E}_{jl}[\sigma]h = h\nabla u^{(j)} \cdot \nabla u^{(l)} + \sigma\left[\nabla u^{(j)} \cdot \nabla v^{(l)} + \nabla u^{(l)} \cdot \nabla v^{(j)}\right];$$

$v^{(j)}$: solution of

$$\begin{cases} \nabla \cdot \sigma\nabla v^{(j)} = -\nabla \cdot h\nabla u^{(j)} & \text{in } \Omega\,, \\ \sigma \dfrac{\partial v^{(j)}}{\partial \nu} = 0 & \text{on } \partial\Omega\,, \\ \displaystyle\int_{\partial\Omega} v^{(j)} = 0\,. \end{cases}$$

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- Minimization problem:

$$\min_{\sigma \in A} J[\sigma] := \frac{1}{2} \sum_{j,l=1}^{2} \int_{\Omega} \left| \mathcal{E}_{jl}[\sigma] - \mathcal{E}_{jl}^{(m)} \right|^2 dx \,,$$

$\mathcal{E}_{jl}^{(m)}$: measurements.

- Fréchet derivative of $J[\sigma]$:

$$dJ[\sigma] = \frac{1}{2} \sum_{j,l=1}^{2} (\mathcal{E}_{jl}[\sigma] - \mathcal{E}_{jl}{}^{(m)}) \nabla u^{(j)} \cdot \nabla u^{(l)} + \sum_{j,l=1}^{2} \nabla u^{(j)} \cdot \nabla p^{(j,l)} \,;$$

- $p^{(j,l)}$: solution of the adjoint problem

$$\begin{cases} \nabla \cdot \sigma \nabla p^{(j,l)} = -\nabla \cdot (\mathcal{E}_{jl}[\sigma] - \mathcal{E}_{jl}^{(m)}) \sigma \nabla u^{(l)} & \text{in } \Omega \,, \\ \sigma \frac{\partial p^{(j,l)}}{\partial \nu} = 0 & \text{on } \partial\Omega \,, \\ \int_{\partial\Omega} p^{(j,l)} = 0 \,. \end{cases}$$

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- For $\sigma \in A$,
  - $h \mapsto \nabla v^{(j)} : W_0^{1,2}(\Omega) \to L^2(\Omega)$: compact.
  - $d\mathcal{E}_{jl}$ takes the form $I + \mathrm{compact}$, up to a multiplication by a continuous function.
- Proof:
  - $h \in H_0^1(\Omega)$. $v^{(j)}$ satisfies

$$\begin{aligned} -\sigma\Delta v^{(j)} &= \nabla\sigma \cdot \nabla v^{(j)} + \nabla \cdot h\nabla u^{(j)}\,, \\ &= \nabla\sigma \cdot \nabla v^{(j)} + h\Delta u^{(j)} + \nabla h \cdot \nabla u^{(j)}\,, \\ &= \nabla\sigma \cdot \nabla v^{(j)} - h\frac{\nabla\sigma \cdot \nabla u^{(j)}}{\sigma} + \nabla h \cdot \nabla u^{(j)} \in L^2(\Omega)\,. \end{aligned}$$

  - $\Rightarrow$ $v^{(j)} \in H^2(\Omega)$ and its H^2-norm: bounded by $\|h\|_{H_0^1(\Omega)}$.
  - Compact embeddings

$$H^2(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^2(\Omega)\,.$$

  - $\mathcal{E}[\sigma]^{-1} d\mathcal{E}[\sigma] = I + \mathrm{compact\,operator}.$

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- $\Rightarrow$

$$\|d\mathcal{E}[\sigma]\|_{\mathcal{L}(H_0^1(\Omega),L^2(\Omega))} \geq C\,.$$

- Convergence of the Landweber iteration scheme:
  - Assume that $\sigma^{(0)}$: good initial guess for σ_*.
  - As $n \to +\infty$, the sequence

$$\sigma^{(n+1)} = T\sigma^{(n)} - \eta d\mathcal{E}^*[T\sigma^{(n)}](\mathcal{E}[\sigma^{(n)}] - \mathcal{E}^{(m)})$$

converges to σ_*; T: Hilbert projection of $H^1(\Omega)$ onto A; σ_*: true conductivity distribution, η: step size;
$\mathcal{E}^{(m)} = (\mathcal{E}_{jl}^{(m)})_{j,l=1,2}$.

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Reconstruct the conductivity distribution knowing the internal energies:

- Linearized versions of the nonlinear (zero-Laplacian) PDE problems.
- Optimal control approach: minimize over the conductivity the discrepancy between the computed and reconstructed internal energies.
- Optimal control approach: more efficient approach specially with incomplete internal measurements of the internal energy densities.
- Resolution of order the size of the focal spot + stability (wrt measurement noise).
- Exact inversion formulas: derivatives of the data $\Rightarrow$ used only to obtain a good initial guess.