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Numerical Analysis II

ETH Zürich
D-MATH

Exam Winter 2018

Problem 1 Convergence Order of the Explicit Mid-Point Rule [24 Marks]

We want to numerically estimate the convergence order of the explicit mid-point rule using the IVP

$$\dot{y} = ty + t^3, \quad y(0) = 0, \tag{1.1}$$

for which the exact solution is given by

$$y(t) = 2\mathrm{e}^{t^2/2} - t^2 - 2. \tag{1.2}$$

(1a) Complete the implementation of the explicit mid-point rule

$$y_{k+1} = y_k + hf\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f(t_k, y_k)\right), \quad k \geq 0. \tag{1.3}$$

using the MATLAB template `ExpMidPoint.m`.

(1b) Implement the right hand side and the solution of the initial value problem (1.1) in the MATLAB file `Plot.m`. Plot the exact solution (1.2) as well as the approximation from the explicit mid-point rule and step size $h = 0.2$ up to end point $T = 1$ in the same figure.

(1c) Complete the function `ExpMidPointConv.m` and plot the global error of the explicit mid-point rule at the end-point $T = 1$ as a function of the step size $h = 2^{-k}$, $k = 2, 3, \ldots, 8$ on a double-logarithmic scale. Compute the convergence order of the method using the MATLAB function `polyfit`. Save the figure as `a1conv.eps`. What is the convergence order of the method?

(1d) Find the order of explicit mid-point rule (1.3). Compare the theoretical result with your experimental result in (1c).

HINT: The order of explicit mid-point rule should be independent of IVP that it applies to. Try to calculate the truncation error of explicit mid-point rule.

Problem 2 Mathematical Pendulum [26 Marks]

The mathematical Pendulum equation is given by

$$\begin{cases} \dot{q} &= p, \\ \dot{p} &= -\sin q. \end{cases} \tag{2.1}$$

(2a) ⚀ Show that, (2.1) is a Hamilton system for the Hamilton function

$$H(p, q) = \frac{1}{2}p^2 - \cos q.$$

(2b) ⚀ Show that, the energy $H(p, q)$ is a conserved quantity of (2.1).

(2c) ⚁ Complete the MATLAB-template `pendelEuler.m` by solving (2.1) using the explicit Euler method. Set the time interval to be $[0, 10]$ with 100 steps, and initial value $(p(0), q(0)) = (\pi/2, 0)$. Plot the evolution of the energy as a function of time. From the plots, can you conclude that the energy is conserved or not?

(2d) ⚂ Complete the MATLAB-template `pendelMpr.m` by solving (2.1) using the implicit midpoint method and `nNewton` iterations of the Newton method. Still set the time interval to be $[0, 10]$ with 100 steps, and initial value $(p(0), q(0)) = (\pi/2, 0)$. Execute the function and plot the result for `nNewton=1` and `nNewton=2`. From the plots, can you conclude that the energy is conserved or not?

(2e) ⚂ Complete the MATLAB-template `pendelLeapfrog.m` by solving (2.1) using the Leapfrog method(also called Verlet method or Strömer method). Again set the time interval to be $[0, 10]$ with 100 steps, and initial value $(p(0), q(0)) = (\pi/2, 0)$. Execute the function and plot the result. From the plots, can you conclude that the energy is conserved or not?

Problem 3 Properties of Lipschitz Equation **[50 Marks]**

Consider

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), & t \in [0,T], \\ x(0) = x_0, & x_0 \in \mathbb{R} \end{cases} \tag{3.1}$$

with $f \in C^\infty$ satisfying the Lipschitz condition

$$|f(t,x) - f(t,y)| \le C_f |x - y|, \quad \text{for all } x, y \in \mathbb{R}, t \in [0,T].$$

(3a) ⚁ Prove that (3.1) has a unique solution $x \in C^\infty([0,T])$.

(3b) ⚄ Let $\Delta t > 0$ be the step size and $t_k = k\Delta t$, $k = 0, \cdots, T/\Delta t$(Suppose $T/\Delta t$ is integer). Consider the trapezoidal rule method

$$\begin{cases} x^{k+1} = x^k + \dfrac{\Delta t}{2}(f(t_{k+1}, x^{k+1}) + f(t_k, x^k)) \\ \quad x^0 = x(0) \end{cases} \tag{3.2}$$

for solving (3.1).

Prove that (3.2) is one-step method and (3.2) is consistent.

Given two different sets of initial value x^0 and $\tilde{x}^0$ and $\Delta t < 2/C_f$, Let x^1 and $\tilde{x}^1$ be the one-step result of (3.2) to x^0 and $\tilde{x}^0$, correspondingly. Prove that there exists a constant K such that

$$|x^1 - \tilde{x}^1| \le K|x^0 - \tilde{x}^0|,$$

i.e. (3.2) is stable.

(3c) ⚄ Let

$$T_k(\Delta t) = \frac{x(t_k + \Delta t) - x(t_k)}{\Delta t} - \frac{1}{2}[f(t_k + \Delta t, x(t_k + \Delta t)) + f(t_k, x(t_k))]$$

be the truncation error, where x is the solution to (3.1). Show that

$$T_k(\Delta t) = -\frac{1}{12}(\Delta t)^2 \frac{\mathrm{d}^3 x}{\mathrm{d}t^3}(\tau)$$

where $\tau \in [t_k, t_{k+1}]$ is one fixed number.

HINT: Try to integrate the following integral by parts

$$\int_{t_k}^{t_{k+1}} (t - t_{k+1})(t - t_k) \frac{\mathrm{d}^3 x}{\mathrm{d}t^3}(t)\mathrm{d}t.$$

(3d) ⚄ Suppose that

$$\left| \frac{\mathrm{d}^3 x}{\mathrm{d}t^3} \right| \le M, \quad \forall t \in [0,T]$$

Show that the global error

$$e_n = x(t_n) - x^n$$

satisfies the inequality

$$|e_{n+1}| \le |e_n| + \frac{\Delta t}{2} C_f(|e_{n+1}| + |e_n|) + \frac{(\Delta t)^3}{12} M$$

For $\Delta t > 0$ such that $\Delta t C_f < 2$ deduce that

$$|e_n| \le \frac{(\Delta t)^2 M}{12 C_f}\left[\left(\frac{1+\Delta t C_f/2}{1-\Delta t C_f/2}\right)^n - 1\right]$$

(3e) ⚁ Now modify (3.2) to

$$\begin{cases} x^{k+1} = x^k + \frac{\Delta t}{2}(f(t_{k+1}, x^k + \Delta t f(t_k, x^k)) + f(t_k, x^k)) \\ x^0 = x(0) \end{cases} \tag{3.3}$$

Deduce that the obtained algorithm is convergent.

(3f) ⚃ Consider $f(t,x) = \log\log(4 + x^2)$, $T = 1$ and $x_0 = 1$.

- Verify that C_f can be chosen equal to $1/(2\log 4)$.
- Implement the algorithm (3.3) and find the numerical approximation $x(1)$ in function `function x1=improvedeuler(N)` using template `improvedeuler.m`. Here N denotes the total step number.
- Now set total step number to be $N = 2^k$, $k = 1, 2, \cdots$. Use your code `improvedeuler.m` and template `globalerror.m` to find numerically the smallest k_0 such that for $k \ge k_0$, $|e_{2^k}| \le 10^{-4}$.