H. Ammari
W. Wu
S. Yu

Numerical Analysis II

ETH Zürich
D-MATH

Mid-term test of Spring 2017

Problem 1 [15 Marks]

(1a) Locally solve the equation

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} + x(t) = 0 \\ x(0) = 1 \end{cases}$$

by **seperation of variables**.

(1b) Locally solve the equation

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = \dfrac{x(t)}{t} \\ x(1) = 1 \end{cases}$$

by **change of variables**.

(1c) Locally solve the equation

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} + x(t) = e^t \\ x(0) = 1 \end{cases}$$

by **the method of integrating factors**.

Problem 2 **[12 Marks]**

(2a) Is the equation

$$tx^2 + x - t^2\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

exact in $\mathbb{R}^2$? Explain why.

(2b) Prove that the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{t} = 0 \tag{2.1}$$

for $t \in [1, 2]$ with $x(1) = 0$ is **exact** with the potential $F(t, x) = xt$. How could (2.1) then be solved?

Problem 3 **[8 Marks]**

(3a) Is the following system of ODEs **Hamiltonian**?

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = -q \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = p \end{cases} \tag{3.1}$$

Find an invariant F for (3.1), i.e., a function F such that $F(p(t), q(t)) = \mathrm{Constant}$ for all t.

Problem 4 [12 Marks]

Consider

$$\begin{cases} \dfrac{dx}{dt} = f(t,x), & t \in [0,T], \\ x(0) = x_0, & x_0 \in \mathbb{R} \end{cases} \tag{4.1}$$

(4a) Does $f(t,x) = x^2$ satisfy the Lipschitz condition?

(4b) Does $f(t,x) = a(t)x$ satisfy the Lipschitz condition? Is there existence and uniqueness of a solution to (4.1)?

Problem 5 [8 Marks]

Let A be a 2×2 matrix.

(5a) Find the solution of

$$\begin{cases} \dfrac{dx}{dt}(t) = Ax(t), & t \in [0,T], \\ x(0) = x_0, & x_0 \in \mathbb{R}^2. \end{cases} \tag{5.1}$$

(5b) Is the solution unique?

Problem 6 **[5 Marks]**

Let A be a 2×2 matrix. Let C be a 2×2 matrix. Assume that C is invertible. Prove that the solution of

$$\begin{cases} \dfrac{dx}{dt}(t) = CAC^{-1}x(t), & t \in [0, T], \\ x(0) = x_0, & x_0 \in \mathbb{R}^2, \end{cases} \tag{6.1}$$

is given by

$$x(t) = Ce^{tA}C^{-1}x_0, \quad t \in [0, T].$$