

Mid-term test of Spring 2017

Problem 1

[15 Marks]

(1a) Locally solve the equation

$$\begin{cases} \frac{dx(t)}{dt} + x(t) = 0 \\ x(0) = 1 \end{cases}$$

by **separation of variables**.

(1b) Locally solve the equation

$$\begin{cases} \frac{dx(t)}{dt} = \frac{x(t)}{t} \\ x(1) = 1 \end{cases}$$

by **change of variables**.

(1c) Locally solve the equation

$$\begin{cases} \frac{dx(t)}{dt} + x(t) = e^t \\ x(0) = 1 \end{cases}$$

by the method of integrating factors.

Problem 2

[12 Marks]

(2a) Is the equation

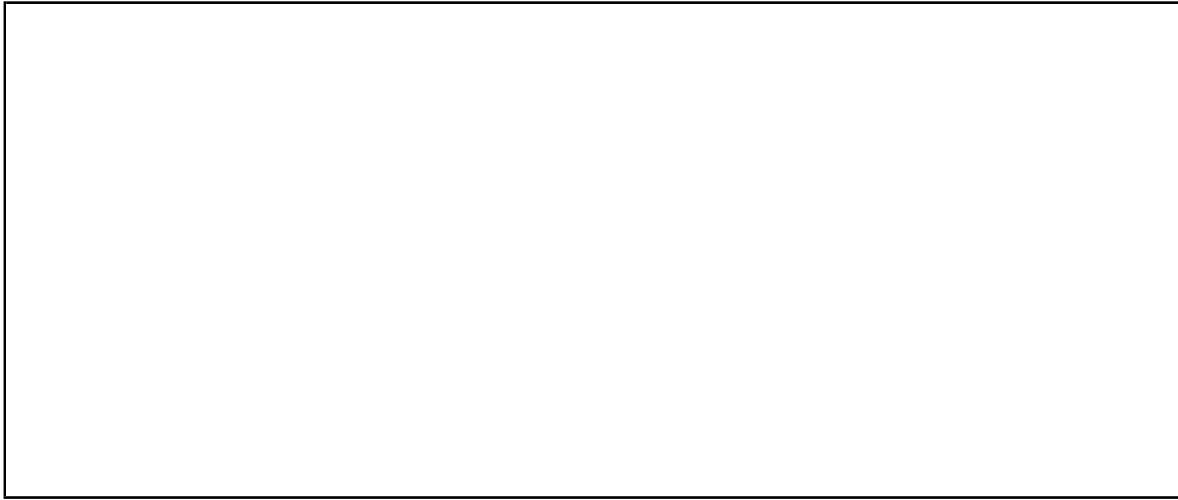
$$tx^2 + x - t^2 \frac{dx}{dt} = 0$$

exact in \mathbb{R}^2 ? Explain why.

(2b) Prove that the equation

$$\frac{dx}{dt} + \frac{x}{t} = 0 \tag{2.1}$$

for $t \in [1, 2]$ with $x(1) = 0$ is **exact** with the potential $F(t, x) = xt$. How could (2.1) then be solved?



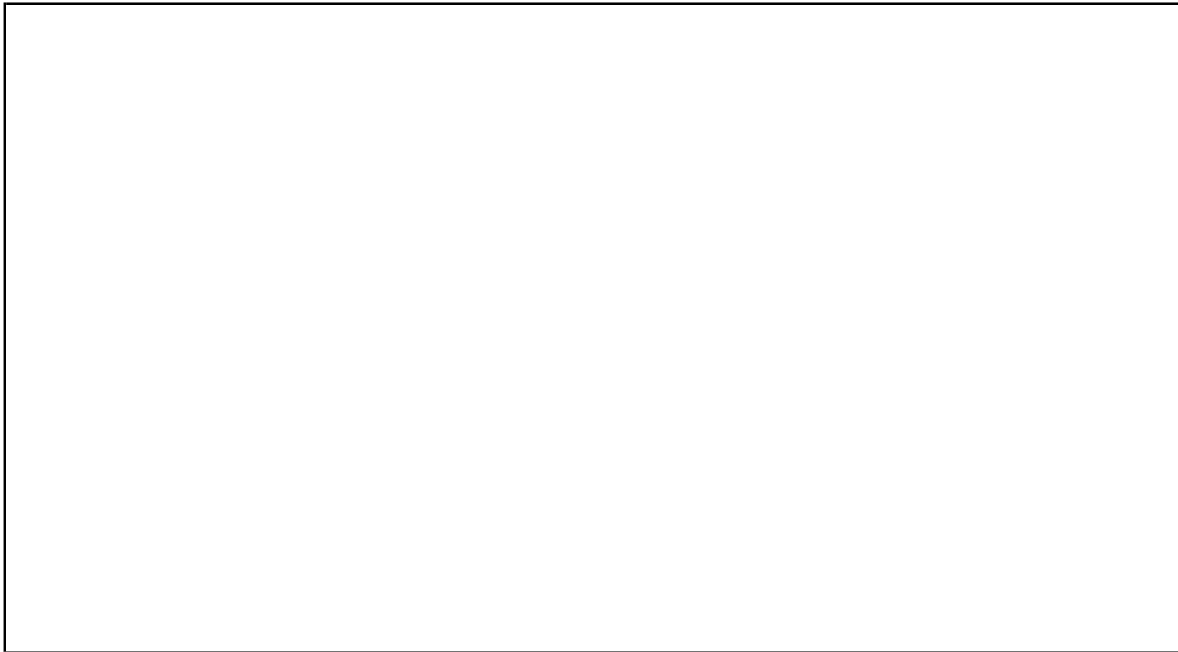
Problem 3

[8 Marks]

(3a) Is the following system of ODEs **Hamiltonian**?

$$\begin{cases} \frac{dp}{dt} = -q \\ \frac{dq}{dt} = p \end{cases} \quad (3.1)$$

Find an invariant F for (3.1), i.e., a function F such that $F(p(t), q(t)) = \text{Constant}$ for all t .



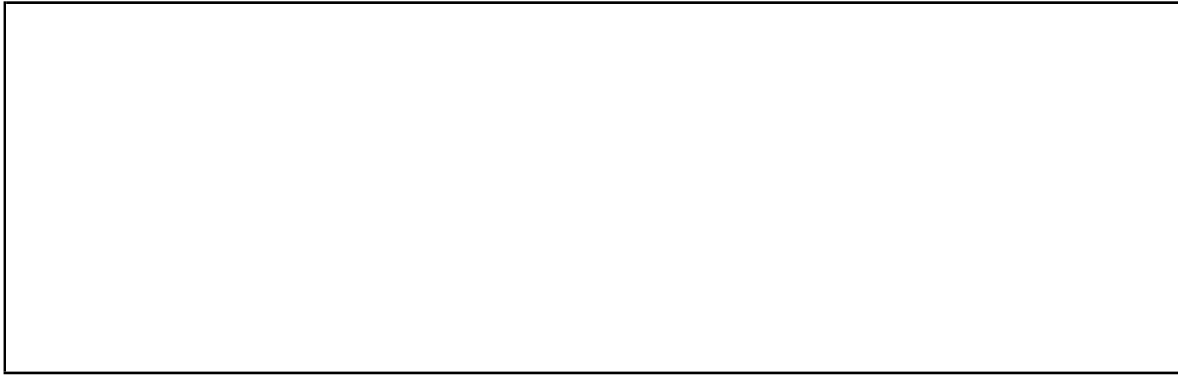
Problem 4**[12 Marks]**

Consider

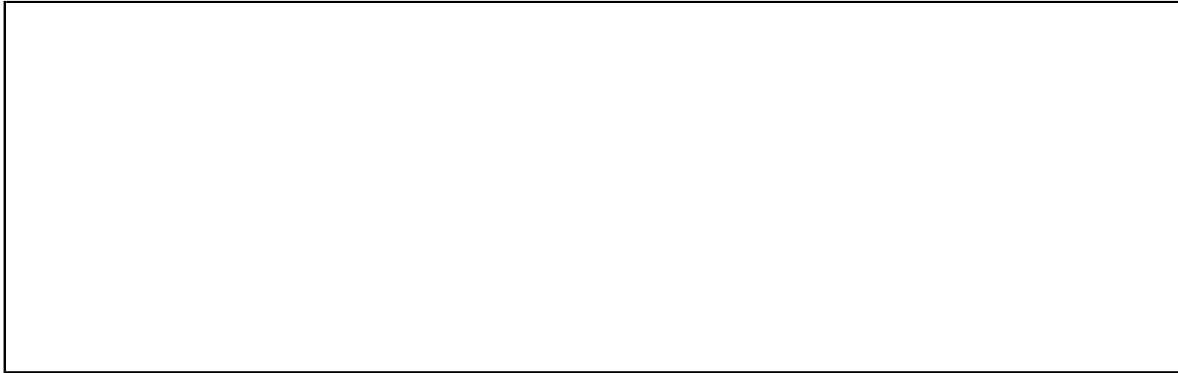
$$\begin{cases} \frac{dx}{dt} = f(t, x), & t \in [0, T], \\ x(0) = x_0, & x_0 \in \mathbb{R} \end{cases} \quad (4.1)$$

(4a) Does $f(t, x) = x^2$ satisfy the Lipschitz condition?**(4b)** Does $f(t, x) = a(t)x$ satisfy the Lipschitz condition? Is there existence and uniqueness of a solution to (4.1)?**Problem 5****[8 Marks]**Let A be a 2×2 matrix.**(5a)** Find the solution of

$$\begin{cases} \frac{dx}{dt}(t) = Ax(t), & t \in [0, T], \\ x(0) = x_0, & x_0 \in \mathbb{R}^2. \end{cases} \quad (5.1)$$



(5b) Is the solution unique?



Problem 6

[5 Marks]

Let A be a 2×2 matrix. Let C be a 2×2 matrix. Assume that C is invertible. Prove that the solution of

$$\begin{cases} \frac{dx}{dt}(t) = CAC^{-1}x(t), & t \in [0, T], \\ x(0) = x_0, & x_0 \in \mathbb{R}^2, \end{cases} \quad (6.1)$$

is given by

$$x(t) = Ce^{tA}C^{-1}x_0, \quad t \in [0, T].$$

