

## Final Exam Spring 2019

### Problem 1

[15 Marks]

(1a) Consider

$$\begin{cases} \frac{dx}{dt} = f(t, x), & t \in [0, T], \\ x(0) = x_0 \in \mathbb{R} \end{cases} \quad (1.1)$$

with  $f \in C^0([0, T] \times \mathbb{R})$  satisfying the Lipschitz condition

$$|f(t, x) - f(t, y)| \leq C_f |x - y|$$

for any  $x, y \in \mathbb{R}$ , any  $t \in [0, T]$  and some positive constant  $C_f$ .

Does (1.1) have a unique solution  $x(t) \in C^1([0, T])$ ? Please explain why.

(1b) Does  $f(t, x) = a(t)x$  with  $a \in C^0([0, T])$  satisfy the Lipschitz condition? Please explain why.

(1c) Locally solve the equation

$$\begin{cases} \frac{dx(t)}{dt} = a(t)x, \\ x(0) = x_0 > 0 \end{cases}$$

with  $a \in C^0([0, T])$  by **the method of integrating factors**.

(1d) Locally solve the equation

$$\begin{cases} \frac{dx(t)}{dt} = a(t)x, \\ x(0) = x_0 \in \mathbb{R} \end{cases}$$

with  $a \in C^0([0, T])$  by **the method of separation of variables**.

## Problem 2

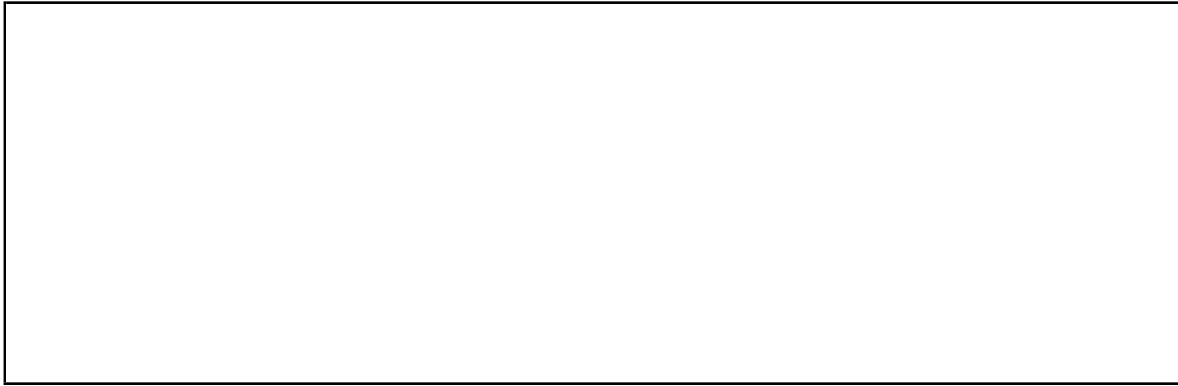
[15 Marks]

(2a) Consider the system of equations

$$\begin{cases} \frac{dp}{dt} = \sin q, & t \geq 0, \\ \frac{dq}{dt} = p, & t \geq 0, \end{cases} \quad (2.1)$$

with initial value  $p(0) = p_0 \in \mathbb{R}$  and  $q(0) = q_0 \in \mathbb{R}$ . Is (2.1) a Hamiltonian system? Please explain why.

(2b) Find an invariant for (2.1), i.e. a function  $F$  such that  $F(p(t), q(t)) = \text{Constant}$  for all  $t \geq 0$ .



(2c) Now consider a different system of equations

$$\begin{cases} \frac{dp}{dt} = q, & t \geq 0, \\ \frac{dq}{dt} = p - p^2, & t \geq 0, \end{cases} \quad (2.2)$$

with initial value  $p(0) = p_0 \in \mathbb{R}$  and  $q(0) = q_0 \in \mathbb{R}$ . Prove that  $I(p, q) := \frac{1}{2}q^2 - \frac{1}{2}p^2 + \frac{1}{3}p^3$  is an invariant for (2.2).



(2d) Is (2.2) a Hamiltonian system? If so, please find its Hamiltonian.

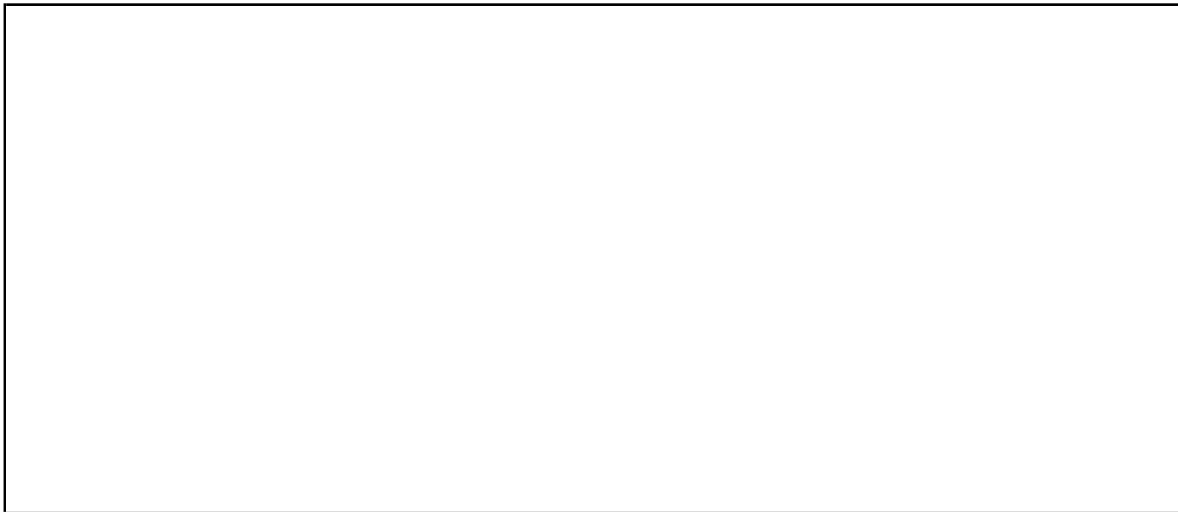


### Problem 3

[12 Marks]

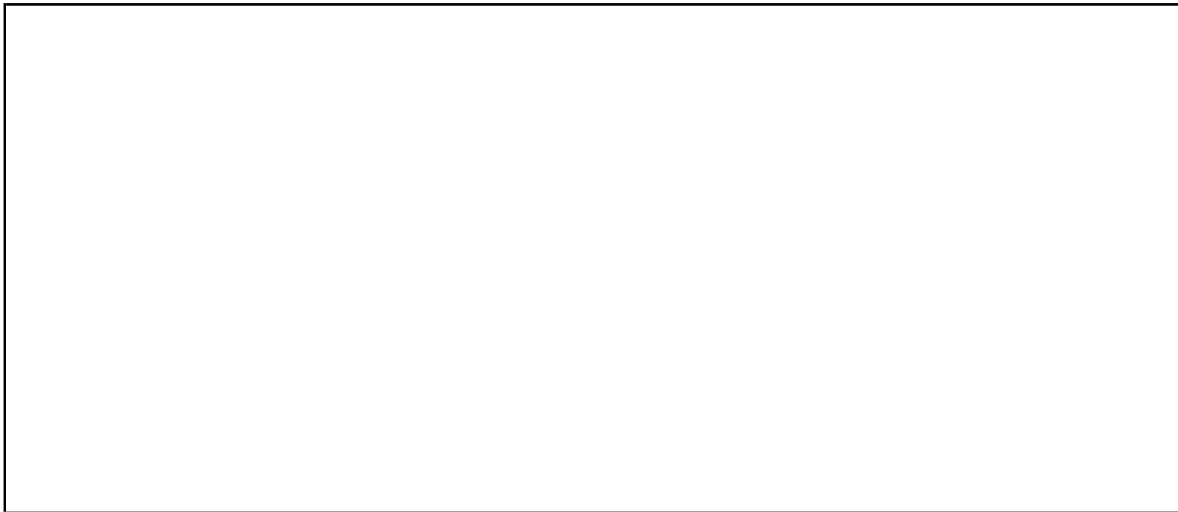
(3a) Verify that  $e^t$  and  $te^t$  are solutions to

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 0. \quad (3.1)$$



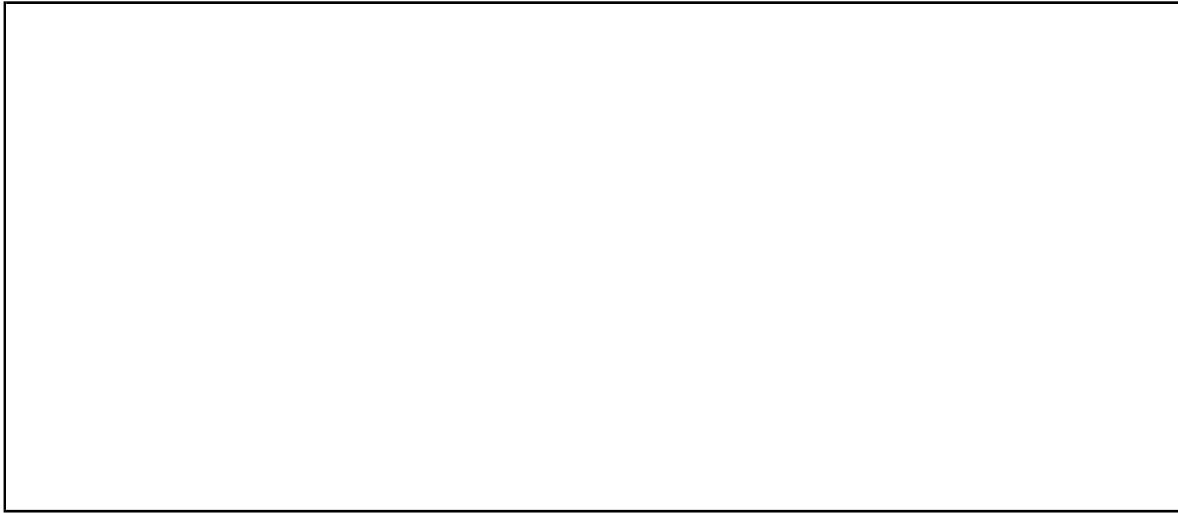
(3b) Solve the following initial value problem

$$\begin{cases} \frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 0, & t \geq 0, \\ x(0) = 0, \\ \frac{dx}{dt}(0) = 1. \end{cases} \quad (3.2)$$

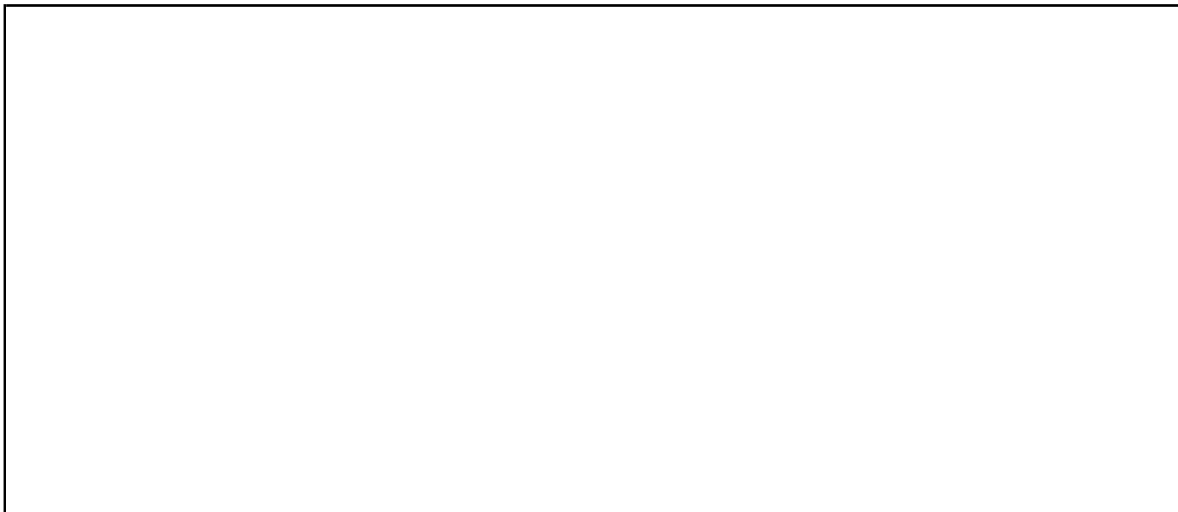


(3c) Verify that  $x(t) = \frac{1}{2}t^2e^t$  satisfies

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t. \quad (3.3)$$



**(3d)** Solve the equation (3.3) with initial value  $x(0) = 0$  and  $\frac{dx}{dt}(0) = 1$ .



**Problem 4****[18 Marks]**

Consider the equation system

$$\begin{cases} \frac{dx}{dt} = 2x(t) + y(t), & t \geq 0, \\ \frac{dy}{dt} = -x(t), & t \geq 0. \end{cases} \quad (4.1)$$

with initial conditions  $x(0) = y(0) = 1$ .**(4a)** Reformulate the problem into the form of

$$\frac{d\mathbf{X}}{dt} = \mathbf{A}\mathbf{X}$$

with initial condition  $\mathbf{X}(0) = \mathbf{X}_0$ . Please specify  $\mathbf{A}$ ,  $\mathbf{X}$  and  $\mathbf{X}_0$ .**(4b)** Try to find matrix  $C$  such that  $C^{-1}AC = D + N$ , where  $D$  is a diagonal matrix, and  $N^2 = 0$ . Then calculate explicitly  $e^{At}$ .

HINT: You may want to recall the knowledge of Jordan decomposition.

**(4c)** Use the result in (4a) and (4b) to explicitly solve (4.1).