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Numerical Analysis II

ETH Zürich
D-MATH

Final Exam Spring 2019

Problem 1 **[15 Marks]**

(1a) Consider

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), \quad t \in [0,T], \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{1.1}$$

with $f \in C^0([0,T] \times \mathbb{R})$ satisfying the Lipschitz condition

$$|f(t,x) - f(t,y)| \leq C_f |x - y|$$

for any $x, y \in \mathbb{R}$, any $t \in [0,T]$ and some positive constant C_f.

Does (1.1) have a unique solution $x(t) \in C^1([0,T])$? Please explain why.

(1b) Does $f(t,x) = a(t)x$ with $a \in C^0([0,T])$ satisfy the Lipschitz condition? Please explain why.

(1c) Locally solve the equation

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = a(t)x, \\ x(0) = x_0 > 0 \end{cases}$$

with $a \in C^0([0,T])$ by **the method of integrating factors**.

(1d) Locally solve the equation

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = a(t)x, \\ x(0) = x_0 \in \mathbb{R} \end{cases}$$

with $a \in C^0([0,T])$ by **the method of separation of variables**.

Problem 2 **[15 Marks]**

(2a) Consider the system of equations

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = \sin q, \quad t \geq 0, \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = p, \qquad t \geq 0, \end{cases} \tag{2.1}$$

with initial value $p(0) = p_0 \in \mathbb{R}$ and $q(0) = q_0 \in \mathbb{R}$. Is (2.1) a Hamiltonian system? Please explain why.

(2b) Find an invariant for (2.1), i.e. a function F such that $F(p(t), q(t)) = \text{Constant}$ for all $t \geq 0$.

(2c) Now consider a different system of equations

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = q, & t \geq 0, \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = p - p^2, & t \geq 0, \end{cases} \tag{2.2}$$

with initial value $p(0) = p_0 \in \mathbb{R}$ and $q(0) = q_0 \in \mathbb{R}$. Prove that $I(p,q) := \frac{1}{2}q^2 - \frac{1}{2}p^2 + \frac{1}{3}p^3$ is an invariant for (2.2).

(2d) Is (2.2) a Hamiltonian system? If so, please find its Hamiltonian.

Problem 3 **[12 Marks]**

(3a) Verify that e^t and te^t are solutions to

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0. \tag{3.1}$$

(3b) Solve the following initial value problem

$$\begin{cases} \dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2} - 2\dfrac{\mathrm{d}x}{\mathrm{d}t} + x = 0, \quad t \geq 0, \\ x(0) = 0, \\ \dfrac{\mathrm{d}x}{\mathrm{d}t}(0) = 1. \end{cases} \tag{3.2}$$

(3c) Verify that $x(t) = \frac{1}{2}t^2 e^t$ satisfies

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}x}{\mathrm{d}t} + x = e^t. \tag{3.3}$$

(3d) Solve the equation (3.3) with initial value $x(0) = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}(0) = 1$.

Problem 4 **[18 Marks]**

Consider the equation system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2x(t) + y(t), & t \geq 0, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x(t), & t \geq 0. \end{cases} \tag{4.1}$$

with initial conditions $x(0) = y(0) = 1$.

(4a) Reformulate the problem into the form of

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{X}$$

with initial condition $\boldsymbol{X}(0) = \boldsymbol{X}_0$. Please specify $\boldsymbol{A}$, $\boldsymbol{X}$ and $\boldsymbol{X}_0$.

(4b) Try to find matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^{-1}\boldsymbol{A}\boldsymbol{C} = \boldsymbol{D} + \boldsymbol{N}$, where $\boldsymbol{D}$ is a diagonal matrix, and $\boldsymbol{N}^2 = \boldsymbol{0}$. Then calculate explicitly $e^{\boldsymbol{A}t}$.

HINT: You may want to recall the knowledge of Jordan decomposition.

(4c) Use the result in (4a) and (4b) to explicitly solve (4.1).