

## Mid Term Test Spring 2019

### Problem 1

[20 Marks]

Consider for  $T > 0$

$$\begin{cases} \frac{dx}{dt} = f(t, x), & t \in [0, T], \\ x(0) = x_0 \in \mathbb{R} \end{cases} \quad (1.1)$$

with  $f \in C^0([0, T] \times \mathbb{R})$  satisfying the Lipschitz condition

$$|f(t, x) - f(t, y)| \leq C_f |x - y|$$

for any  $x, y \in \mathbb{R}$ , any  $t \in [0, T]$  and some positive constant  $C_f$ .

**(1a)** Does (1.1) have a unique solution  $x(t) \in C^1([0, T])$ ? Please explain why.

**(1b)** For  $k$  a positive integer, if  $f \in C^k([0, T] \times \mathbb{R})$ , does  $x \in C^{k+1}([0, T])$ ? If  $f \in C^\infty([0, T] \times \mathbb{R})$ , does  $x \in C^\infty([0, T])$ ?

**(1c)** Does  $f(t, x) = t^2x$  satisfy the Lipschitz condition? Justify.

(1d) Solve the equation

$$\begin{cases} \frac{dx(t)}{dt} = t^2x, & t \in [0, T], & x \in \mathbb{R}, \\ x(0) = x_0 \end{cases}$$

by the method of integrating factors.

(1e) Solve the equation

$$\begin{cases} \frac{dx(t)}{dt} = t^2x, & t \in [0, T], & x \in \mathbb{R}, \\ x(0) = x_0 \end{cases}$$

by the method of separation of variables.

(1f) If we regard  $x(t)$  also as a function of the initial value  $x_0$ , what is the differential equation satisfied by the derivative with respect to  $t$  of  $\partial x(t)/\partial x_0$ ? Is it a linear equation?

## Problem 2

[20 Marks]

We say that a system of equations is Hamiltonian if there exists a Hamiltonian function  $H(p, q)$  such that for  $T > 0$

$$\begin{cases} \frac{dp}{dt} = -\frac{\partial H}{\partial q}, & t \in [0, T], \\ \frac{dq}{dt} = \frac{\partial H}{\partial p}, & t \in [0, T]. \end{cases} \quad (2.1)$$

(2a) Consider the system of equations for  $T > 0$

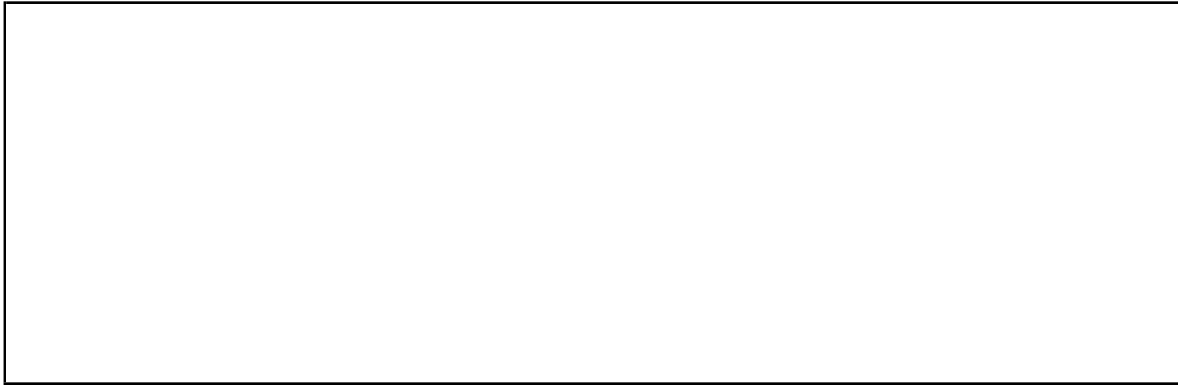
$$\begin{cases} \frac{dp}{dt} = \cos q, & t \in [0, T], \\ \frac{dq}{dt} = p, & t \in [0, T], \end{cases} \quad (2.2)$$

with initial value  $p(0) = p_0 \in \mathbb{R}$  and  $q(0) = q_0 \in \mathbb{R}$ . Is (2.2) a Hamiltonian system? Please explain why.

(2b) An invariant for (2.2) is a function  $F$  such that  $F(p(t), q(t)) = \text{Constant}$  for all  $t \geq 0$ . Prove that  $H(p, q) = \frac{1}{2}p^2 - \sin q$  is an invariant.

(2c) Now consider a different Hamiltonian  $I(p, q) = \frac{1}{2}q^2 - \frac{1}{2}p^2 + \frac{1}{3}p^3$ . Find a system of equation for which  $I(p, q)$  is an invariant.

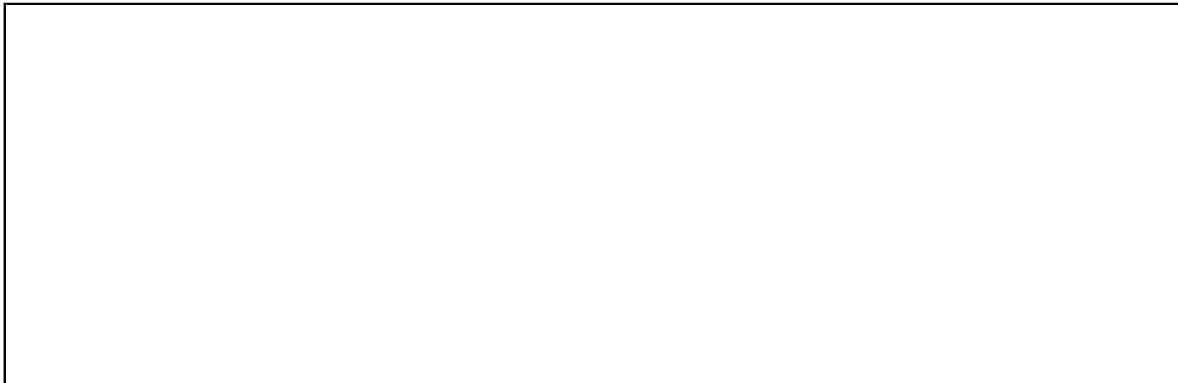
(2d) Is the system from (2c) a Hamiltonian system? If so, please find its Hamiltonian.



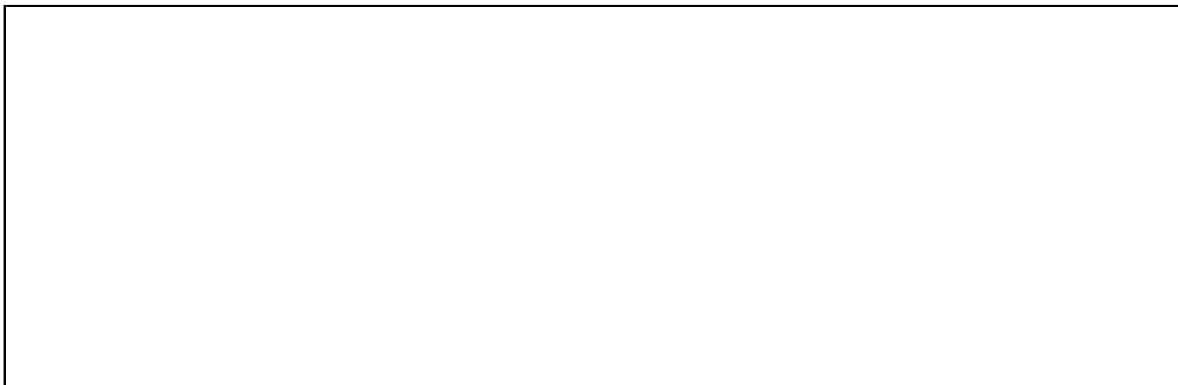
(2e) Let  $J$  denote the matrix  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Consider the system of linear equations

$$\begin{cases} \frac{dx}{dt} = J^{-1}x, \\ x(0) = x_0 \in \mathbb{R}^2. \end{cases} \quad (2.3)$$

Prove that (2.3) is a Hamilton system associated with the Hamiltonian  $H(x) = \frac{1}{2}\|x\|^2$ , where  $\|x\|$  denotes the norm of  $x$ .



(2f) Define the flow  $\Phi_t$  associated with (2.3) by  $\Phi_t(x_0) = x(t)$ , where  $x(t)$  is the solution to (2.3). Prove that  $\Phi_t(x_0) = e^{tJ^{-1}}x_0$ .



**Problem 3****[20 Marks]**Let  $A, B$  be  $2 \times 2$  matrices.**(3a)** Is  $e^{t(A+B)} = e^{tA}e^{tB}$ ?**(3b)** Let

$$A = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Show that  $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$  solves  $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ .**(3c)** Show that  $A$  is nilpotent, ie that there exists  $N \in \mathbb{N}$  such that  $A^n = 0$  for  $n \geq N$ **(3d)** Calculate  $e^{tA}$



(3e) Use the result in (3b) and (3d) to explicitly solve

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}.$$

