H. Ammari
A. Scapin
A. Vanel

ETH Zürich
D-MATH

Mid Term Test Spring 2019

Problem 1 **[20 Marks]**

Consider for $T > 0$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), \quad t \in [0,T], \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{1.1}$$

with $f \in C^0([0,T] \times \mathbb{R})$ satisfying the Lipschitz condition

$$|f(t,x) - f(t,y)| \leq C_f |x - y|$$

for any $x, y \in \mathbb{R}$, any $t \in [0,T]$ and some positive constant C_f.

(1a) Does (1.1) have a unique solution $x(t) \in C^1([0,T])$? Please explain why.

(1b) For k a positive integer, if $f \in C^k([0,T] \times \mathbb{R})$, does $x \in C^{k+1}([0,T])$? If $f \in C^\infty([0,T] \times \mathbb{R})$, does $x \in C^\infty([0,T])$?

(1c) Does $f(t,x) = t^2 x$ satisfy the Lipschitz condition? Justify.

(1d) Solve the equation

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = t^2 x, \quad t \in [0,T], \quad x \in \mathbb{R}, \\ x(0) = x_0 \end{cases}$$

by **the method of integrating factors**.

(1e) Solve the equation

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = t^2 x, \quad t \in [0,T], \quad x \in \mathbb{R}, \\ x(0) = x_0 \end{cases}$$

by **the method of separation of variables**.

(1f) If we regard $x(t)$ also as a function of the initial value x_0, what is the differential equation satisfied by the derivative with respect to t of $\partial x(t)/\partial x_0$? Is it a linear equation?

Problem 2 **[20 Marks]**

We say that a system of equations is Hamiltonian if there exists a Hamiltonian function $H(p,q)$ such that for $T > 0$

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = -\dfrac{\partial H}{\partial q}, \quad t \in [0,T], \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = \dfrac{\partial H}{\partial p}, \quad t \in [0,T]. \end{cases} \tag{2.1}$$

(2a) Consider the system of equations for $T > 0$

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = \cos q, & t \in [0, T], \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = p, & t \in [0, T], \end{cases} \tag{2.2}$$

with initial value $p(0) = p_0 \in \mathbb{R}$ and $q(0) = q_0 \in \mathbb{R}$. Is (2.2) a Hamiltonian system? Please explain why.

(2b) An invariant for (2.2) is a function F such that $F(p(t), q(t)) = \text{Constant}$ for all $t \geq 0$. Prove that $H(p, q) = \frac{1}{2}p^2 - \sin q$ is an invariant.

(2c) Now consider a different Hamiltonian $I(p, q) = \frac{1}{2}q^2 - \frac{1}{2}p^2 + \frac{1}{3}p^3$. Find a system of equation for which $I(p, q)$ is an invariant.

(2d) Is the system from (2c) a Hamiltonian system? If so, please find its Hamiltonian.

(2e) Let J denote the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Consider the system of linear equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = J^{-1}x, \\ x(0) = x_0 \in \mathbb{R}^2. \end{cases} \tag{2.3}$$

Prove that (2.3) is a Hamilton system associated with the Hamiltonian $H(x) = \frac{1}{2}||x||^2$, where $||x||$ denotes the norm of x.

(2f) Define the flow Φ_t associated with (2.3) by $\Phi_t(x_0) = x(t)$, where $x(t)$ is the solution to (2.3). Prove that $\Phi_t(x_0) = e^{tJ^{-1}}x_0$.

Problem 3 **[20 Marks]**

Let $\boldsymbol{A}$,$\boldsymbol{B}$ be 2×2 matrices.

(3a) Is $e^{t(\boldsymbol{A}+\boldsymbol{B})} = e^{t\boldsymbol{A}}e^{t\boldsymbol{B}}$?

(3b) Let

$$\boldsymbol{A} = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Show that $\boldsymbol{x}(t) = e^{t\boldsymbol{A}}\boldsymbol{x}_0$ solves $\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x}$.

(3c) Show that $\boldsymbol{A}$ is nilpotent, ie that there exists $N \in \mathbb{N}$ such that $\boldsymbol{A}^n = 0$ for $n \geq N$

(3d) Calculate $e^{t\boldsymbol{A}}$

(3e) Use the result in (3b) and (3d) to explicitly solve

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x}.$$