

## End Term Spring 2017

### Problem 1

[28 Marks]

Consider

$$\begin{cases} \frac{dx}{dt} = f(t, x), & t \in [0, T] \\ x(0) = x_0 \in \mathbb{R} \end{cases} \quad (1.1)$$

with  $f \in C^\infty$  subject to the Lipschitz condition  $|f(t, x) - f(t, y)| \leq C|x - y|$  for  $\forall x, y \in \mathbb{R}$ ,  $\forall t \in [0, T]$ .

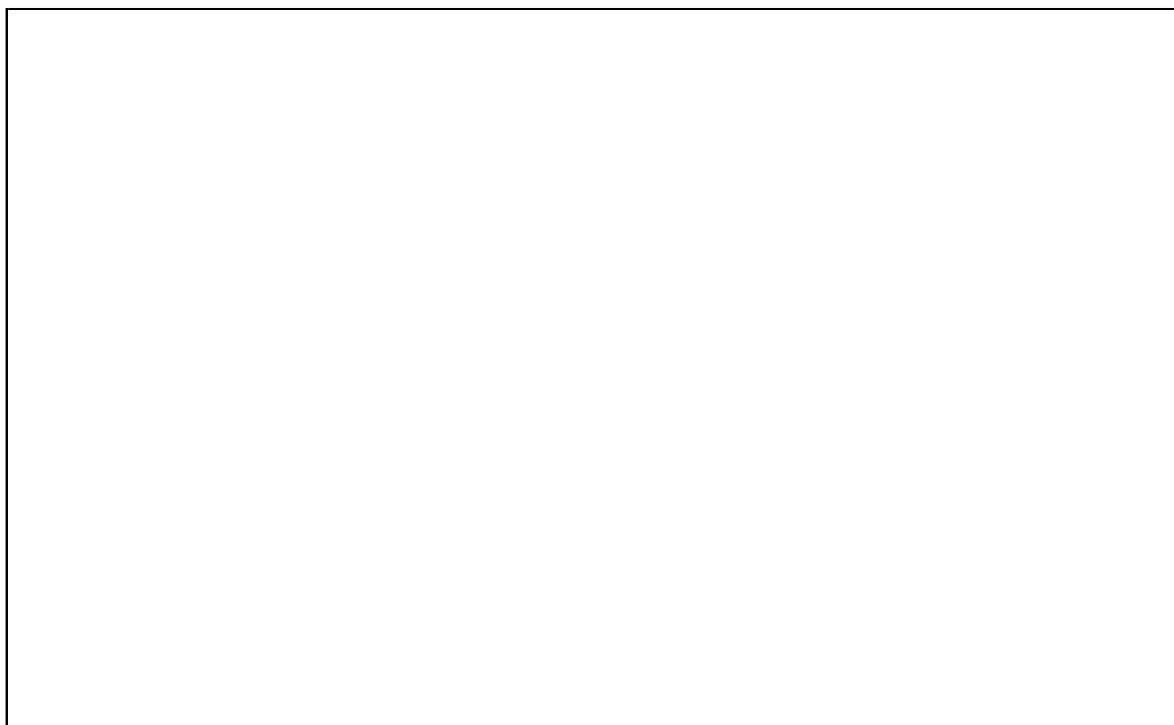
We use the following numerical scheme for (1.1):

$$x^{k+1} = x^{k-1} + 2\Delta t f(t_k, x^k) \quad (1.2)$$

(1a) Define the Truncation error by

$$T_k(\Delta t) = \frac{x(t + \Delta t) - x(t - \Delta t) - 2\Delta t f(t, x)}{2\Delta t},$$

prove that the scheme (1.2) is of order 2.



**(1b)** What kind of method (1.2) is?

Explicit One-step       Explicit Two-step   
Implicit One-step       Implicit Two-step

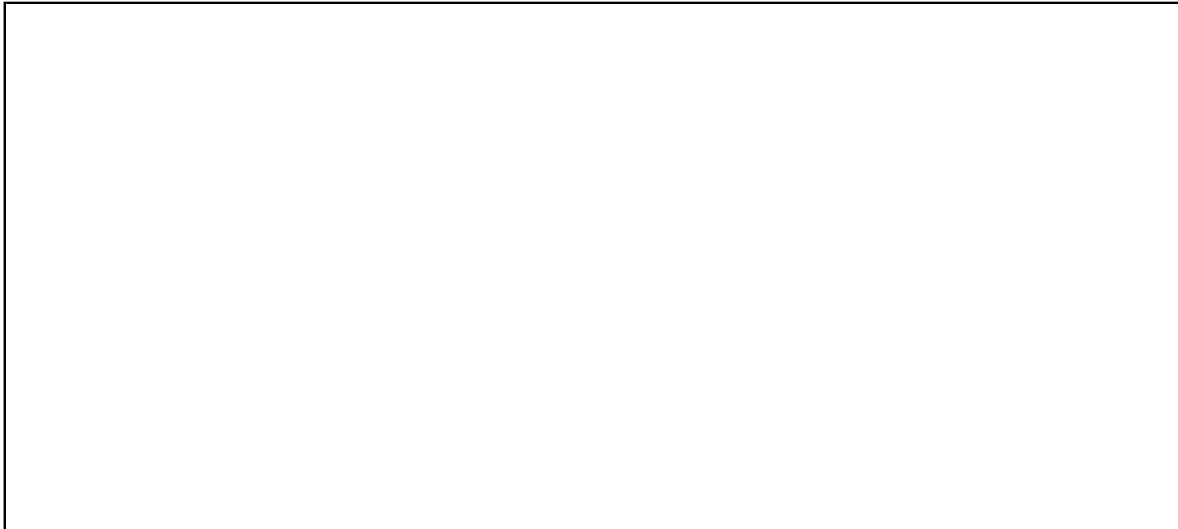
**(1c)** Is scheme (1.2) consistent with (1.1)? Prove that.

**(1d)** Suppose that  $\{x^2, \dots, x^k\}$  and  $\{\tilde{x}^2, \dots, \tilde{x}^k\}$  have been generated by (1.2) but with different initial data  $x^0, x^1$  and  $\tilde{x}^0, \tilde{x}^1$ . Prove by induction that

$$|x^k - \tilde{x}^k| \leq C \max\{|x^0 - \tilde{x}^0|, |x^1 - \tilde{x}^1|, \dots, |x^{k-1} - \tilde{x}^{k-1}|\}.$$

Is (1.2) stable? Prove that.

(1e) Is (1.2) convergent? Prove that.



## Problem 2

[13 Marks]

Consider the scheme

$$x^{k+1} = x^k + \frac{\Delta t}{2}(\kappa_1 + \kappa_2) \quad (2.1)$$

where

$$\begin{aligned} \kappa_1 &= f(t_k, x^k), \\ \kappa_2 &= f(t_{k+1}, x^k + \Delta t \kappa_1). \end{aligned}$$

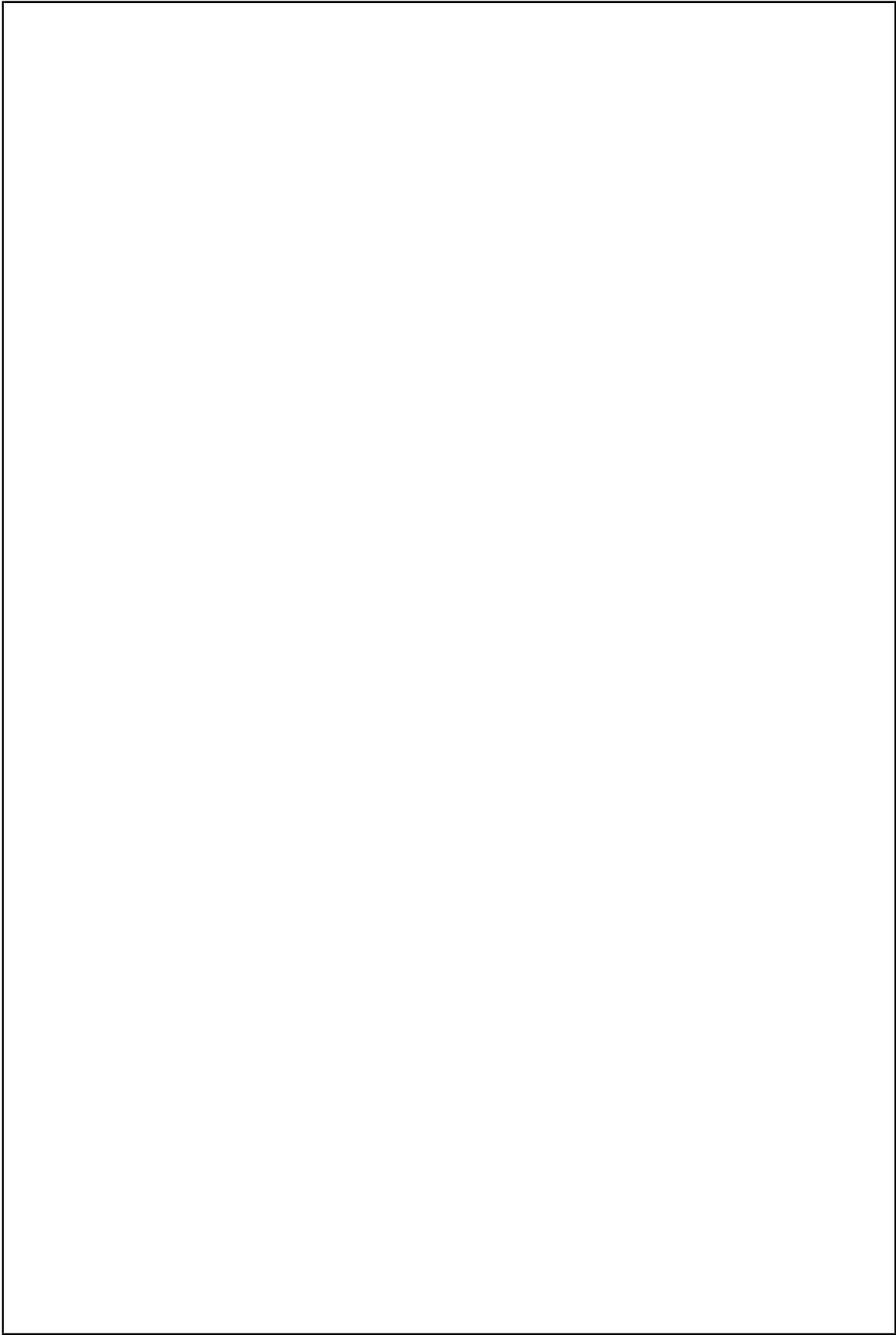
(2a) Is (2.1) a one-step method or two-step method?

One-step method       Two-step Method

(2b) Define the truncation error by

$$T_k(\Delta t) = \frac{x(t + \Delta t) - x(t)}{\Delta t} - \frac{1}{2}[f(t, x(t)) + f(t + \Delta t, x(t) + \Delta t f(t, x(t)))].$$

Prove that (2.1) is consistent and of at least order 2 by either direct calculation or properties from Runge-Kutta method. Attention: Please write out coefficients  $\{a_{ij}\}$ ,  $\{b_i\}$ ,  $\{c_i\}$  and order conditions explicitly (no need to prove) if choosing to use RK properties.



**Problem 3****[19 Mark]**

Consider the Hamiltonian system

$$\begin{cases} \frac{dp}{dt} = -q, \\ \frac{dq}{dt} = p, \end{cases} \quad (3.1)$$

where  $p(t), q(t) \in \mathbb{R}$  and the numerical scheme

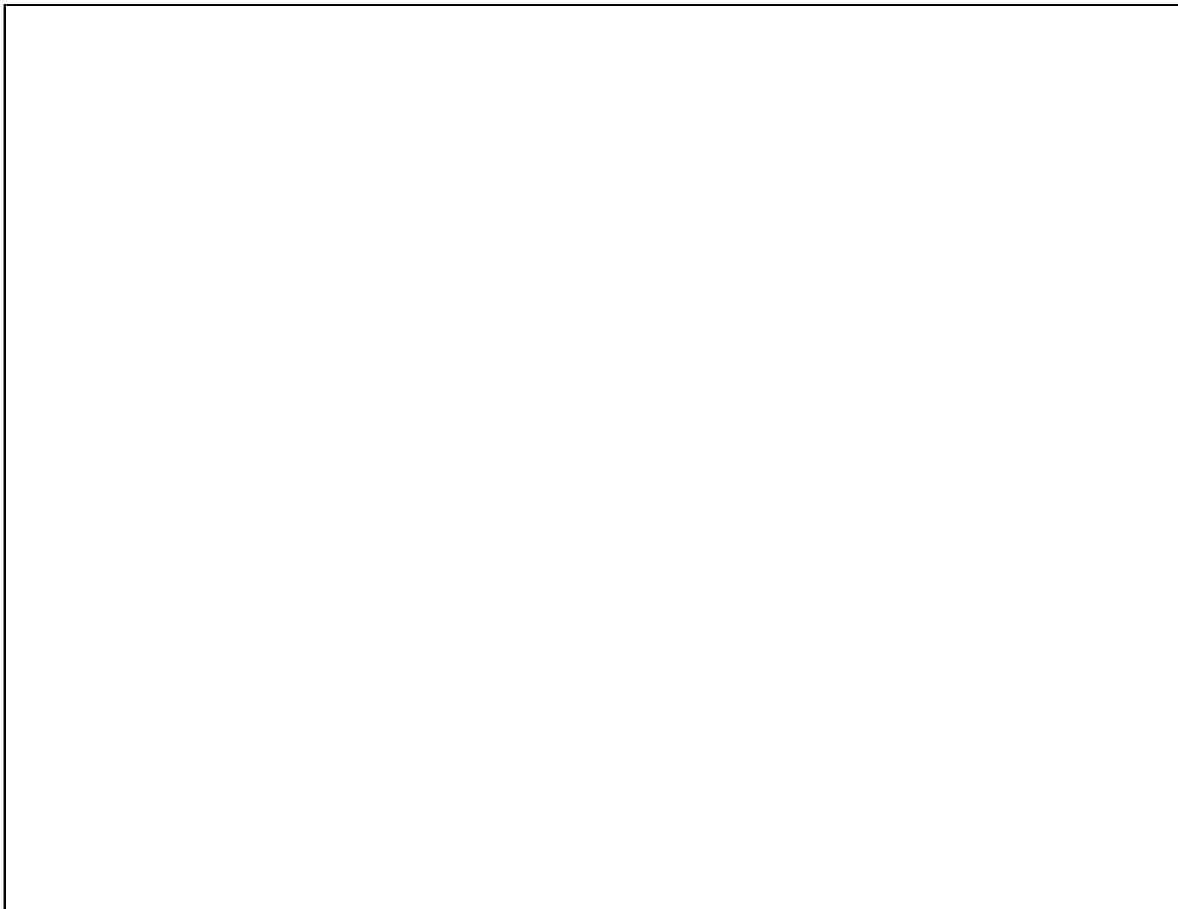
$$\begin{cases} p^{k+1} = p^k - \Delta t q^k \\ q^{k+1} = q^k + \Delta t p^{k+1}. \end{cases} \quad (3.2)$$

**(3a)** What kind of method (3.2) is?

- Explicit Euler method     Implicit Euler method   
Explicit Leapfrog method     Implicit Leapfrog method

**(3b)** Define the numerical flow  $\Phi_{\Delta t} : (p^k, q^k) \mapsto (p^{k+1}, q^{k+1})$ . Prove that (3.2) is symplectic, i.e., the numerical flow satisfies

$$(\Phi'_{\Delta t})^T J \Phi'_{\Delta t} = J, \quad \text{where } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$



(3c) Is the Hamiltonian  $H$  associated with (3.1) preserved by the scheme (3.2)? Prove that.

