H. Ammari
W. Wu
S. Yu

Numerical Analysis II

ETH Zürich
D-MATH

End Term Spring 2017

Problem 1 **[28 Marks]**

Consider

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), \quad t \in [0,T] \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{1.1}$$

with $f \in C^\infty$ subject to the Lipschitz condition $|f(t,x) - f(t,y)| \leq C|x-y|$ for $\forall x, y \in \mathbb{R}$, $\forall t \in [0,T]$.

We use the following numerical scheme for (1.1):

$$x^{k+1} = x^{k-1} + 2\Delta t f(t_k, x^k) \tag{1.2}$$

(1a) Define the Truncation error by

$$T_k(\Delta t) = \frac{x(t+\Delta t) - x(t-\Delta t) - 2\Delta t f(t,x)}{2\Delta t},$$

prove that the scheme (1.2) is of order 2.

(1b) What kind of method (1.2) is?

Explicit One-step ☐ Explicit Two-step ☐
Implicit One-step ☐ Implicit Two-step ☐

(1c) Is scheme (1.2) consistent with (1.1)? Prove that.

(1d) Suppose that $\{x^2, \cdots, x^k\}$ and $\{\tilde{x}^2, \cdots, \tilde{x}^k\}$ have been generated by (1.2) but with different initial data x^0, x^1 and $\tilde{x}^0, \tilde{x}^1$. Prove by induction that

$$|x^k - \tilde{x}^k| \leq C \max\{|x^0 - \tilde{x}^0|, |x^1 - \tilde{x}^1|, \cdots, |x^{k-1} - \tilde{x}^{k-1}|\}.$$

Is (1.2) stable? Prove that.

(1e) Is (1.2) convergent? Prove that.

Problem 2 **[13 Marks]**

Consider the scheme

$$x^{k+1} = x^k + \frac{\Delta t}{2}(\kappa_1 + \kappa_2) \tag{2.1}$$

where

$$\kappa_1 = f(t_k, x^k),$$
$$\kappa_2 = f(t_{k+1}, x^k + \Delta t \kappa_1).$$

(2a) Is (2.1) a one-step method or two-step method?

One-step method ☐ Two-step Method ☐

(2b) Define the truncation error by

$$T_k(\Delta t) = \frac{x(t+\Delta t) - x(t)}{\Delta t} - \frac{1}{2}[f(t, x(t)) + f(t+\Delta t, x(t) + \Delta t f(t, x(t)))].$$

Prove that (2.1) is consistent and of at least order 2 by either direct calculation or properties from Runge-Kutta method. Attention: Please write out coefficients $\{a_{ij}\}$, $\{b_i\}$, $\{c_i\}$ and order conditions explicitly(no need to prove) if choosing to use RK properties.

Problem 3 **[19 Mark]**

Consider the Hamiltonian system

$$\begin{cases} \dfrac{dp}{dt} = -q, \\ \dfrac{dq}{dt} = p, \end{cases} \tag{3.1}$$

where $p(t), q(t) \in \mathbb{R}$ and the numerical scheme

$$\begin{cases} p^{k+1} = p^k - \Delta t q^k \\ q^{k+1} = q^k + \Delta t p^{k+1}. \end{cases} \tag{3.2}$$

(3a) What kind of method (3.2) is?

Explicit Euler method ☐ Implicit Euler method ☐
Explicit Leapfrog method ☐ Implicit Leapfrog method ☐

(3b) Define the numerical flow $\Phi_{\Delta t} : (p^k, q^k) \mapsto (p^{k+1}, q^{k+1})$. Prove that (3.2) is symplectic, i.e., the numerical flow satisfies

$$(\Phi'_{\Delta t})^T J \Phi'_{\Delta t} = J, \quad \text{where } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

(3c) Is the Hamiltonian H associated with (3.1) preserved by the scheme (3.2)? Prove that.