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Numerical Analysis II

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D-MATH

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Problem 1 [35 Marks]

Consider the Hamiltonian system

$$\begin{cases} \frac{\mathrm{d}p}{\mathrm{d}t} = -q, \\ \frac{\mathrm{d}q}{\mathrm{d}t} = p, \end{cases} \tag{1.1}$$

with $p, q \in \mathbb{R}$ and the numerical scheme (for $\Delta t > 0$)

$$\begin{cases} p^{k+1} = p^k - \Delta t q^{k+1}, \\ q^{k+1} = q^k + \Delta t p^k. \end{cases} \tag{1.2}$$

Define the numerical flow

$$\Phi_{\Delta t} : (p^k, q^k) \mapsto (p^{k+1}, q^{k+1}). \tag{1.3}$$

(1a) Compute the Jacobian $\Phi'_{\Delta t}$ of the numerical flow $\Phi_{\Delta t}$ defined by

$$\Phi'_{\Delta t}(p^k, q^k) = \frac{\partial(p^{k+1}, q^{k+1})}{\partial(p^k, q^k)}.$$

(1b) Is (1.2) symplectic? i.e. Does the numerical flow satisfy

$$(\Phi'_{\Delta t})^\top J \Phi'_{\Delta t} = J$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}?$$

(1c) Is (1.2) symmetric, i.e. does

$$\Phi^*_{\Delta t} = \Phi_{\Delta t}?$$

Here $\Phi^*_{\Delta t}$ is defined by $\Phi^*_{\Delta t} = (\Phi_{-\Delta t})^{-1}$, i.e. by replacing Δt by $-\Delta t$ and exchanging k and $k+1$.

(1d) (1.2) is a

Explicit Euler method. ☐ Implicit Euler method. ☐

Symplectic Euler method. ☐ Leapfrog method. ☐

(1e) The composition $\Phi^*_{\Delta t/2} \circ \Phi_{\Delta t/2}$ is

symmetric. ☐ not symmetric. ☐

(1f) The composition $\Phi^*_{\Delta t/2} \circ \Phi_{\Delta t/2}$ is

symplectic. ☐ not symplectic. ☐

(1g) The composition $\Phi^*_{\Delta t/2} \circ \Phi_{\Delta t/2}$ is

a Symplectic Euler method. ☐ a Leapfrog method. ☐

(1h) Let $x = (p, q)^\top$. Rewrite (1.1) in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = J^{-1}\nabla H(x).$$

Is $f(x) = J^{-1}\nabla H(x)$ divergence-free? Is $\Phi_{\Delta t}$ defined by (1.3) volume preserving? Explain why.

Problem 2 **[25 Marks]**

Consider

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), \quad t \in [0,T] \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{2.1}$$

with $f \in C^\infty$ subject to the Lipschitz condition

$$|f(t,x) - f(t,y)| \le C|x-y|, \ \forall x, y \in \mathbb{R}, \ \forall t \in [0,T].$$

Consider the midpoint scheme

$$x^{k+1} = x^k + \Delta t f(t_k + \frac{\Delta t}{2}, x^k + \frac{\Delta t}{2} f(t_k, x^k)), \tag{2.2}$$

where $\Delta t > 0$ is small enough and $t_k = k\Delta t$ for $k \in \mathbb{N}$.

(2a) (2.2) is

explicit one-step method. ☐ implicit two-step method. ☐

(2b) Let $\Phi(t_k, x^k, \Delta t)$ be defined by

$$\Phi(t_k, x^k, \Delta t) = f(t_k + \frac{\Delta t}{2}, x^k + \frac{\Delta t}{2} f(t_k, x^k))$$

so that (2.2) can be rewritten in the form

$$x^{k+1} = x^k + \Delta t \Phi(t_k, x^k, \Delta t).$$

Prove that (2.2) is consistent with (2.1).

(2c) Define the truncation error by

$$T_k(\Delta t) = \frac{x(t_{k+1}) - x(t_k)}{\Delta t} - \Phi(t_k, x(t_k), \Delta t).$$

Prove that (2.2) is of order two, *i.e.*, $T_k(\Delta t) = O((\Delta t)^2)$.

(2d) Prove that (2.2) is stable, *i.e.*, there exists positive constants h_0 and c_Φ such that

$$|\Phi(t, x, h) - \Phi(t, y, h)| \leq c_\Phi |x - y|$$

for $\forall t \in [0, T]$, $\forall x, y \in \mathbb{R}$, $h \in [0, h_0]$.

(2e) Is (2.2) for solving (2.1) convergent? Explain why.