

End Term Exam Spring 2018

Problem 1

[35 Marks]

Consider the Hamiltonian system

$$\begin{cases} \frac{dp}{dt} = -q, \\ \frac{dq}{dt} = p, \end{cases} \quad (1.1)$$

with $p, q \in \mathbb{R}$ and the numerical scheme (for $\Delta t > 0$)

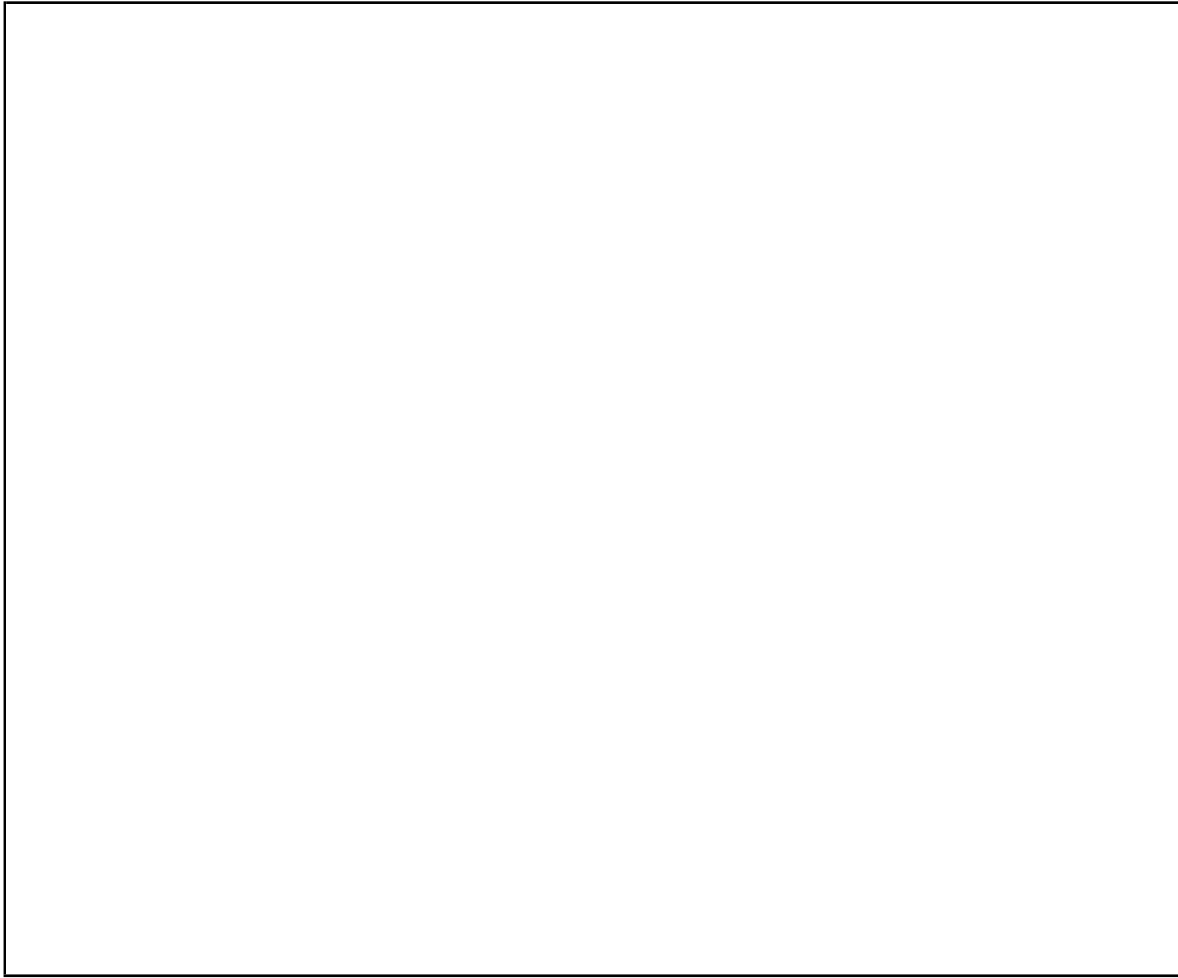
$$\begin{cases} p^{k+1} = p^k - \Delta t q^{k+1}, \\ q^{k+1} = q^k + \Delta t p^k. \end{cases} \quad (1.2)$$

Define the numerical flow

$$\Phi_{\Delta t} : (p^k, q^k) \mapsto (p^{k+1}, q^{k+1}). \quad (1.3)$$

(1a) Compute the Jacobian $\Phi'_{\Delta t}$ of the numerical flow $\Phi_{\Delta t}$ defined by

$$\Phi'_{\Delta t}(p^k, q^k) = \frac{\partial(p^{k+1}, q^{k+1})}{\partial(p^k, q^k)}.$$

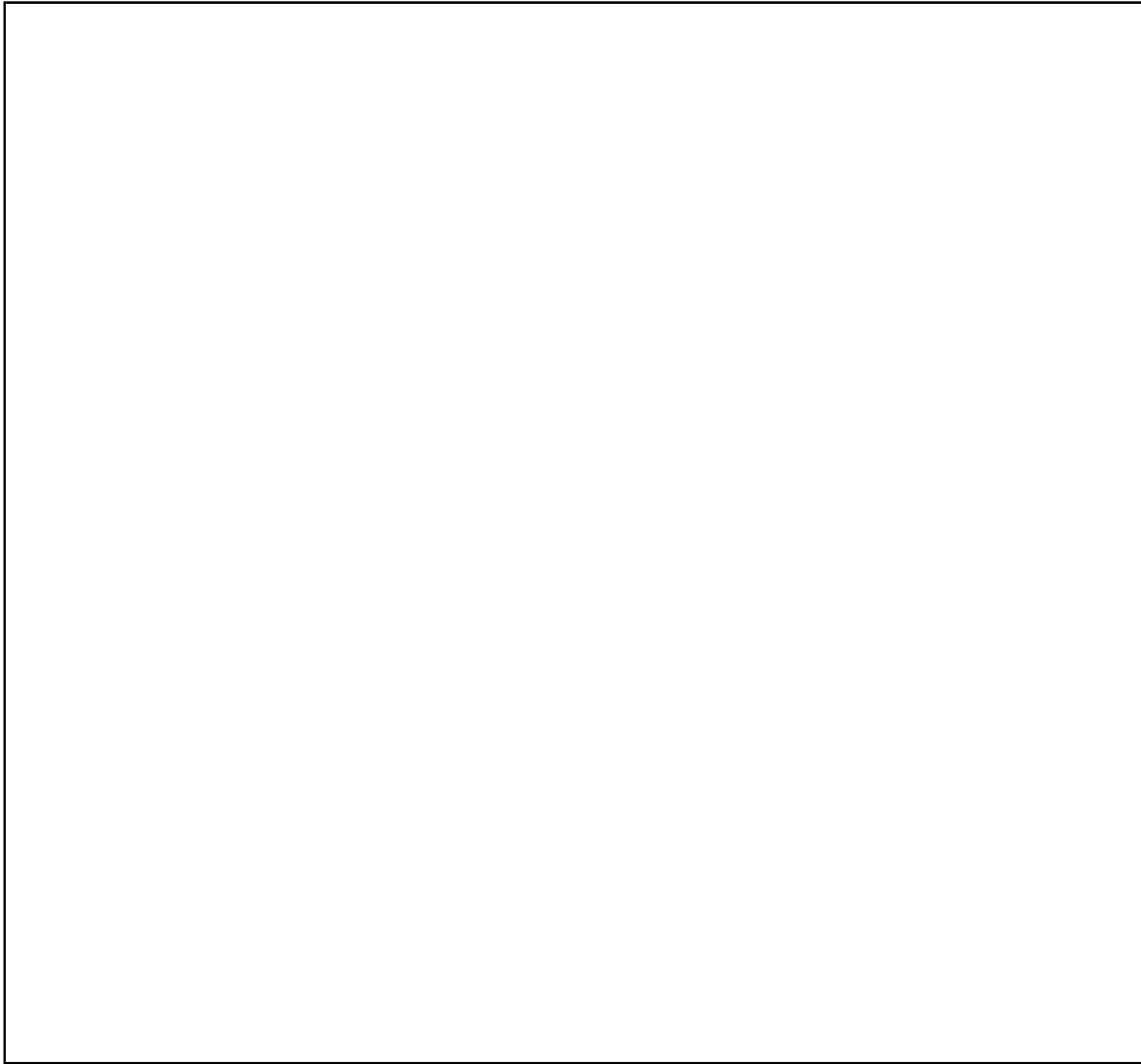


(1b) Is (1.2) symplectic? i.e. Does the numerical flow satisfy

$$(\Phi'_{\Delta t})^\top J \Phi'_{\Delta t} = J$$

where

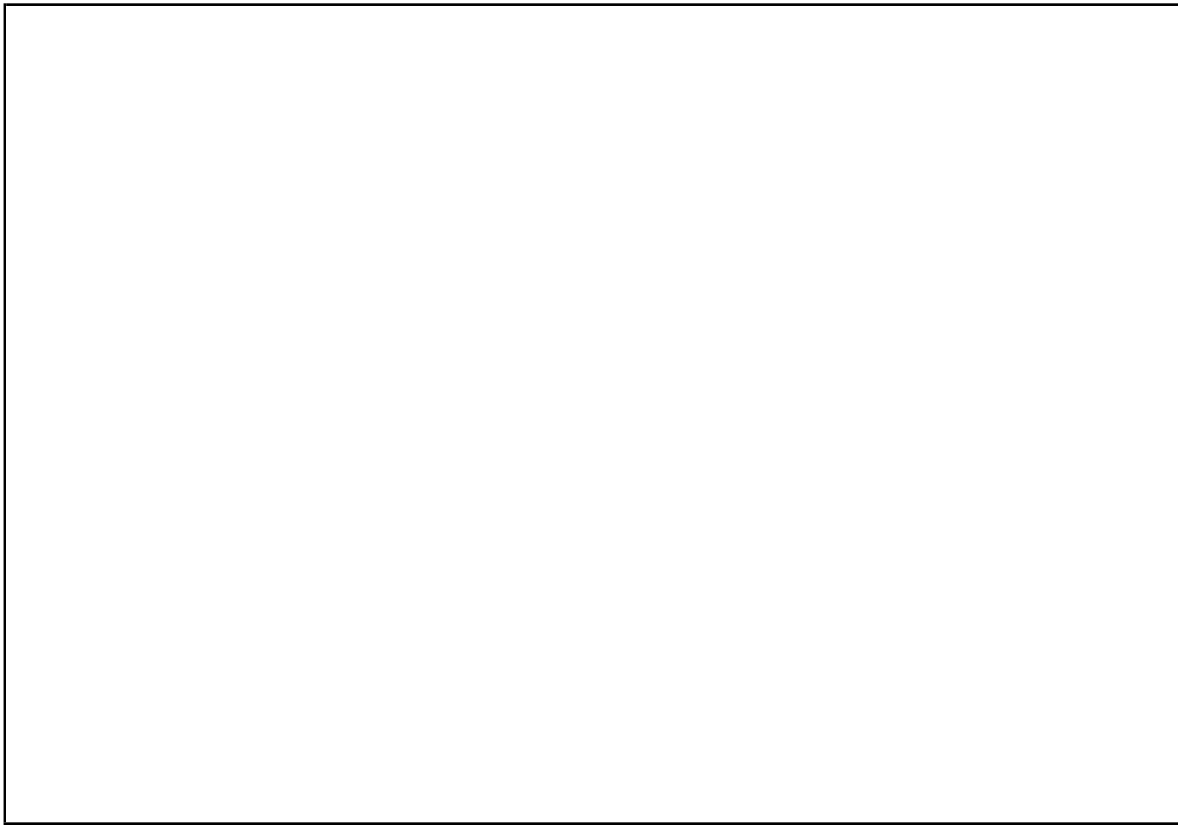
$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}?$$



(1c) Is (1.2) symmetric, i.e. does

$$\Phi_{\Delta t}^* = \Phi_{\Delta t}?$$

Here $\Phi_{\Delta t}^*$ is defined by $\Phi_{\Delta t}^* = (\Phi_{-\Delta t})^{-1}$, i.e. by replacing Δt by $-\Delta t$ and exchanging k and $k + 1$.



(1d) (1.2) is a

Explicit Euler method. Implicit Euler method.

Symplectic Euler method. Leapfrog method.

(1e) The composition $\Phi_{\Delta t/2}^* \circ \Phi_{\Delta t/2}$ is

symmetric. not symmetric.

(1f) The composition $\Phi_{\Delta t/2}^* \circ \Phi_{\Delta t/2}$ is

symplectic. not symplectic.

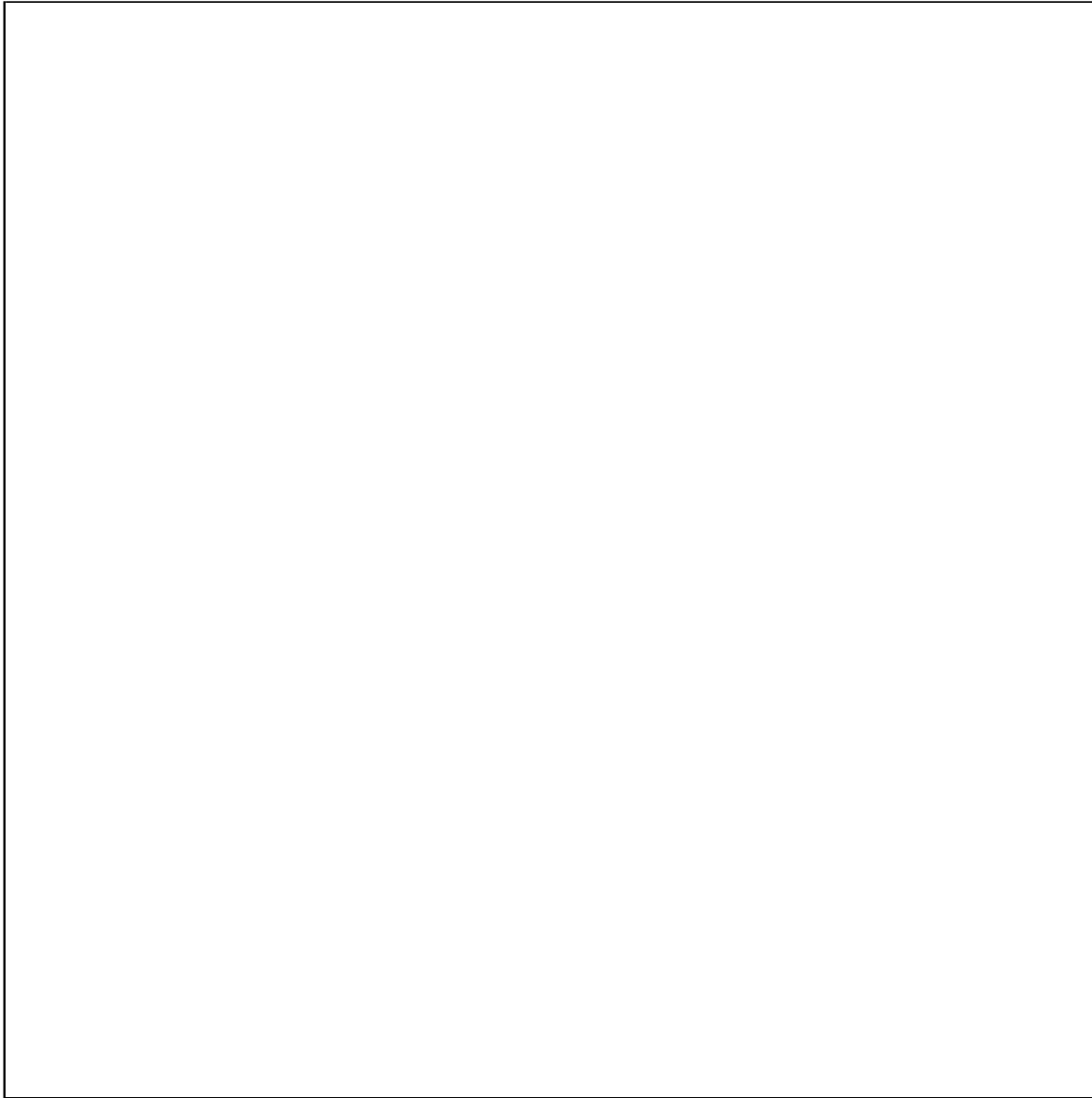
(1g) The composition $\Phi_{\Delta t/2}^* \circ \Phi_{\Delta t/2}$ is

a Symplectic Euler method. a Leapfrog method.

(1h) Let $x = (p, q)^\top$. Rewrite (1.1) in the form

$$\frac{dx}{dt} = J^{-1} \nabla H(x).$$

Is $f(x) = J^{-1} \nabla H(x)$ divergence-free? Is $\Phi_{\Delta t}$ defined by (1.3) volume preserving? Explain why.



Problem 2

[25 Marks]

Consider

$$\begin{cases} \frac{dx}{dt} = f(t, x), & t \in [0, T] \\ x(0) = x_0 \in \mathbb{R} \end{cases} \quad (2.1)$$

with $f \in C^\infty$ subject to the Lipschitz condition

$$|f(t, x) - f(t, y)| \leq C|x - y|, \quad \forall x, y \in \mathbb{R}, \quad \forall t \in [0, T].$$

Consider the midpoint scheme

$$x^{k+1} = x^k + \Delta t f\left(t_k + \frac{\Delta t}{2}, x^k + \frac{\Delta t}{2} f(t_k, x^k)\right), \quad (2.2)$$

where $\Delta t > 0$ is small enough and $t_k = k\Delta t$ for $k \in \mathbb{N}$.

(2a) (2.2) is

explicit one-step method. implicit two-step method.

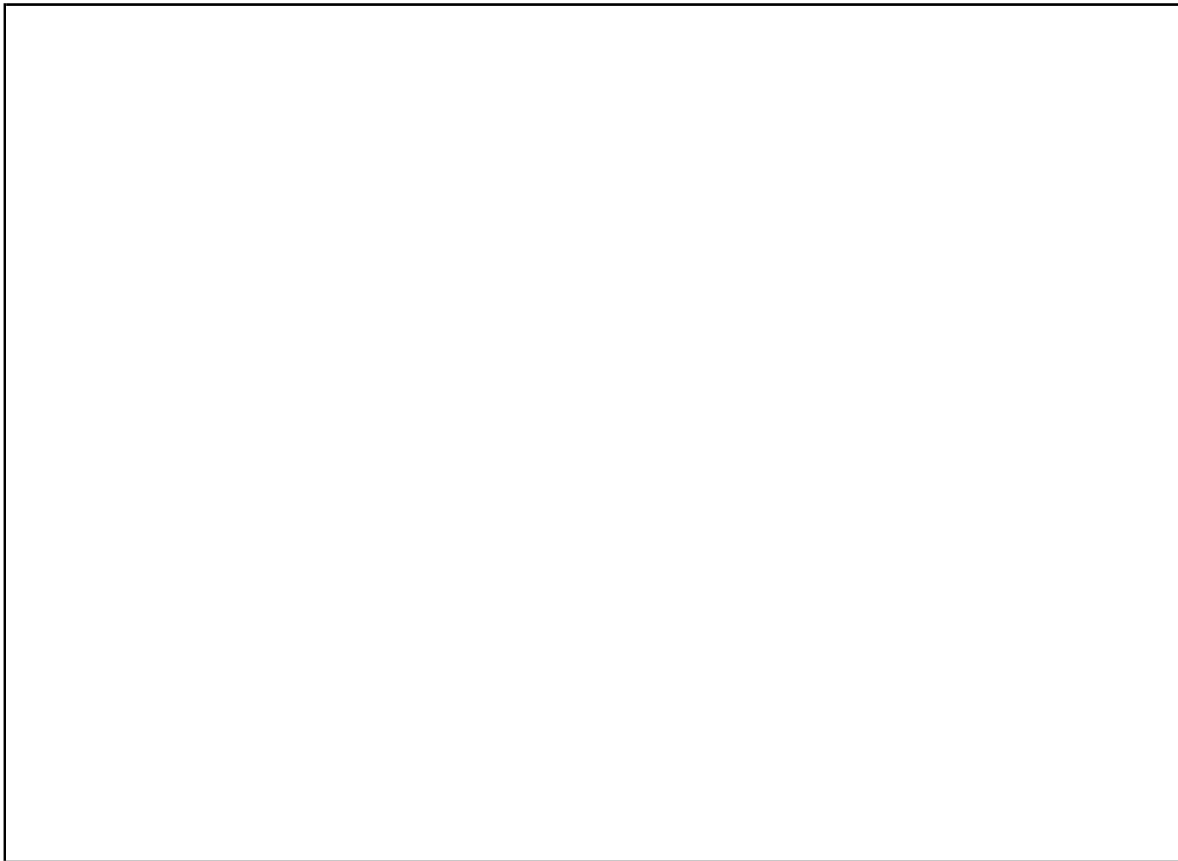
(2b) Let $\Phi(t_k, x^k, \Delta t)$ be defined by

$$\Phi(t_k, x^k, \Delta t) = f\left(t_k + \frac{\Delta t}{2}, x^k + \frac{\Delta t}{2} f(t_k, x^k)\right)$$

so that (2.2) can be rewritten in the form

$$x^{k+1} = x^k + \Delta t \Phi(t_k, x^k, \Delta t).$$

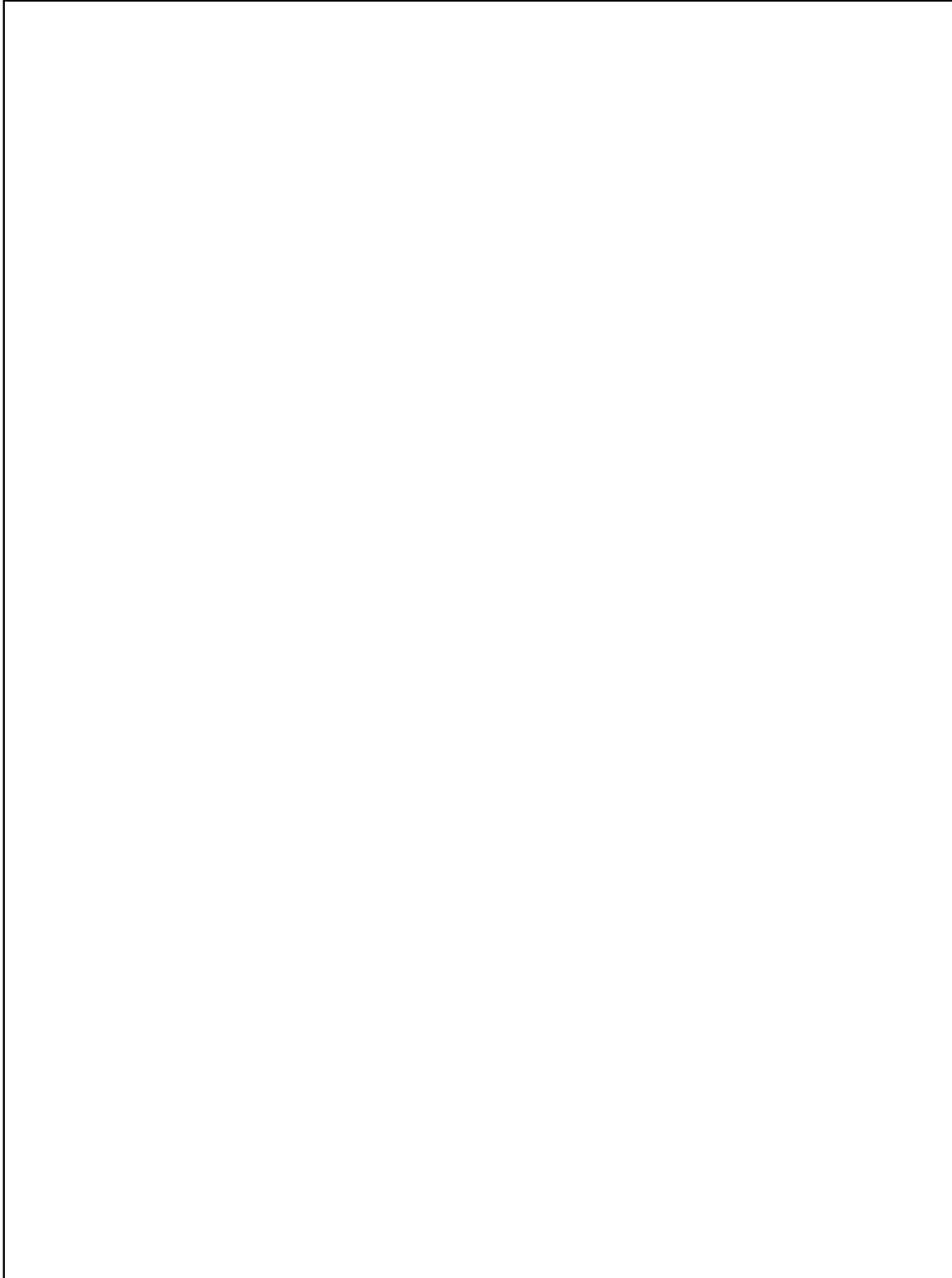
Prove that (2.2) is consistent with (2.1).



(2c) Define the truncation error by

$$T_k(\Delta t) = \frac{x(t_{k+1}) - x(t_k)}{\Delta t} - \Phi(t_k, x(t_k), \Delta t).$$

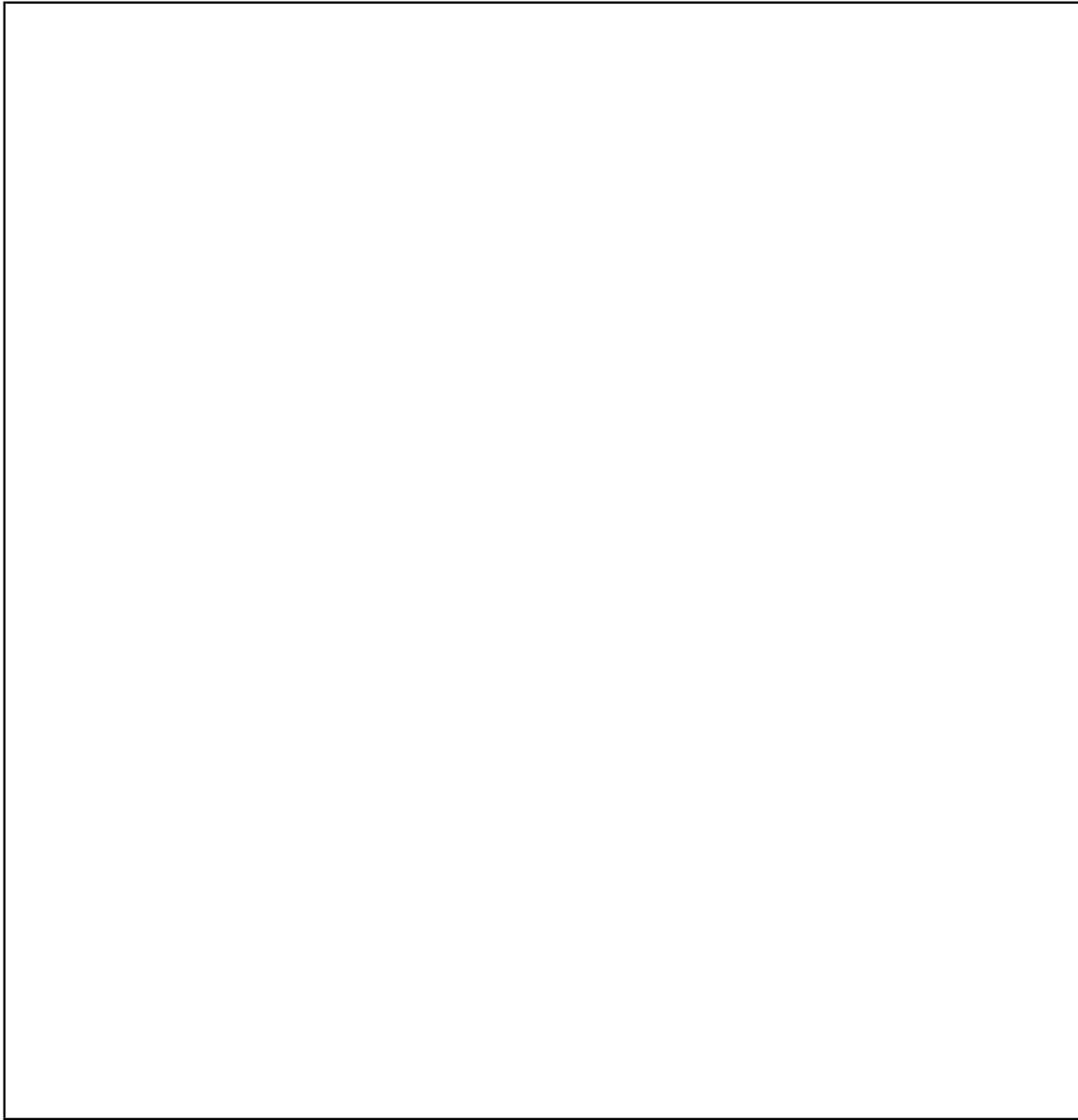
Prove that (2.2) is of order two, *i.e.*, $T_k(\Delta t) = O((\Delta t)^2)$.



(2d) Prove that (2.2) is stable, *i.e.*, there exists positive constants h_0 and c_Φ such that

$$|\Phi(t, x, h) - \Phi(t, y, h)| \leq c_\Phi |x - y|$$

for $\forall t \in [0, T], \forall x, y \in \mathbb{R}, h \in [0, h_0]$.



(2e) Is (2.2) for solving (2.1) convergent? Explain why.

