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Numerical Analysis II

ETH Zürich
D-MATH

Exam Autumn 2018

Problem 1 [50 points]

Let $x : [0, T] \to \mathbb{R}^d$ be the solution to the following ODE:

$$\begin{cases} \dfrac{dx}{dt} = A(x)x, t \in [0, T], \\ x(0) = x_0 \in \mathbb{R}^d \end{cases} \tag{1.1}$$

where $A = A(x)$ is a $d \times d$ matrix valued function satisfying

$$A(x)^T = -A(x) \quad \text{for all } x.$$

In other words, $A(x)$ is skew-symmetric.

(1a) Show that the Euclidean vector norm $\|x\|$ is an invariant for every solution x.

(1b) Show that $\|x_1\| \geq \|x_0\|$ for the explicit Euler method

$$x_1 = x_0 + hA(x_0)x_0.$$

(1c) Show that $\|x_1\| \leq \|x_0\|$ for the implicit Euler method

$$x_1 = x_0 + hA(x_1)x_1.$$

(1d) Show that $\|x_1\| = \|x_0\|$ for the implicit single step method defined by

$$x_1 = x_0 + hA(x_0 + x_1)(x_0 + x_1) \ . \tag{1.2}$$

Consider now the explicit exponential single step method

$$\begin{aligned} x^* &= x_0 + \tfrac{1}{2}hA(x_0)x_0 \ , \\ x_1 &= \exp(hA(x^*))x_0 \ . \end{aligned} \tag{1.3}$$

Here $\exp$ means the exponential of a matrix.

(1e) Show that $\|x_1\| = \|x_0\|$ for (1.3).

HINT: You may use the fact that $\exp(\mathbf{M})$ is orthogonal if $\mathbf{M}$ is skew symmetric.

(1f) Prove that the single step method (1.3) is of order 2, *i.e.*,

$$\frac{\|x(h) - x_1\|}{h} = O(h^2).$$

(1g) Write a MATLAB function

```
function X = exprot(Afn,x0,T,N),
```

which carries out $N \in \mathbb{N}$ equidistant steps of the method (1.3) for (1.1) on the time interval $[0, T]$ with initial value $x_0 \in \mathbb{R}^d$ and which outputs a $d \times (N + 1)$-matrix with the components x_k. The input argument `Afn` is a function handle for the matrix function $A(x)$.

HINT: The matrix exponential function is available in MATLAB as `expm()`.

(1h) Write a MATLAB program `exprotsim.m`, which solves the initial value problem

$$\frac{dx}{dt} = \frac{Qx}{\|x\|^2} \times x \quad , \quad x(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; , \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix}, \; \varphi = \frac{\pi}{4},$$

using the function `exprot` from subproblem subproblem (1g) on the time interval $[0, 10]$ and which plots the for $N = 100, 200, 400$ obtained approximated trajectories with the help of the MATLAB function `plot3` in a single plot.

HINT: The function handle for the matrix function $A(x)$ is already contained in `exprotsim.m`.

$$x \times y = \begin{pmatrix} y_2 x_3 - y_3 x_2 \\ y_3 x_1 - y_1 x_3 \\ y_1 x_2 - y_2 x_1 \end{pmatrix}$$

Problem 2 [50 points]

Consider the autonomous Lotka-Volterra differential equation

$$\begin{aligned} \dot{u} &= f_u(u, v) := (1 - v)u, \\ \dot{v} &= f_v(u, v) := (u - 1)v. \end{aligned} \tag{2.1}$$

(2a) Show that, $I(u, v) := u - \log u + v - \log v$ is an invariant of the differential equation (2.1).

(2b) Show that, the evolution belonging to (2.1) is not symplectic.

HINT: Show that, (2.1) is not a Hamiltonian differential equation and state a proper theorem from the lecture.

(2c) Show that, the autonomous differential equation

$$\begin{aligned} \dot{p} &= 1 - \exp(q) \\ \dot{q} &= \exp(p) - 1 \end{aligned} \tag{2.2}$$

is solved by the function

$$p(t) = \log u(t) \quad , \quad q(t) = \log v(t) \tag{2.3}$$

if $u(t)$ and $v(t)$ are solutions of (2.1).

(2d) Show that, the differential equation ((2.2)) from subproblem (2c) is a Hamiltonian differential equation and state the corresponding Hamiltonian function $H(p, q)$.

(2e) Following MATLAB function implements an explicit 1-step method to solve the initial value problem Equation 2.1 with initial values $u_0, v_0 \in \mathbb{R}^+$ on the time interval $[0, T]$, $T > 0$. The argument N specifies the number of equidistant time steps.

```
function [u,v] = trsv(u0,v0,N,T)
h = T/N;
v = v0;
u = u0*exp(0.5*h*(1-v));
for k=2:N
  v = v*exp(h*(u-1));
  u = u*exp(h*(1-v));
end
v = v*exp(h*(u-1));
u = u*exp(0.5*h*(1-v));
```

This function is accessible in the file `trsv.m`. Determine experimentally using an appropriate MATLAB script `trsvcvg.m` the convergence rate of the error $|u(T) - u_N| + |v(T) - v_N|$ for the end time $T = 100$ and the starting values $u_0 = v_0 = 0.5$. Here u_N, v_N stand for the approximated solutions calculated by `trsv`.

HINT: You can use a reference solution calculated by `ode45` with absolute and relative tolerance equal to 10^{-12}.

(2f) Let Ψ_h be the associated flow of the 1-step method `trsv` given in subproblem (2e). In other words,

$$\begin{pmatrix} u_{k+1} \\ v_{k+1} \end{pmatrix} = \Psi_h \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

Write down $\Psi_h \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ explcitly. Then prove that the flow Ψ_h is symmetric.