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Numerical Analysis II

ETH Zürich
D-MATH

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Problem 1 **[45 Marks]**

Let J be the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = J^{-1}x, \; t \in [0,T], \\ x(0) = x_0 \in \mathbb{R}^2. \end{cases} \tag{1.1}$$

For $\Delta t > 0$ small enough, consider the following schemes

$$x^{k+1} = \psi_{\Delta t}^{(1)}(x^k) := (I + \Delta t J^{-1})x^k, \tag{1.2}$$

and

$$x^{k+1} = \psi_{\Delta t}^{(2)}(x^k) := (I - \Delta t J^{-1})^{-1}x^k, \tag{1.3}$$

where I is the 2×2 identity matrix.

(1a)

(i) Is (1.1) a Hamiltonian system?

(ii) Is the quantity

$$H(x) = \frac{1}{2}||x||^2 = \frac{1}{2}x^\top x,$$

where $^\top$ is the transpose, an invariant of (1.1)?

(1b)

(i) Is $\psi_{\Delta t}^{(2)}$ the adjoint of $\psi_{\Delta t}^{(1)}$? Justify.

(ii) Prove that (1.2) and (1.3) are consistent with (1.1).

(iii) Prove that (1.2) and (1.3) are of order one.

(1c) Let $\Psi_{\Delta t} := \psi^{(2)}_{\frac{\Delta t}{2}} \circ \psi^{(1)}_{\frac{\Delta t}{2}}$

(i) Prove that

$$\Psi_{\Delta t}(x^k) = \left(I - \frac{\Delta t}{2} J^{-1}\right)^{-1} \left(I + \frac{\Delta t}{2} J^{-1}\right) x^k.$$

(ii) Is $\Psi_{\Delta t}$ symmetric? Justify.

(iii) Is $\Psi_{\Delta t}$ consistent with (1.1)? Justify.

(iv) Compute the Jacobians $\left(\psi^{(1)}_{\Delta t}\right)' := \dfrac{\partial x^{k+1}}{\partial x^k}$, $\left(\psi^{(2)}_{\Delta t}\right)'$ and $(\Psi_{\Delta t})'$.

(v) Are the schemes defined by $\psi_{\Delta t}^{(1)}$, $\psi_{\Delta t}^{(2)}$ and $\Psi_{\Delta t}$ symplectic?

(1d) Prove that $\Psi_{\Delta t}$ is of order two. Hint: compute $\int_{t_k}^{t_{k+1}}(t-t_{k+1})(t-t_k)x'''(t)dt$ in two different ways, using integration by part and the integral mean value theorem.

(1e) Are the schemes defined by $\psi^{(1)}_{\Delta t}$, $\psi^{(2)}_{\Delta t}$ and $\Psi_{\Delta t}$ volume preserving?

(1f) Is H preserved by $\Psi_{\Delta t}$?

Problem 2 **[15 Marks]**

Consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), \quad t \in [0,T], \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is arbitrarily smooth.

Let us consider the following Runge-Kutta method:

$$\begin{aligned} y^{n+1} &= y^n + h\left(\frac{1}{6}k_1 + \frac{2}{3}k_2 + \frac{1}{6}k_3\right) \\ k_1 &= f(t_n, y_n) \\ k_2 &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \\ k_3 &= f(t_n + h, y_n - hk_1 + 2hk_2) \end{aligned} \tag{2.2}$$

(2a) Is the Runge-Kutta method (2.2) explicit or implicit?

explicit ☐ implicit ☐

(2b) Show that the method (2.2) is consistent.

(2c) Show that the method (2.2) for solving (2.1) is stable.

(2d) Find the order of the method (2.2) for solving (2.1). Justify.