H. Ammari
B. Davies
A. Vanel

Spring Term 2020

Numerical Analysis II

ETH Zürich
D-MATH

Mid-term Assignment 2020

Problem 1 **[30 Marks]**

(1a)

(i) For $x, y > 0$, show that

$$\left|\sqrt{x} - \sqrt{y}\right| = \frac{|x-y|}{\sqrt{x} + \sqrt{y}}.$$

(ii) Does

$$f(t,x) = \sqrt{t^2 + x^2}, \qquad t, x \in]0, +\infty[,$$

satisfy a Lipschitz condition? If so give a Lipschitz constant.

(iii) Does

$$f(t,x) = \frac{\sin t}{3 + t^3}(1 + e^{-3x}), \quad t, x \in [0, +\infty[,$$

satisfy a Lipschitz condition? If so give a Lipschitz constant.

(1b) Consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), \quad t \in [0,T], \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where we will assume that $f \in C^\infty$ is an arbitrary function subject to the Lipschitz condition

$$|f(t,x) - f(t,y)| \le C|x-y|, \ \forall x, y \in \mathbb{R}, \ \forall t \in [0,T].$$

(i) Does (1.1) have a unique solution $x(t) \in C^\infty([0,T])$? Justify.

(ii) If we regard $x(t)$ also as a function of the initial value x_0, what is the equation satisfied by the derivative with respect to t of $\partial x(t)/\partial x_0$? Is it a linear equation? Justify.

(1c) Consider the midpoint scheme

$$x^{k+1} = x^k + \Delta t f\left(t_k + \frac{\Delta t}{2}, x^k + \frac{\Delta t}{2} f(t_k, x^k)\right), \tag{1.2}$$

where $\Delta t > 0$ is small enough and $t_k = k\Delta t$ for $k \in \mathbb{N}$.

(i) The scheme (1.2) is an...

explicit one-step method. ☐	explicit two-step method. ☐
implicit one-step method. ☐	implicit two-step method. ☐

(ii) Let $\Phi(t_k, x^k, \Delta t)$ be defined by

$$\Phi(t_k, x^k, \Delta t) = f\left(t_k + \frac{\Delta t}{2}, x^k + \frac{\Delta t}{2} f(t_k, x^k)\right),$$

so that (1.2) can be rewritten in the form

$$x^{k+1} = x^k + \Delta t \Phi(t_k, x^k, \Delta t).$$

Prove that (1.2) is consistent with (1.1).

(iii) Define the truncation error by

$$T_k(\Delta t) = \frac{x(t_{k+1}) - x(t_k)}{\Delta t} - \Phi(t_k, x(t_k), \Delta t).$$

Prove that (1.2) is of order two, *i.e.*, that $T_k(\Delta t) = O((\Delta t)^2)$.

(iv) Prove that (1.2) is stable, *i.e.*, there exist positive constants h_0 and c_Φ such that

$$|\Phi(t,x,h) - \Phi(t,y,h)| \le c_\Phi |x-y|$$

for all $t \in [0,T]$, $x, y \in \mathbb{R}$, $h \in [0,h_0]$.

(v) Is (1.2) for solving (1.1) convergent? Explain why.

Problem 2 **[30 Marks]**

(2a) Consider the system of equations

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = \sin q, & t \geq 0, \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = p, & t \geq 0, \end{cases} \tag{2.1}$$

with initial values $p(0) = p_0 \in \mathbb{R}$ and $q(0) = q_0 \in \mathbb{R}$.

(i) Is (2.1) a Hamiltonian system? You should justify your answer.

(ii) Find an invariant for (2.1), *i.e.* a function F such that $F(p(t), q(t))$ is constant for all $t \geq 0$.

(2b) Let A be a real symmetric 2×2 matrix and let J denote the unitary matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Consider the system of linear equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = J^{-1}Ax, \\ x(0) = x_0 \in \mathbb{R}^2. \end{cases} \tag{2.2}$$

(i) Prove that (2.2) is a Hamilton system associated with the Hamiltonian $H(x) = \frac{1}{2}x^\top Ax$, where $x^\top$ denotes the transpose of x.

(ii) Define the flow Φ_t associated with (2.2) by $\Phi_t(x_0) = x(t)$. Prove that $\Phi_t(x_0) = e^{tJ^{-1}A}x_0$.

(iii) Let $\Phi_t'(x_0)$ be the Jacobian of $\Phi_t(x_0)$, *i.e.* $\Phi_t'(x_0) = \frac{\partial \Phi_t(x_0)}{\partial x_0}$. Compute $\Phi_t'(x_0)$ and prove that $(\Phi_t'(x_0))^\top J \, \Phi_t'(x_0) = J$ for all $x_0 \in \mathbb{R}^2$.

(2c) Let A be a real symmetric 2×2 matrix and let J denote the unitary matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Suppose the function $f : [0, T] \to \mathbb{R}^2$ is continuous. Consider the inhomogeneous system of linear equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = J^{-1}Ax + f(t), \\ x(0) = x_0 \in \mathbb{R}^2. \end{cases} \tag{2.3}$$

(i) Prove that the flow is given by

$$\Phi_t(x_0) = e^{tJ^{-1}A}x_0 + \int_0^t e^{(t-s)J^{-1}A} f(s)\, ds,$$

for $t \in [0, T]$.

(ii) In this case, does the Jacobian $\Phi_t'(x_0)$ satisfy $(\Phi_t'(x_0))^\top J\, \Phi_t'(x_0) = J$ for all $x_0 \in \mathbb{R}^2$? Justify your answer.