

## Mid-term Assignment 2020

### Problem 1

[30 Marks]

(1a)

(i) For  $x, y > 0$ , show that

$$|\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}}.$$



(ii) Does

$$f(t, x) = \sqrt{t^2 + x^2}, \quad t, x \in ]0, +\infty[,$$

satisfy a Lipschitz condition? If so give a Lipschitz constant.

(iii) Does

$$f(t, x) = \frac{\sin t}{3 + t^3}(1 + e^{-3x}), \quad t, x \in [0, +\infty[,$$

satisfy a Lipschitz condition? If so give a Lipschitz constant.

**(1b)** Consider the initial value problem

$$\begin{cases} \frac{dx}{dt} = f(t, x), & t \in [0, T], \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \quad (1.1)$$

where we will assume that  $f \in C^\infty$  is an arbitrary function subject to the Lipschitz condition

$$|f(t, x) - f(t, y)| \leq C|x - y|, \quad \forall x, y \in \mathbb{R}, \quad \forall t \in [0, T].$$

(i) Does (1.1) have a unique solution  $x(t) \in C^\infty([0, T])$ ? Justify.

(ii) If we regard  $x(t)$  also as a function of the initial value  $x_0$ , what is the equation satisfied by the derivative with respect to  $t$  of  $\partial x(t)/\partial x_0$ ? Is it a linear equation? Justify.

**(1c)** Consider the midpoint scheme

$$x^{k+1} = x^k + \Delta t f\left(t_k + \frac{\Delta t}{2}, x^k + \frac{\Delta t}{2} f(t_k, x^k)\right), \quad (1.2)$$

where  $\Delta t > 0$  is small enough and  $t_k = k\Delta t$  for  $k \in \mathbb{N}$ .

(i) The scheme (1.2) is an...

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implicit one-step method. <input type="checkbox"/>	implicit two-step method. <input type="checkbox"/>

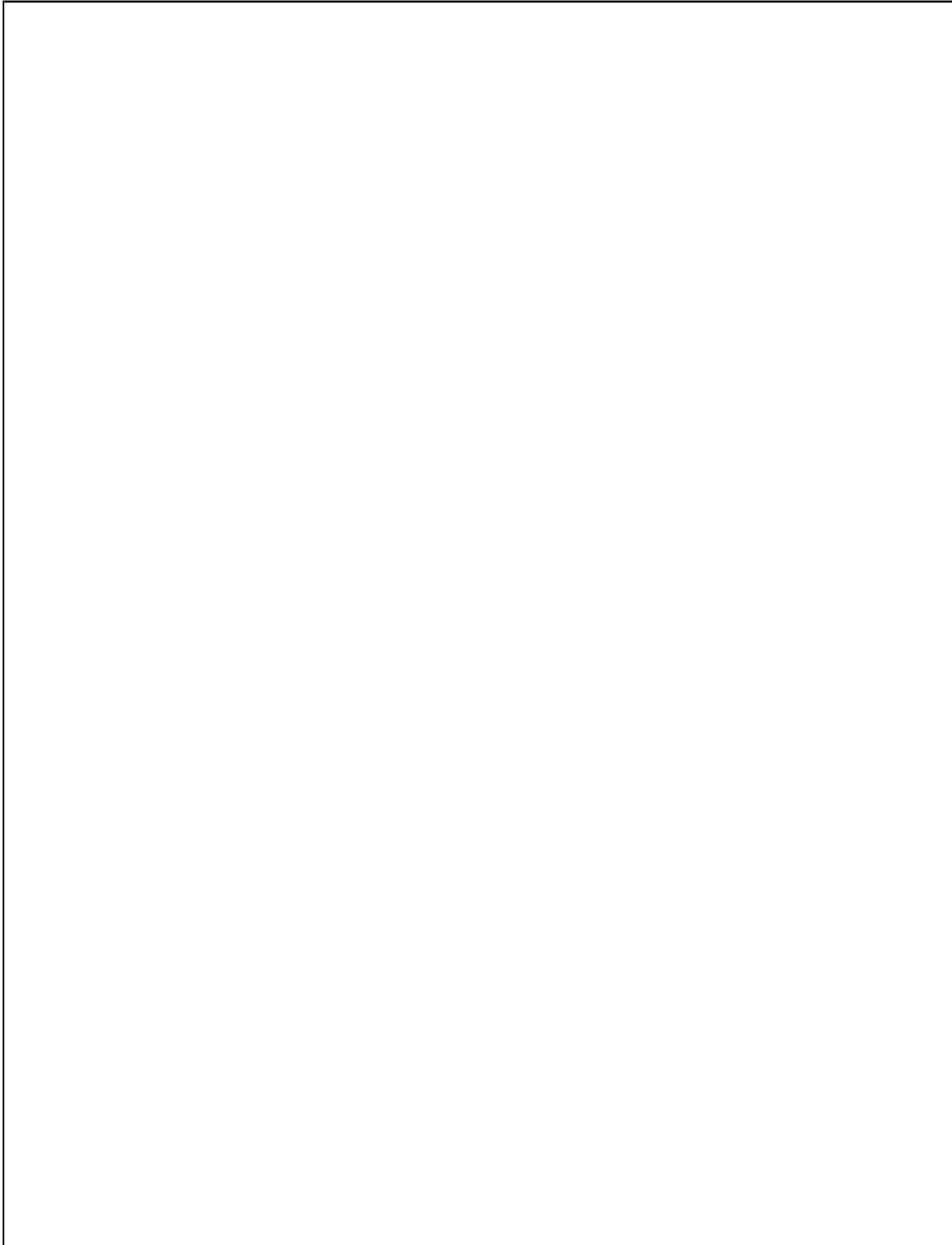
(ii) Let  $\Phi(t_k, x^k, \Delta t)$  be defined by

$$\Phi(t_k, x^k, \Delta t) = f\left(t_k + \frac{\Delta t}{2}, x^k + \frac{\Delta t}{2} f(t_k, x^k)\right),$$

so that (1.2) can be rewritten in the form

$$x^{k+1} = x^k + \Delta t \Phi(t_k, x^k, \Delta t).$$

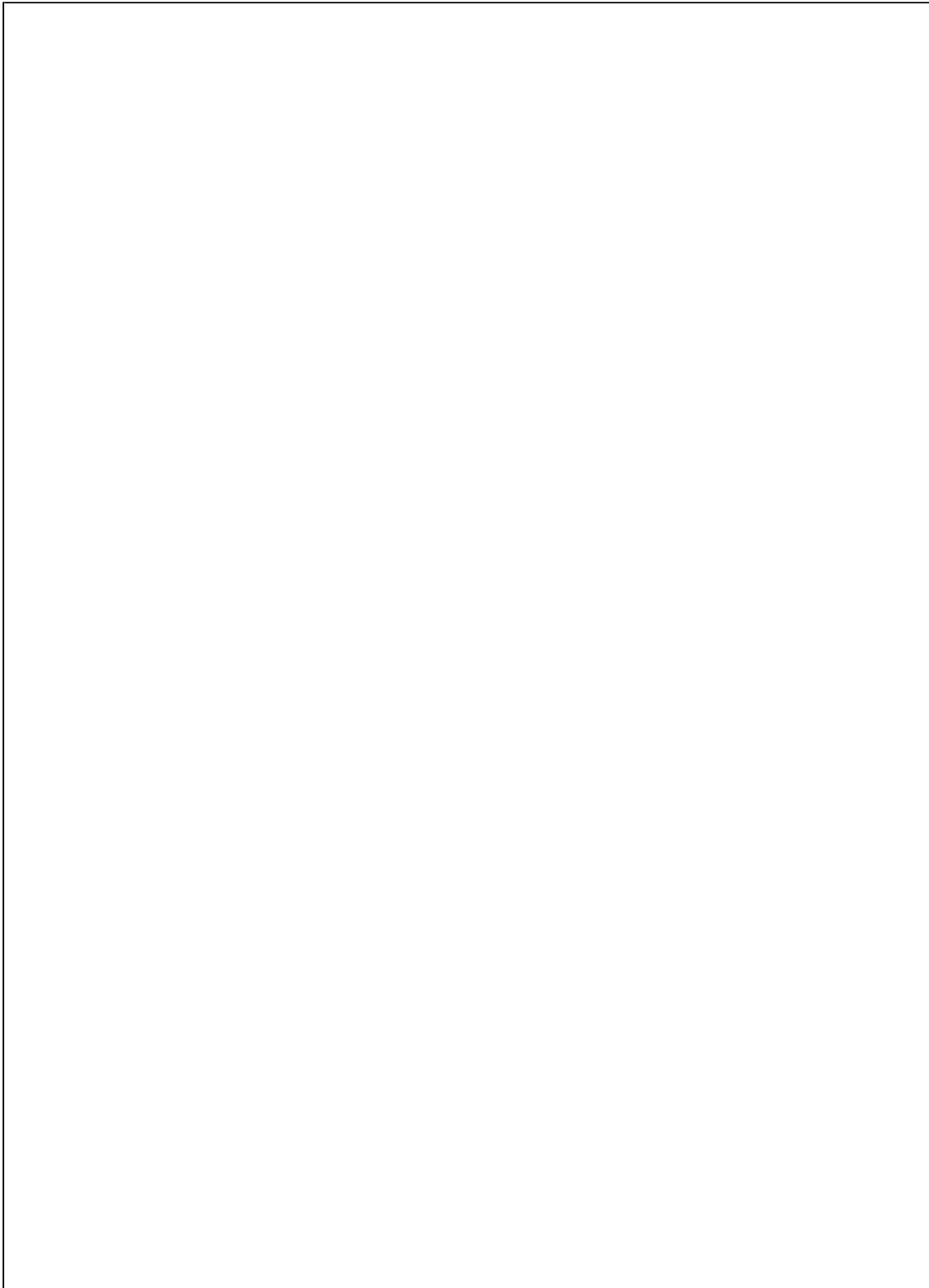
Prove that (1.2) is consistent with (1.1).



(iii) Define the truncation error by

$$T_k(\Delta t) = \frac{x(t_{k+1}) - x(t_k)}{\Delta t} - \Phi(t_k, x(t_k), \Delta t).$$

Prove that (1.2) is of order two, *i.e.*, that  $T_k(\Delta t) = O((\Delta t)^2)$ .



(iv) Prove that (1.2) is stable, *i.e.*, there exist positive constants  $h_0$  and  $c_\Phi$  such that

$$|\Phi(t, x, h) - \Phi(t, y, h)| \leq c_\Phi |x - y|$$

for all  $t \in [0, T]$ ,  $x, y \in \mathbb{R}$ ,  $h \in [0, h_0]$ .



(v) Is (1.2) for solving (1.1) convergent? Explain why.



**Problem 2****[30 Marks]**

(2a) Consider the system of equations

$$\begin{cases} \frac{dp}{dt} = \sin q, & t \geq 0, \\ \frac{dq}{dt} = p, & t \geq 0, \end{cases} \quad (2.1)$$

with initial values  $p(0) = p_0 \in \mathbb{R}$  and  $q(0) = q_0 \in \mathbb{R}$ .

(i) Is (2.1) a Hamiltonian system? You should justify your answer.

(ii) Find an invariant for (2.1), *i.e.* a function  $F$  such that  $F(p(t), q(t))$  is constant for all  $t \geq 0$ .

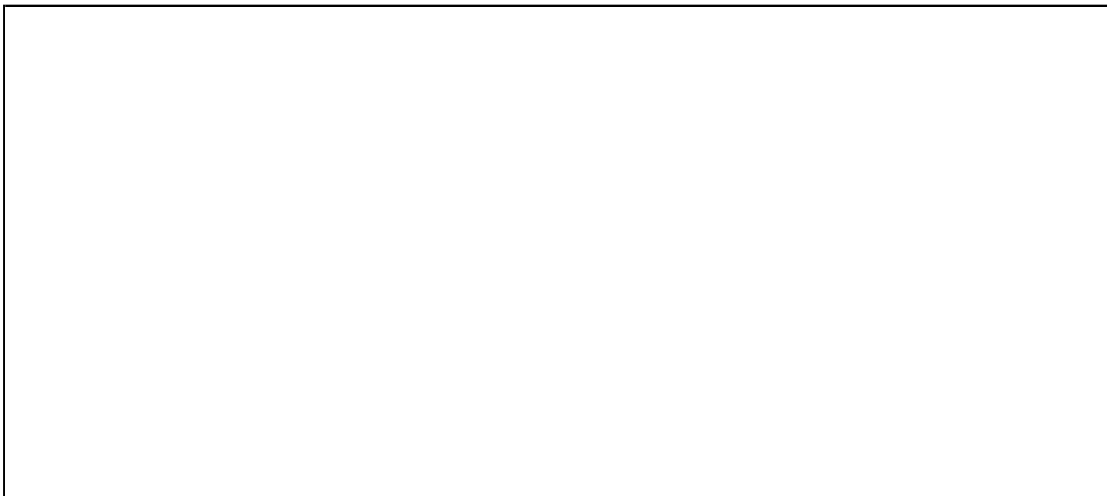
**(2b)** Let  $A$  be a real symmetric  $2 \times 2$  matrix and let  $J$  denote the unitary matrix  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Consider the system of linear equations

$$\begin{cases} \frac{dx}{dt} = J^{-1}Ax, \\ x(0) = x_0 \in \mathbb{R}^2. \end{cases} \quad (2.2)$$

(i) Prove that (2.2) is a Hamilton system associated with the Hamiltonian  $H(x) = \frac{1}{2}x^\top Ax$ , where  $x^\top$  denotes the transpose of  $x$ .



(ii) Define the flow  $\Phi_t$  associated with (2.2) by  $\Phi_t(x_0) = x(t)$ . Prove that  $\Phi_t(x_0) = e^{tJ^{-1}A}x_0$ .





- (iii) Let  $\Phi'_t(x_0)$  be the Jacobian of  $\Phi_t(x_0)$ , i.e.  $\Phi'_t(x_0) = \frac{\partial \Phi_t(x_0)}{\partial x_0}$ . Compute  $\Phi'_t(x_0)$  and prove that  $(\Phi'_t(x_0))^\top J \Phi'_t(x_0) = J$  for all  $x_0 \in \mathbb{R}^2$ .



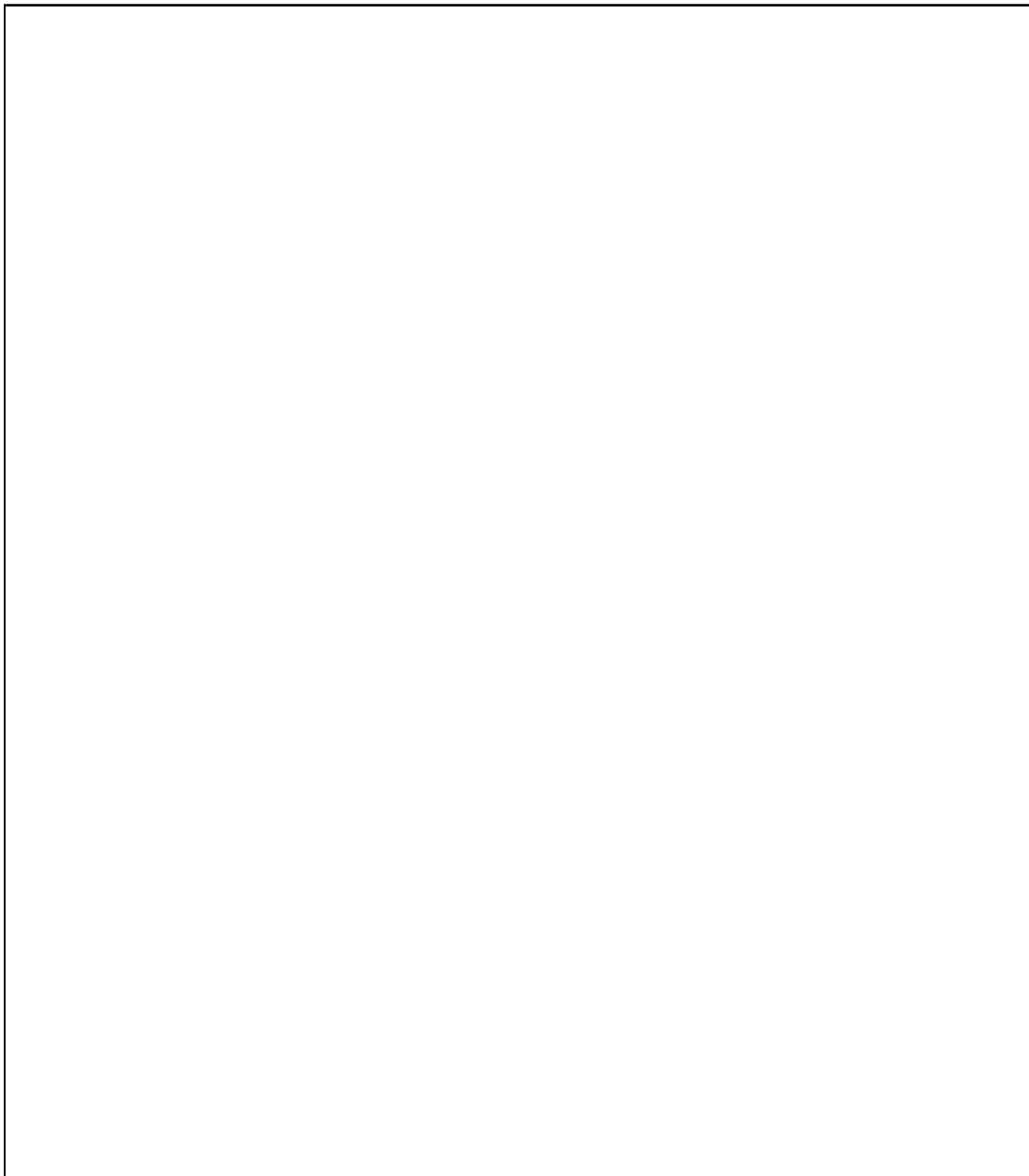
(2c) Let  $A$  be a real symmetric  $2 \times 2$  matrix and let  $J$  denote the unitary matrix  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Suppose the function  $f : [0, T] \rightarrow \mathbb{R}^2$  is continuous. Consider the inhomogeneous system of linear equations

$$\begin{cases} \frac{dx}{dt} = J^{-1}Ax + f(t), \\ x(0) = x_0 \in \mathbb{R}^2. \end{cases} \quad (2.3)$$

(i) Prove that the flow is given by

$$\Phi_t(x_0) = e^{tJ^{-1}A}x_0 + \int_0^t e^{(t-s)J^{-1}A}f(s) ds,$$

for  $t \in [0, T]$ .



(ii) In this case, does the Jacobian  $\Phi'_t(x_0)$  satisfy  $(\Phi'_t(x_0))^\top J \Phi'_t(x_0) = J$  for all  $x_0 \in \mathbb{R}^2$ ?  
Justify your answer.

