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Summer Term 2021

Numerical Analysis II

ETH Zürich
D-MATH

Final Exam Winter 2021

A total of **80 points** can be obtained in this exam. There are three problems of similar length.

Problem 1 **[28 points]**

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be an $n \times n$ matrix with real entries. We denote by $\mathbf{A}^\top$ the transpose of $\mathbf{A}$. We say that A is skew symmetric if $\mathbf{A}^\top = -\mathbf{A}$. We also say that a matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$ is orthogonal if $\mathbf{B}\mathbf{B}^\top = \mathbf{I}$, with $\mathbf{I}$ being the $n \times n$ identity matrix.

(1a) Let the matrix-valued function $\mathbf{Y} : \mathbb{R} \to \mathbb{R}^{n\times n}$ satisfy the linear matrix differential equation

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}t} = \mathbf{A}\mathbf{Y}, \tag{1.1}$$

with $\mathbf{A} \in \mathbb{R}^{n\times n}$ being skew symmetric.

(i) **[4 points]** Compute $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{Y}^\top\mathbf{Y})$ and show that

$$\mathbf{Y}(0) \text{ orthogonal} \quad \Longrightarrow \quad \mathbf{Y}(t) \text{ orthogonal}.$$

(ii) **[6 points]** Consider the implicit mid-point method

$$\mathbf{Y}_{k+1} = \mathbf{Y}_k + \frac{\Delta t}{2}\mathbf{A}(\mathbf{Y}_k + \mathbf{Y}_{k+1}), \quad k = 0, \dots, K-1, \tag{1.2}$$

for solving (1.1) on $[0, T]$ with the initial condition $\mathbf{Y}_0 = \mathbf{Y}(0)$. Here, $\Delta t > 0$, $K = T/\Delta t$, $t_k = k\Delta t$ for $k = 0, \dots, K$ and $\mathbf{Y}_k = \mathbf{Y}(t_k)$. We assume that K is an integer, for simplicity. Show that the solution $\mathbf{Y}_k$ computed via the implicit mid-point method (1.2) satisfies

$$\mathbf{Y}_0 \text{ orthogonal} \quad \Longrightarrow \quad \mathbf{Y}_k \text{ orthogonal for all } 1 \le k \le K.$$

HINT: Use the substitution $\mathbf{Y}_{k+1} - \mathbf{Y}_k = \frac{\Delta t}{2}\mathbf{A}(\mathbf{Y}_k + \mathbf{Y}_{k+1})$ to show that $2\mathbf{Y}_{k+1}^\top\mathbf{Y}_{k+1} - 2\mathbf{Y}_k^\top\mathbf{Y}_k = 0$.

(1b) In addition to the implicit mid-point method, recall the explicit Euler method

$$\mathbf{Y}_{k+1} = \mathbf{Y}_k + \Delta t\mathbf{A}\mathbf{Y}_k \tag{1.3}$$

and the implicit Euler method

$$\mathbf{Y}_{k+1} = \mathbf{Y}_k + \Delta t\mathbf{A}\mathbf{Y}_{k+1}. \tag{1.4}$$

(i) **[8 points]** Complete the python templates `ExplEulStep.py`, `ImplEulStep.py` and `ImplMidpStep.py` to implement each of the explicit Euler method (1.3), the implicit Euler method (1.4) and the implicit mid-point method (1.2).

(ii) **[10 points]** Let $n = 2$, $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{Y}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Compute, using the three methods from part (1b)(i), discrete approximations $\mathbf{Y}_k$ of $\mathbf{Y}(t_k)$, for $k = 1, \ldots, 20$ with $\Delta t = 1/20$. Compare the norms $\left\|\mathbf{Y}_k^\top \mathbf{Y}_k - \mathbf{I}\right\|_F$, for $k = 1, \ldots, 20$. Briefly explain which methods you observe to preserve orthogonality, referring to the result from subproblem (1a)(ii).

HINT: The Frobenius norm $\|\mathbf{M}\|_F$ of an $m \times n$ matrix $\mathbf{M}$ is defined as the square root of the sum of the absolute squares of its elements:

$$\|\mathbf{M}\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|m_{ij}|^2}. \tag{1.5}$$

It can be computed in Python using the command `linalg.norm(M,'fro')` from the `numpy` library.

Problem 2 **[26 points]**

Let $\omega \in \mathbb{R}$, $\omega \neq 0$. Consider the system of equations

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = -\omega q, \quad t \in [0,T], \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = \omega p, \\ p(t=0) = p_0 \in \mathbb{R}, \\ q(t=0) = q_0 \in \mathbb{R}. \end{cases} \tag{2.1}$$

(2a)

(i) **[2 points]** Prove that (2.1) is a Hamiltonian system associated with the Hamiltonian function $H(p,q) = \omega(p^2+q^2)/2$.

(ii) **[3 points]** Prove that p^2+q^2 is an invariant of (2.1).

(2b) **[3 points]** Formulate the Leapfrog method (section 5.2.5 in the lecture notes) for (2.1).

(2c) **[5 points]** Let $\mathbf{y}(t) = (p(t), q(t))^\top$ and $\mathbf{y}(t=0) = \mathbf{y}_0 = (p_0, q_0)^\top$. Prove that

$$\mathbf{y}(t) = \mathbf{W}(t\omega)\mathbf{y}_0,$$

is the analytic solution to (2.1) with

$$\mathbf{W}(t\omega) := \exp\left(t \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \right).$$

Check that the eigenvalues of $\mathbf{W}(t\omega)$ are $\exp(\pm \mathrm{i}\, \omega\, t)$.

(2d) **[4 points]** Transform the Leapfrog scheme from subproblem (2b) into the form

$$\mathbf{y}_{k+1} = \mathbf{S}(\Delta t\, \omega)\mathbf{y}_k,$$

with

$$\mathbf{S}(\Delta t\, \omega) = \begin{pmatrix} 1 - \frac{1}{2}(\Delta t\omega)^2 & -\Delta t\omega + \frac{1}{4}(\Delta t\omega)^3 \\ \Delta t\omega & 1 - \frac{1}{2}(\Delta t\omega)^2 \end{pmatrix},$$

which has the eigenvalues $\lambda_\pm = \frac{1}{2}(2 - (\Delta t)^2\,\omega^2 \pm \sqrt{(\Delta t)^4\,\omega^4 - 4(\Delta t)^2\omega^2})$.

(2e) **[9 points]** Implement the spectra of the exact and approximate evolution operators $\mathbf{W}(\Delta t\,\omega)$ and $\mathbf{S}(\Delta t\,\omega)$. What do you observe for $\Delta t\omega < 1$ and for $\Delta t \to +\infty$? What can you deduce about the stability of the numerical method?

Problem 3 **[26 points]**

Let the continuously differentiable matrix-valued mapping $\mathbf{A} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ satisfy

$$\mathbf{A}(\mathbf{y})^\top = -\mathbf{A}(\mathbf{y}) \quad \forall \mathbf{y} \in \mathbb{R}^d.$$

(3a) Let $\mathbf{M} \in \mathbb{R}^{d\times d}$ be a $d \times d$ matrix with real entries. We denote by $\mathbf{M}^\top$ the transpose of $\mathbf{M}$. We say that $\mathbf{M}$ is skew symmetric if $\mathbf{M}^\top = -\mathbf{M}$. We also say that a matrix $\mathbf{B} \in \mathbb{R}^{d\times d}$ is orthogonal if $\mathbf{B}\mathbf{B}^\top = \mathbf{I}$, with $\mathbf{I}$ being the $d \times d$ identity matrix.

(i) **[2 points]** Prove that if $\mathbf{M}$ is skew symmetric then $\mathbf{M}x \cdot x = 0$ for all $x \in \mathbb{R}^d$.

(ii) **[2 points]** Prove that if $\mathbf{M}$ is skew symmetric then $e^{\mathbf{M}}$ is orthogonal.

(3b) **[2 points]** Show that the Euclidean vector norm $\|\mathbf{y}\|$ is an invariant of every solution of the autonomous differential equation

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{A}(\mathbf{y})\mathbf{y}. \tag{3.1}$$

HINT: Compute $\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{y}\|^2$.

(3c) Consider the explicit Euler method,

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \Delta t \mathbf{A}(\mathbf{y}_k)\mathbf{y}_k \tag{3.2}$$

the implicit Euler method

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \Delta t \mathbf{A}(\mathbf{y}_{k+1})\mathbf{y}_{k+1} \tag{3.3}$$

and the implicit single step method

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \Delta t \mathbf{A}(\mathbf{y}_k + \mathbf{y}_{k+1})(\mathbf{y}_k + \mathbf{y}_{k+1}) \ . \tag{3.4}$$

each applied to (3.1).

(i) **[2 points]** Show that $\|\mathbf{y}_{k+1}\| \geq \|\mathbf{y}_k\|$ for the explicit Euler method (3.2) applied to (3.1).

(ii) **[2 points]** Show that $\|\mathbf{y}_{k+1}\| \leq \|\mathbf{y}_k\|$ for the implicit Euler method (3.3) applied to (3.1).

(iii) **[2 points]** Show that $\|\mathbf{y}_{k+1}\| = \|\mathbf{y}_k\|$ for the implicit single step method (3.4) applied to (3.1).

(3d) Consider now the explicit exponential single step method defined by

$$\begin{aligned} \mathbf{y}_{k+\frac{1}{2}} &= \mathbf{y}_k + \tfrac{1}{2}\Delta t \mathbf{A}(\mathbf{y}_k)\mathbf{y}_k \ , \\ \mathbf{y}_{k+1} &= \exp(\Delta t \mathbf{A}(\mathbf{y}_{k+\frac{1}{2}}))\mathbf{y}_k \ . \end{aligned} \tag{3.5}$$

(i) **[2 points]** Show that $\|\mathbf{y}_{k+1}\| = \|\mathbf{y}_k\|$ for (3.5).

HINT: Use the result from subproblem (3a)(i).

(ii) **[4 points]** Prove that, provided $t \mapsto y(t)$ is smooth enough,

$$\mathbf{y}(t+\Delta t) = \mathbf{y}(t)+\Delta t \mathbf{A}(\mathbf{y}(t))\mathbf{y}(t)+\frac{(\Delta t)^2}{2}\big[\nabla \mathbf{A}(\mathbf{y}(t))(\mathbf{A}(\mathbf{y})\mathbf{y},\mathbf{y}) + \mathbf{A}(\mathbf{y}(t))^2\mathbf{y}(t)\big]+\mathcal{O}\big((\Delta t)^3\big)$$

HINT: Use a Taylor expansion.

(iii) **[1 point]** Rewrite (3.5) in the form

$$\mathbf{y}^{k+1} = \mathbf{y}^k + \Delta t \Phi(t_k, \mathbf{y}^k, \Delta t)$$

with $\Phi(t, \mathbf{y}, \Delta t)$ defined by

$$\Phi(t, \mathbf{y}, \Delta t) = \frac{1}{\Delta t}\left(\exp\left(\Delta t \mathbf{A}\left(\mathbf{y} + \frac{\Delta t}{2}\mathbf{A}(\mathbf{y})\mathbf{y}\right)\right) - \mathbf{I}\right)\mathbf{y}$$

(iv) **[5 points]** Deduce that the truncation error

$$T_k(\Delta t) := \frac{\mathbf{y}(t_{k+1}) - \mathbf{y}(t_k)}{\Delta t} - \Phi(t_k, \mathbf{y}(t_k), \Delta t)$$

is of order $\mathcal{O}((\Delta t)^2)$ as $\Delta t \to 0$ and therefore, the explicit exponential single step scheme is consistent and is of order at least 2.

(3e) **[2 points]** Suppose that $\mathbf{A}(\mathbf{y}) = \mathbf{A}$, a constant skew-symmetric matrix, for all $\mathbf{y} \in \mathbb{R}^d$. Show that (3.5) yields the exact solution in this case.