

End-term Assignment 2021

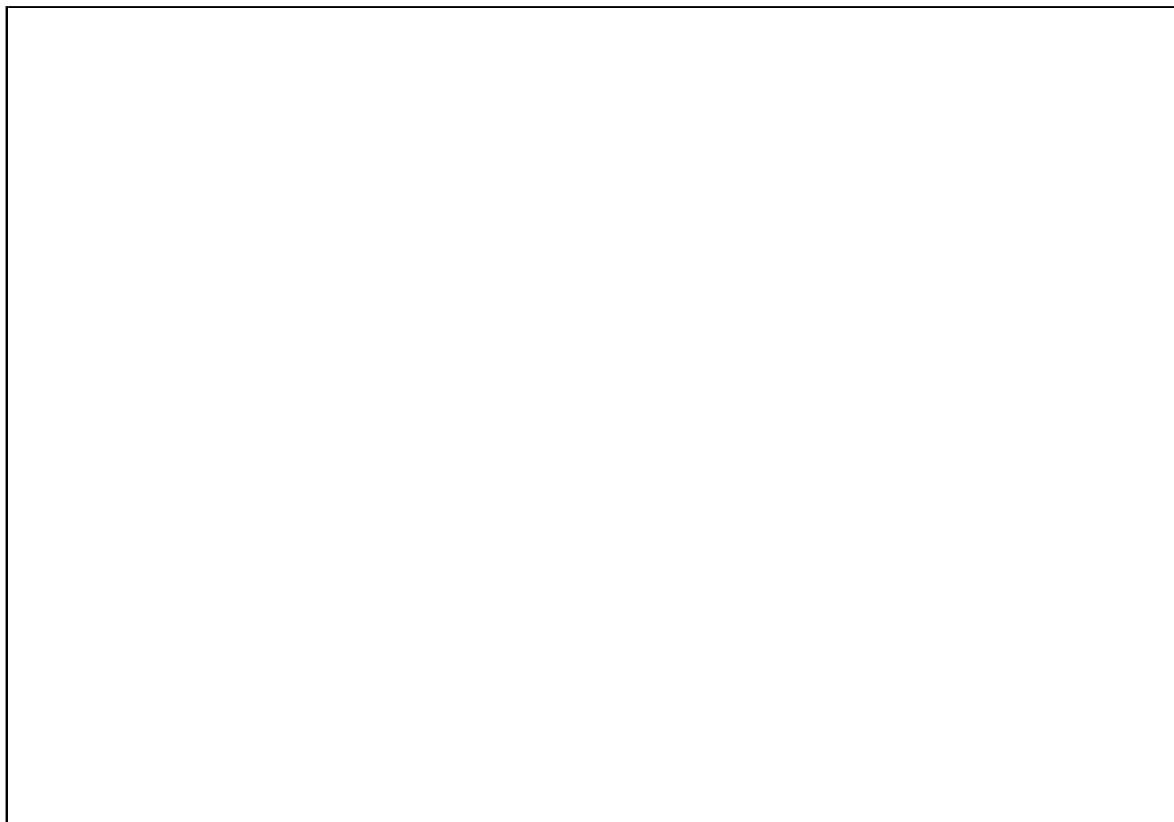
Problem 1

[60 Marks]

For $T > 0$, consider the system of equations

$$\begin{cases} \frac{dp}{dt} = -\sin q, & t \in [0, T], \\ \frac{dq}{dt} = p & t \in [0, T], \\ p(0) = p_0 \in \mathbb{R}, q(0) = q_0 \in \mathbb{R}. \end{cases} \quad (1.1)$$

(1a) Check that (1.1) is a Hamiltonian system and find its associated Hamiltonian function H .



Consider (1.1) in the form

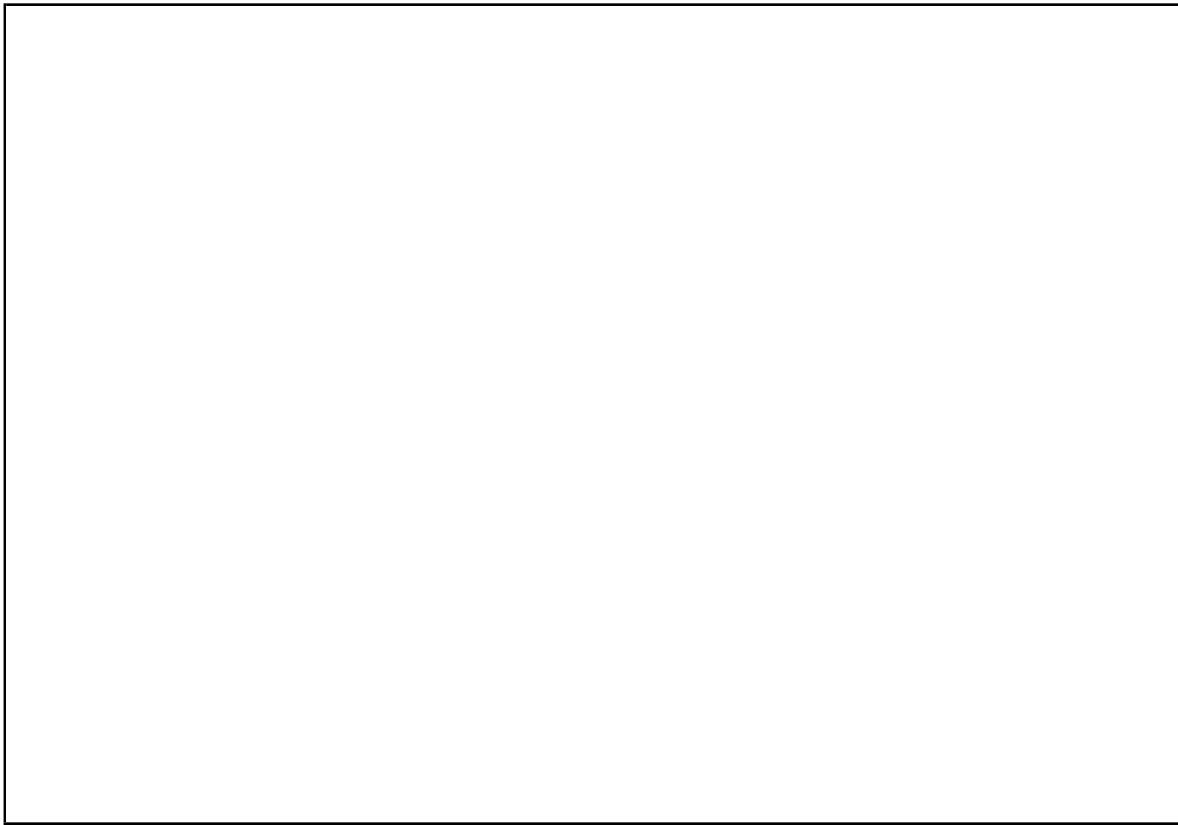
$$\begin{cases} \frac{dx}{dt} = f(x), \\ x(0) = x_0 = (p_0, q_0)^\top, \end{cases} \quad (1.2)$$

where

$$x = (p, q)^\top, \quad f(x) = J^{-1}\nabla H(x), \quad \text{and } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (1.3)$$

(1b) Is f divergence-free?

(1c) Is the flow φ_t associated with (1.1) volume preserving? Explain why.



Consider the numerical scheme (for $\Delta t > 0$)

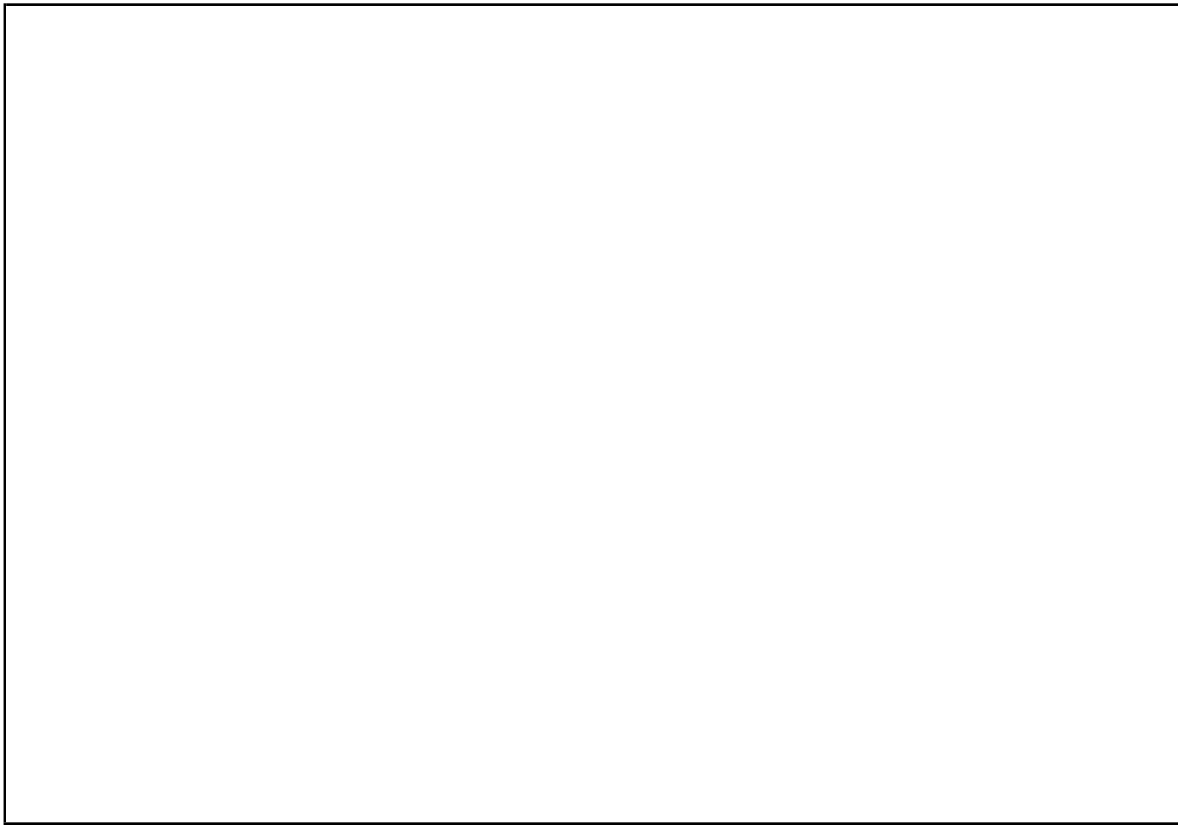
$$\begin{cases} p^{k+1} = p^k - \Delta t \sin q^{k+1}, \\ q^{k+1} = q^k + \Delta t p^k. \end{cases} \quad (1.4)$$

Define the numerical flow

$$\Phi_{\Delta t} : (p^k, q^k) \mapsto (p^{k+1}, q^{k+1}). \quad (1.5)$$

(1d) Compute the Jacobian $\Phi'_{\Delta t}$ of the numerical flow $\Phi_{\Delta t}$ defined by

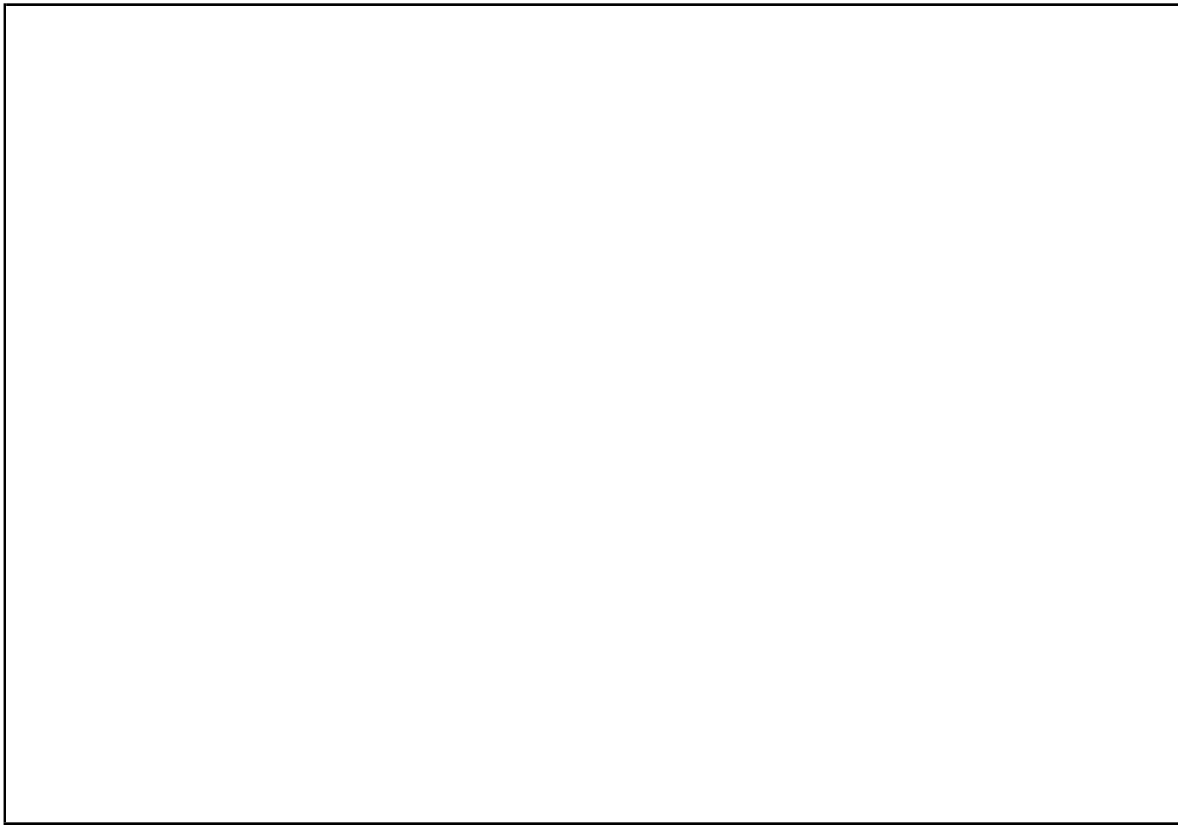
$$\Phi'_{\Delta t}(p^k, q^k) = \frac{\partial(p^{k+1}, q^{k+1})}{\partial(p^k, q^k)}.$$



(1e) Is (1.4) symplectic, i.e. does $\Phi'_{\Delta t}$ satisfies

$$(\Phi'_{\Delta t})^\top J \Phi'_{\Delta t} = J?$$

Explain why. Here J^\top denotes the transpose of J .



(1f) (1.4) is a...

Symplectic Euler method. Leapfrog method.

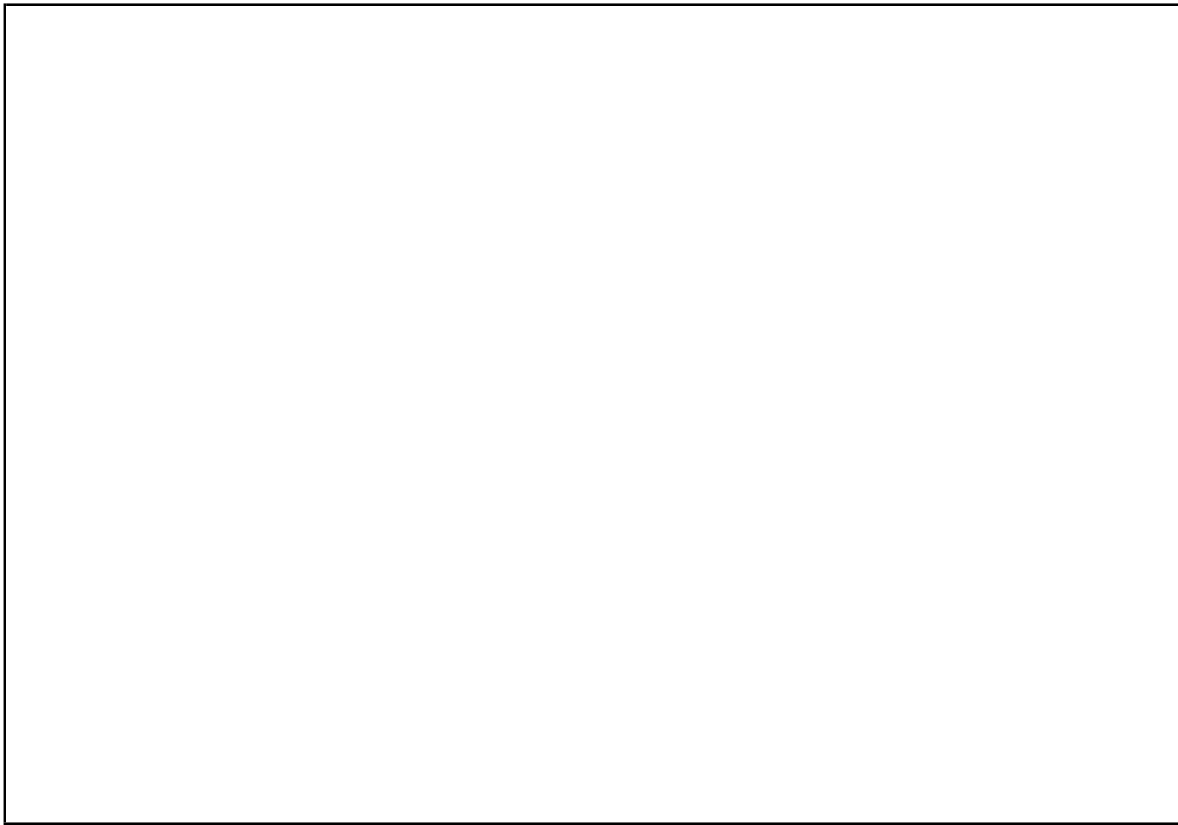
(1g) Is (1.4) symmetric, i.e. does

$$\Phi_{\Delta t}^* = \Phi_{\Delta t}?$$

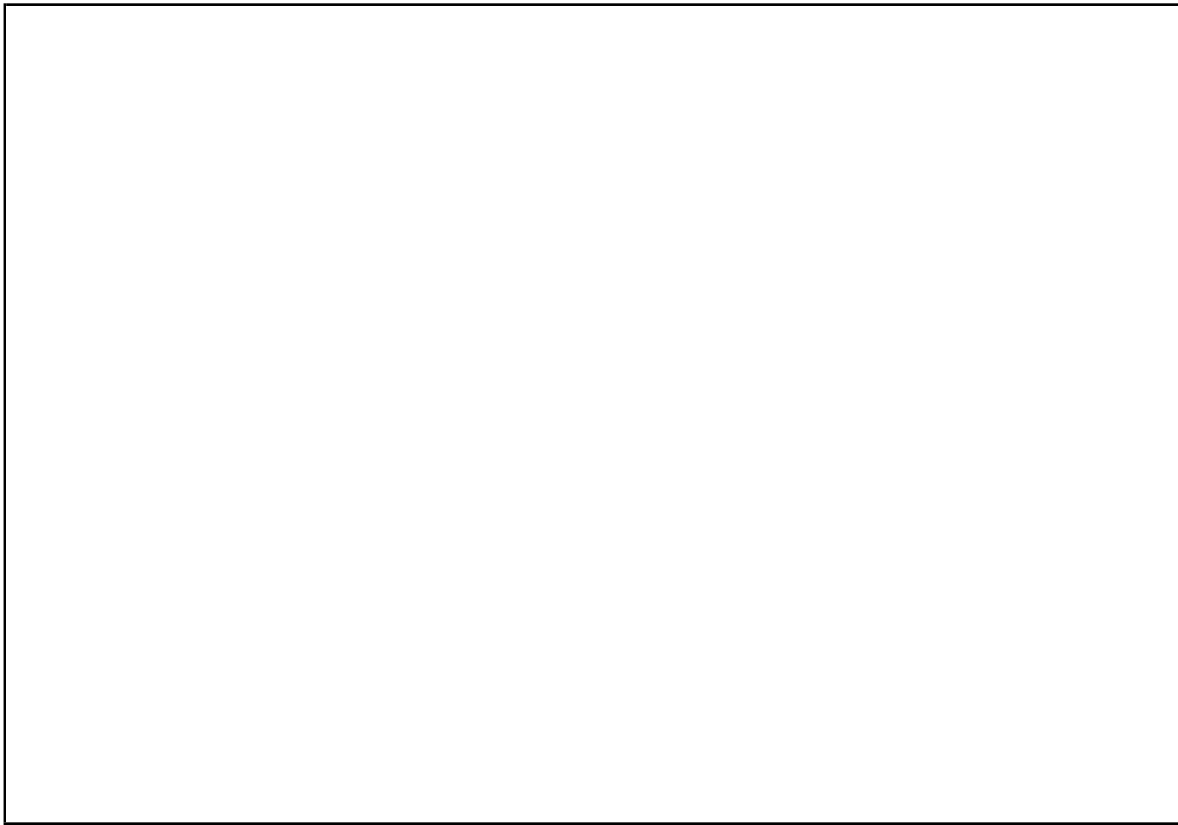
Justify your answer. Here $\Phi_{\Delta t}^*$ is defined by $\Phi_{\Delta t}^* = (\Phi_{-\Delta t})^{-1}$, i.e., by replacing Δt by $-\Delta t$ and exchanging k and $k + 1$.



(1h) Is the composition $\Phi_{\Delta t/2}^* \circ \Phi_{\Delta t/2}$ symmetric? Explain why.



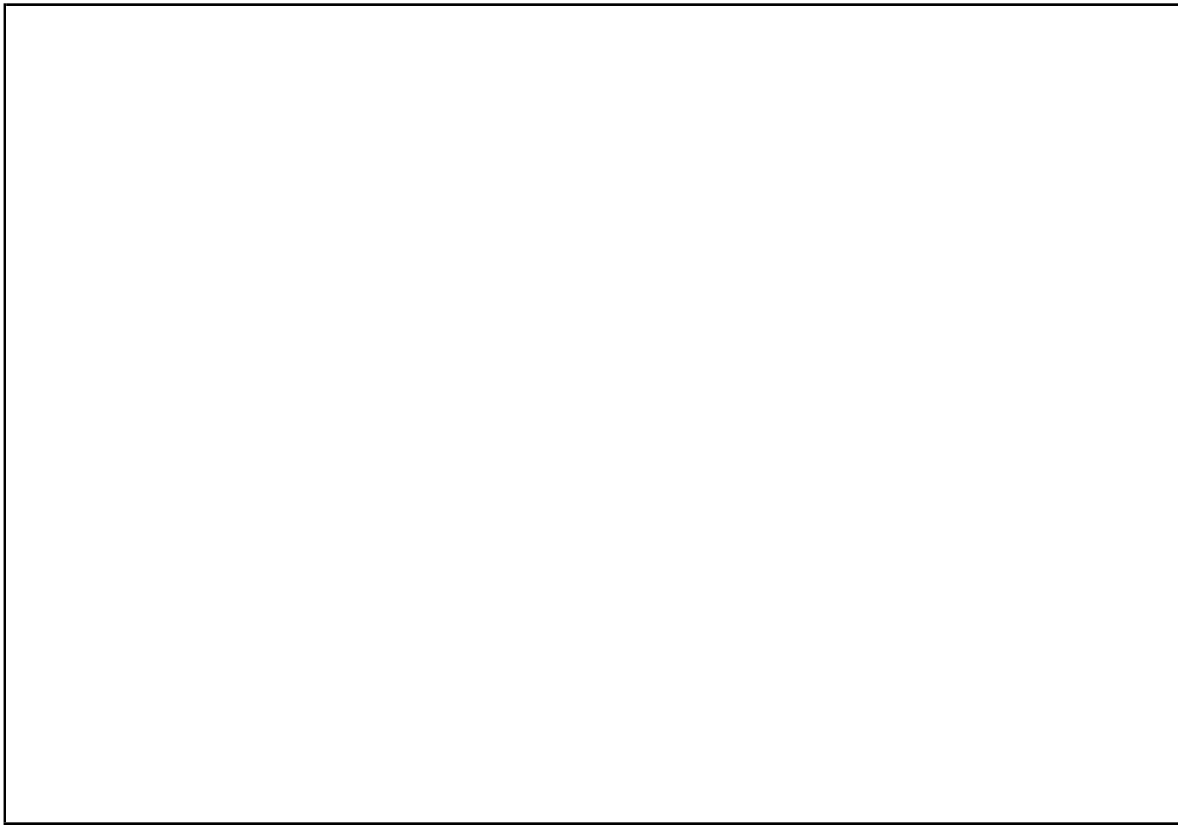
(1i) Is the composition $\Phi_{\Delta t/2}^* \circ \Phi_{\Delta t/2}$ symplectic? Explain why.



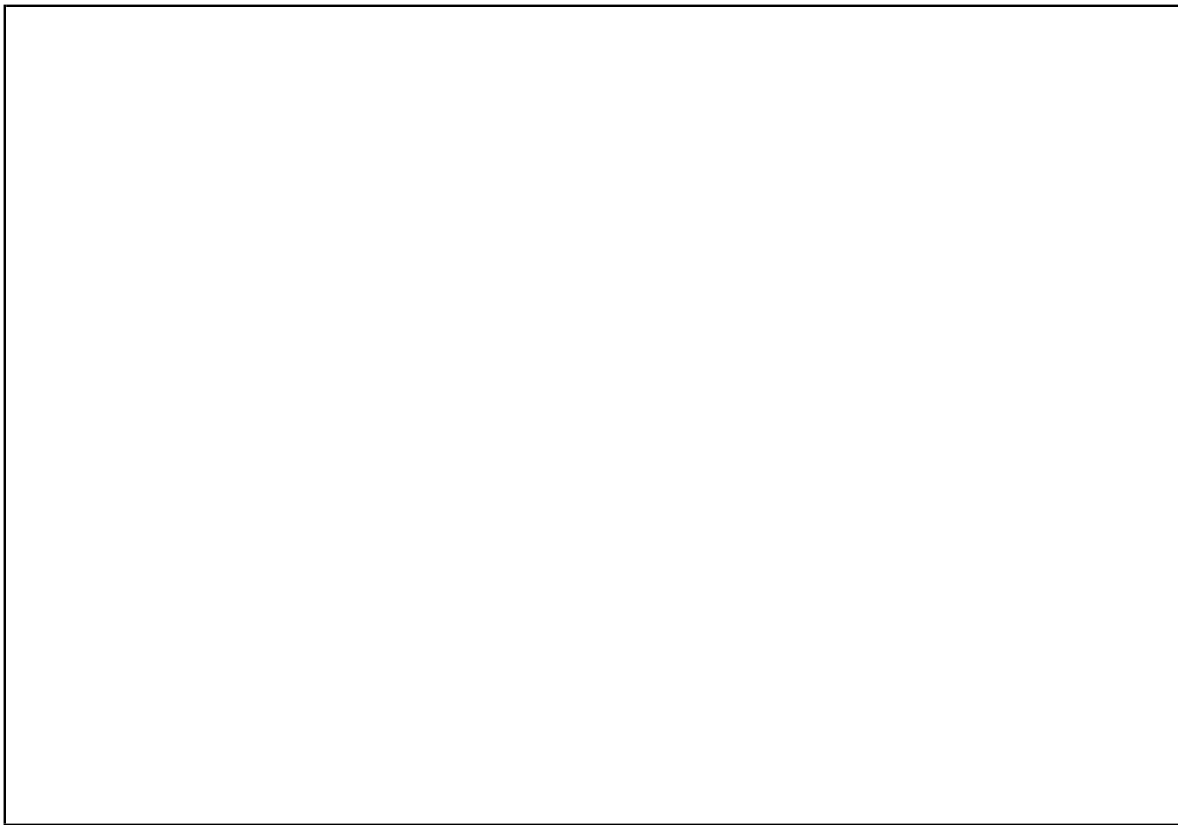
(1j) The composition $\Phi_{\Delta t/2}^* \circ \Phi_{\Delta t/2}$ is...

a Symplectic Euler method. a Leapfrog method.

(1k) Is the composition $\Phi_{\Delta t/2}^* \circ \Phi_{\Delta t/2}$ of order two? Explain why.



(11) Is (1.4) volume preserving? Explain why.

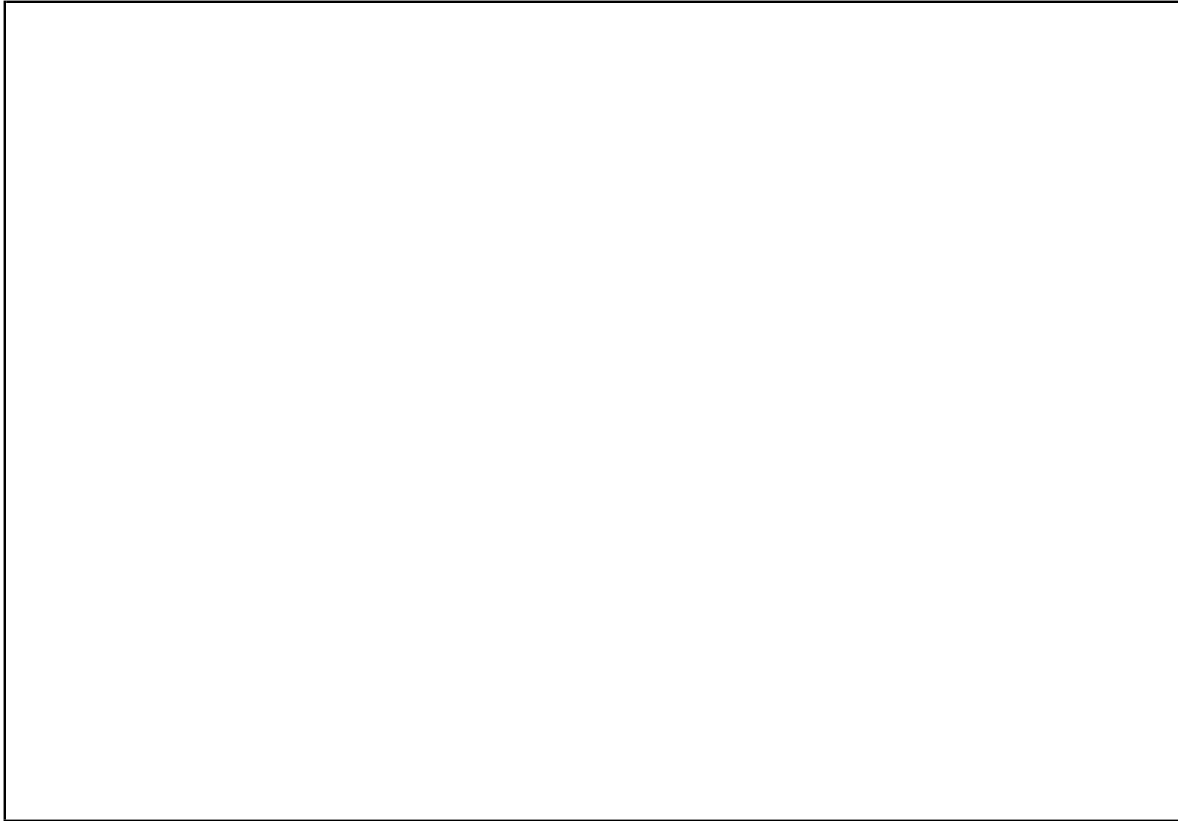


Consider the midpoint scheme

$$\begin{cases} x^{k+1} = x^k + \Delta t f\left(\frac{x^k + x^{k+1}}{2}\right), \\ x^0 = x_0, \end{cases} \quad (1.6)$$

for solving (1.2) where Δt is the step size and f is defined by (1.3).

(1m) Prove that the method is symplectic.



Suppose that there exists a matrix $S \in \mathbb{R}^{2 \times 2}$ (symmetric and positive definite) and a positive constant c such that

$$x(t)^\top S x(t) = c, \quad \forall t \geq 0, \quad \text{where } x \text{ is the solution to (1.2)}. \quad (1.7)$$

(1n) Prove that $(x^k)^\top S x^k = c$ for $k \geq 0$, where x^k is from (1.6).



(10) Let us denote $x^k = (p^k, q^k)$. Does the midpoint scheme (1.6) preserve the quantity

$$\frac{1}{2}p_k^2 - \cos q_k ?$$



Problem 2 [Bonus]

[20 Marks]

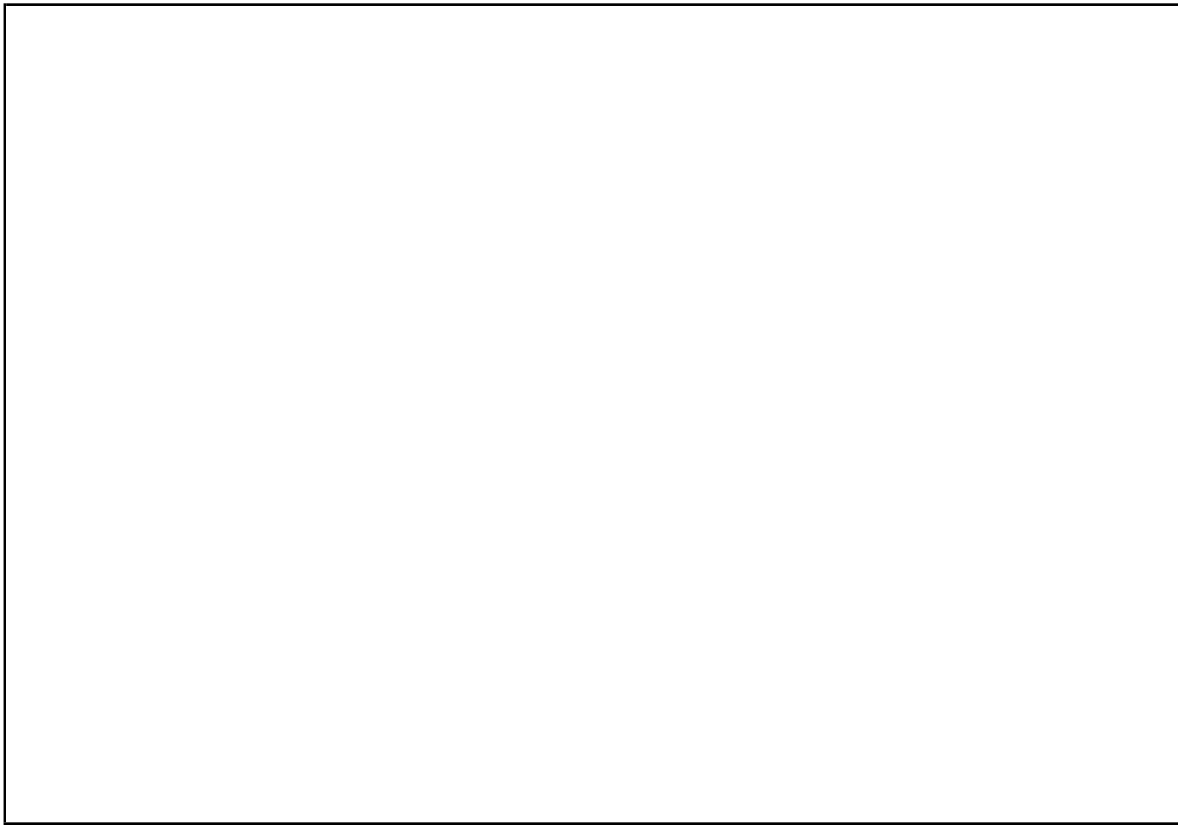
Consider the advection equation (with periodic boundary conditions):

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = -\frac{\partial u}{\partial x}(t, x), & 0 \leq x \leq 1, t \geq 0, \\ u(t, x+1) = u(t, x), & 0 \leq x \leq 1, t \geq 0, \\ u(0, x) = u_0(x), & 0 \leq x \leq 1. \end{cases} \quad (2.1)$$

The initial datum $u_0(x)$ is periodic with period one.

(2a) Use the finite difference approximations to derive the following scheme for solving (2.1):

$$\frac{u_j^{h+1} - u_j^h}{\Delta t} + \frac{u_{j+1}^h - u_{j-1}^h}{2\Delta x} = 0. \quad (2.2)$$



(2b) Prove that (2.2) is consistent with (2.1).



(2c) Using Fourier analysis prove that (2.2) is unconditionally stable in L^2 .

