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Numerical Analysis II

ETH Zürich
D-MATH

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Problem 1 **[60 Marks]**

For $T > 0$, consider the system of equations

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = -\sin q, & t \in [0,T], \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = p & t \in [0,T], \\ p(0) = p_0 \in \mathbb{R},\ q(0) = q_0 \in \mathbb{R}. \end{cases} \tag{1.1}$$

(1a) Check that (1.1) is a Hamiltonian system and find its associated Hamiltonian function H.

Consider (1.1) in the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x), \\ x(0) = x_0 = (p_0, q_0)^\top, \end{cases} \tag{1.2}$$

where

$$x = (p, q)^\top, \quad f(x) = J^{-1}\nabla H(x), \quad \text{and } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.3}$$

(1b) Is f divergence-free?

(1c) Is the flow φ_t associated with (1.1) volume preserving? Explain why.

Consider the numerical scheme (for $\Delta t > 0$)

$$\begin{cases} p^{k+1} = p^k - \Delta t \sin q^{k+1}, \\ q^{k+1} = q^k + \Delta t p^k. \end{cases} \tag{1.4}$$

Define the numerical flow

$$\Phi_{\Delta t} : (p^k, q^k) \mapsto (p^{k+1}, q^{k+1}). \tag{1.5}$$

(1d) Compute the Jacobian Φ'_{Δ} of the numerical flow $\Phi_{\Delta t}$ defined by

$$\Phi'_{\Delta t}(p^k, q^k) = \frac{\partial(p^{k+1}, q^{k+1})}{\partial(p^k, q^k)}.$$

(1e) Is (1.4) symplectic,. i.e. does $\Phi'_{\Delta t}$ satisfies

$$(\Phi'_{\Delta t})^\top J \Phi'_{\Delta t} = J?$$

Explain why. Here $J^\top$ denotes the transpose of J.

(1f) (1.4) is a...

Symplectic Euler method. ☐ Leapfrog method. ☐

(1g) Is (1.4) symmetric, i.e. does

$$\Phi^*_{\Delta t} = \Phi_{\Delta t}?$$

Justify your answer. Here $\Phi^*_{\Delta t}$ is defined by $\Phi^*_{\Delta t} = (\Phi_{-\Delta t})^{-1}$, i.e., by replacing Δt by $-\Delta t$ and exchanging k and $k+1$.

(1h) Is the composition $\Phi^*_{\Delta t/2} \circ \Phi_{\Delta t/2}$ symmetric? Explain why.

(1i) Is the composition $\Phi^*_{\Delta t/2} \circ \Phi_{\Delta t/2}$ symplectic? Explain why.

(1j) The composition $\Phi^*_{\Delta t/2} \circ \Phi_{\Delta t/2}$ is...

a Symplectic Euler method. ☐ a Leapfrog method. ☐

(1k) Is the composition $\Phi^*_{\Delta t/2} \circ \Phi_{\Delta t/2}$ of order two? Explain why.

(1l) Is (1.4) volume preserving? Explain why.

Consider the midpoint scheme

$$\begin{cases} x^{k+1} = x^k + \Delta t f\Big(\dfrac{x^k + x^{k+1}}{2}\Big), \\ x^0 = x_0, \end{cases} \tag{1.6}$$

for solving (1.2) where Δt is the step size and f is defined by (1.3).

(1m) Prove that the method is symplectic.

Suppose that there exists a matrix $S \in \mathbb{R}^{2\times 2}$ (symmetric and positive definite) and a positive constant c such that

$$x(t)^\top S x(t) = c, \quad \forall t \geq 0, \quad \text{where } x \text{ is the solution to } (1.2). \tag{1.7}$$

(1n) Prove that $(x^k)^\top S x^k = c$ for $k \geq 0$, where x^k is from (1.6).

(1o) Let us denote $x^k = (p^k, q^k)$. Does the midpoint scheme (1.6) preserve the quantity

$$\frac{1}{2}p_k^2 - \cos q_k \text{ ?}$$

Problem 2 [Bonus] [20 Marks]

Consider the advection equation (with periodic boundary conditions):

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = -\dfrac{\partial u}{\partial x}(t,x), & 0 \le x \le 1,\ t \ge 0, \\ u(t,x+1) = u(t,x), & 0 \le x \le 1,\ t \ge 0, \\ u(0,x) = u_0(x), & 0 \le x \le 1. \end{cases} \tag{2.1}$$

The initial datum $u_0(x)$ is periodic with period one.

(2a) Use the finite difference approximations to derive the following scheme for solving (2.1):

$$\frac{u_j^{h+1} - u_j^h}{\Delta t} + \frac{u_{j+1}^h - u_{j-1}^h}{2\Delta x} = 0. \tag{2.2}$$

(2b) Prove that (2.2) is consistent with (2.1).

(2c) Using Fourier analysis prove that (2.2) is unconditionally stable in L^2.