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Numerical Analysis II

ETH Zürich
D-MATH

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Problem 1 [40 Marks]

(1a) Let $f \in C^2$ be an arbitrary function subject to the Lipschitz condition

$$|f(x) - f(y)| \leq C|x - y|, \ \forall x, y \in \mathbb{R}.$$

Consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad t \in [0, T], \\ x(0) = x_0 \in \mathbb{R}. \end{cases} \tag{1.1}$$

(i) Does (1.1) have a unique solution $x(t) \in C^3([0, T])$? Justify.

(ii) If we regard $x(t)$ also as a function of the initial value x_0, what is the equation satisfied by the derivative with respect to t of $\partial x(t)/\partial x_0$? Is it a linear equation? Justify.

(1b) Consider the scheme

$$x^{k+1} = x^k + \Delta t\big[\alpha f(x^k) + \beta f\big(x^k + \gamma \Delta t f(x^k)\big)\big], \tag{1.2}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$, $\Delta t > 0$ is small enough and let $t_k = k\Delta t$ for $k \in \mathbb{N}$.

(i) The scheme (1.2) is an...

explicit one-step method. ☐	explicit two-step method. ☐
implicit one-step method. ☐	implicit two-step method. ☐

(ii) Let $\Phi(t_k, x^k, \Delta t)$ be defined by

$$\Phi(t_k, x^k, \Delta t) = \alpha f(x^k) + \beta f\big(x^k + \gamma \Delta t f(x^k)\big),$$

so that (1.2) can be rewritten in the form

$$x^{k+1} = x^k + \Delta t \Phi(t_k, x^k, \Delta t).$$

Prove that (1.2) is consistent with (1.1) if and only if $\alpha + \beta = 1$.

(iii) Define the truncation error by

$$T_k(\Delta t) = \frac{x(t_{k+1}) - x(t_k)}{\Delta t} - \Phi(t_k, x(t_k), \Delta t).$$

Find α, β, γ such that (1.2) is of order two, *i.e.*, that $T_k(\Delta t) = O((\Delta t)^2)$.

(iv) Prove that (1.2) is stable, *i.e.*, there exist positive constants h_0 and c_Φ such that

$$|\Phi(t,x,h) - \Phi(t,y,h)| \leq c_\Phi |x-y|$$

for all $t \in [0,T]$, $x, y \in \mathbb{R}$, $h \in [0, h_0]$.

(v) When $\alpha + \beta = 1$, is scheme (1.2) for solving (1.1) convergent? Explain why.

(1c) Consider the following particular II order differential equation:

$$\begin{cases} \dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -x, \qquad t \in [0, +\infty[, \\ x(0) = x_0, \\ \dfrac{\mathrm{d}x}{\mathrm{d}t}(0) = x_1, \end{cases} \tag{1.3}$$

with initial data $x_0, x_1 \in \mathbb{R}$.

(i) Rewrite (1.3) in the form of a first order linear equation:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = AX,$$

with A being a 2×2 matrix.

(ii) Assuming $\alpha + \beta = 1$, apply (1.2) to solve this first-order linear equation and find the 2×2 matrix $B(\Delta t)$ such that

$$X^{k+1} = B(\Delta t) X^k.$$

(iii) Discuss the behavior of X^k as $k \to +\infty$.

Problem 2 **[20 Marks]**

Let k be a positive constant and consider the II order differential equation

$$\begin{cases} \dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -k \sin x \quad \text{on } [0,T], \\ x(0) = x_0, \\ \dfrac{\mathrm{d}x}{\mathrm{d}t}(0) = x_1, \end{cases} \tag{2.1}$$

with initial data $x_0, x_1 \in \mathbb{R}$.

(2a)

(i) Set $p(t) = x(t)$, $q(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$ and $p_0 = x_0$, $q_0 = x_1$. Rewrite (2.1) in the following form

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = q & \text{on } [0,T], \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = -k \sin p & \text{on } [0,T], \\ q(0) = q_0, \\ p(0) = p_0. \end{cases} \tag{2.2}$$

(ii) Is (2.2) a Hamiltonian system? You should justify your answer.

(iii) Find an invariant for (2.2), i.e., a function F such that $F(p(t), q(t))$ is constant for all $t \in [0, T]$.

(iv) Define the flow ϕ_t associated with (2.2) by $\phi_t(p_0, q_0) = (p(t), q(t))$.

Show that $\phi_t'(p_0, q_0)$ the jacobian of $\phi_t(p_0, q_0)$, i.e., $\phi_t'(p_0, q_0) = \frac{\partial \phi_t(p_0,q_0)}{\partial(p_0,q_0)}$, satisfies a linear differential equation and write it down.

(v) Is ϕ_t symplectic? Justify.