

Mid-term Assingment 2021

Problem 1

[40 Marks]

(1a) Let $f \in C^2$ be an arbitrary function subject to the Lipschitz condition

$$|f(x) - f(y)| \leq C|x - y|, \quad \forall x, y \in \mathbb{R}.$$

Consider the initial value problem

$$\begin{cases} \frac{dx}{dt} = f(x), & t \in [0, T], \\ x(0) = x_0 \in \mathbb{R}. \end{cases} \quad (1.1)$$

(i) Does (1.1) have a unique solution $x(t) \in C^3([0, T])$? Justify.

(ii) If we regard $x(t)$ also as a function of the initial value x_0 , what is the equation satisfied by the derivative with respect to t of $\partial x(t)/\partial x_0$? Is it a linear equation? Justify.

(1b) Consider the scheme

$$x^{k+1} = x^k + \Delta t [\alpha f(x^k) + \beta f(x^k + \gamma \Delta t f(x^k))], \quad (1.2)$$

where $\alpha, \beta, \gamma \in \mathbb{R}$, $\Delta t > 0$ is small enough and let $t_k = k\Delta t$ for $k \in \mathbb{N}$.

(i) The scheme (1.2) is an...

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implicit one-step method. <input type="checkbox"/>	implicit two-step method. <input type="checkbox"/>

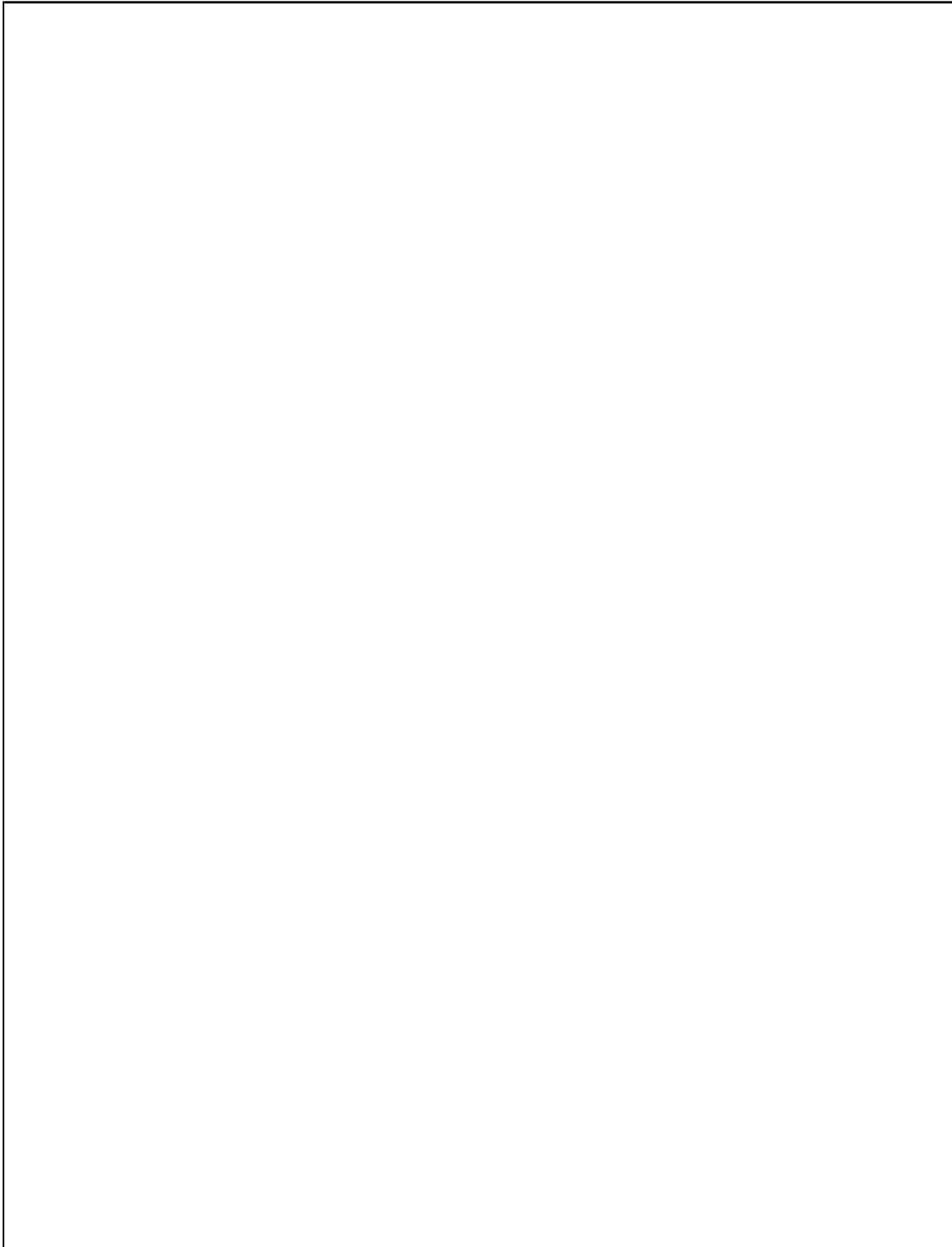
(ii) Let $\Phi(t_k, x^k, \Delta t)$ be defined by

$$\Phi(t_k, x^k, \Delta t) = \alpha f(x^k) + \beta f(x^k + \gamma \Delta t f(x^k)),$$

so that (1.2) can be rewritten in the form

$$x^{k+1} = x^k + \Delta t \Phi(t_k, x^k, \Delta t).$$

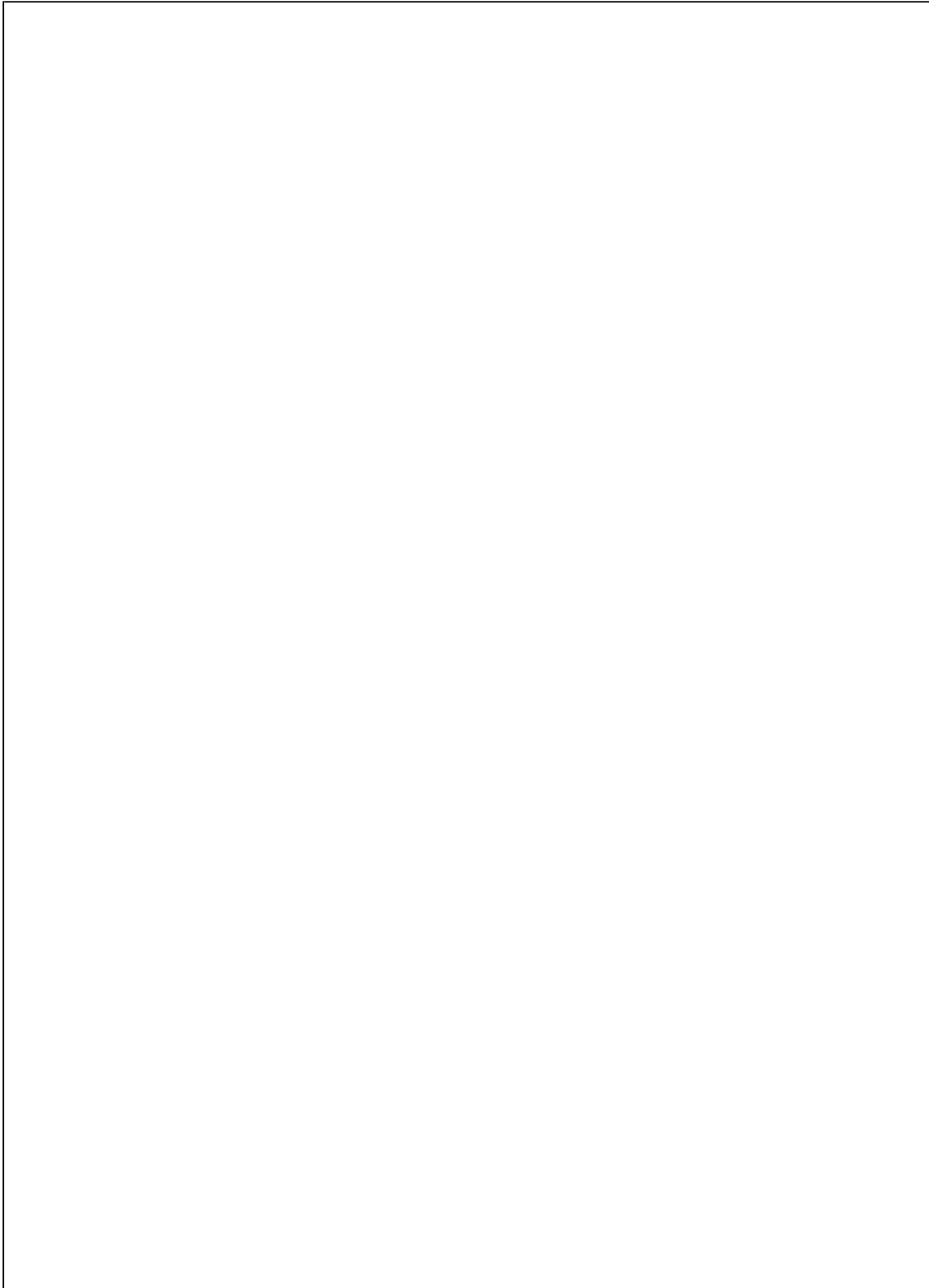
Prove that (1.2) is consistent with (1.1) if and only if $\alpha + \beta = 1$.



(iii) Define the truncation error by

$$T_k(\Delta t) = \frac{x(t_{k+1}) - x(t_k)}{\Delta t} - \Phi(t_k, x(t_k), \Delta t).$$

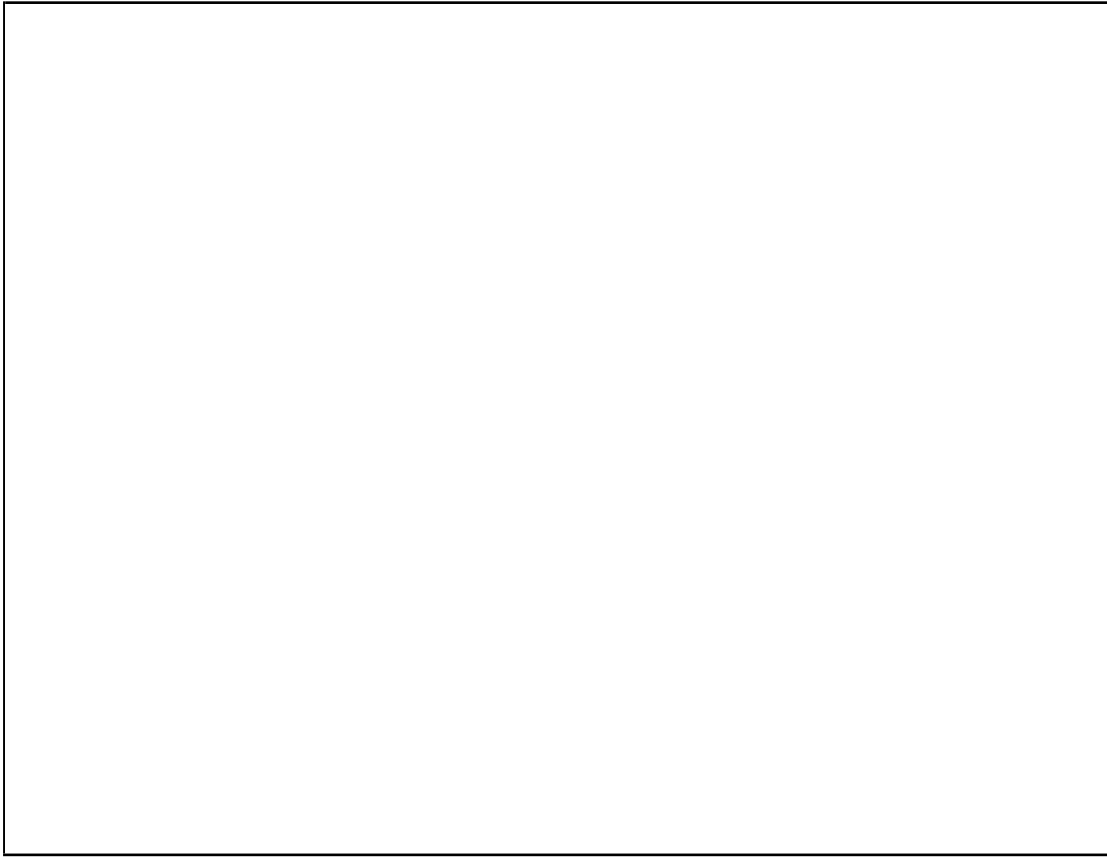
Find α, β, γ such that (1.2) is of order two, *i.e.*, that $T_k(\Delta t) = O((\Delta t)^2)$.



(iv) Prove that (1.2) is stable, *i.e.*, there exist positive constants h_0 and c_Φ such that

$$|\Phi(t, x, h) - \Phi(t, y, h)| \leq c_\Phi |x - y|$$

for all $t \in [0, T]$, $x, y \in \mathbb{R}$, $h \in [0, h_0]$.



(v) When $\alpha + \beta = 1$, is scheme (1.2) for solving (1.1) convergent? Explain why.



(1c) Consider the following particular II order differential equation:

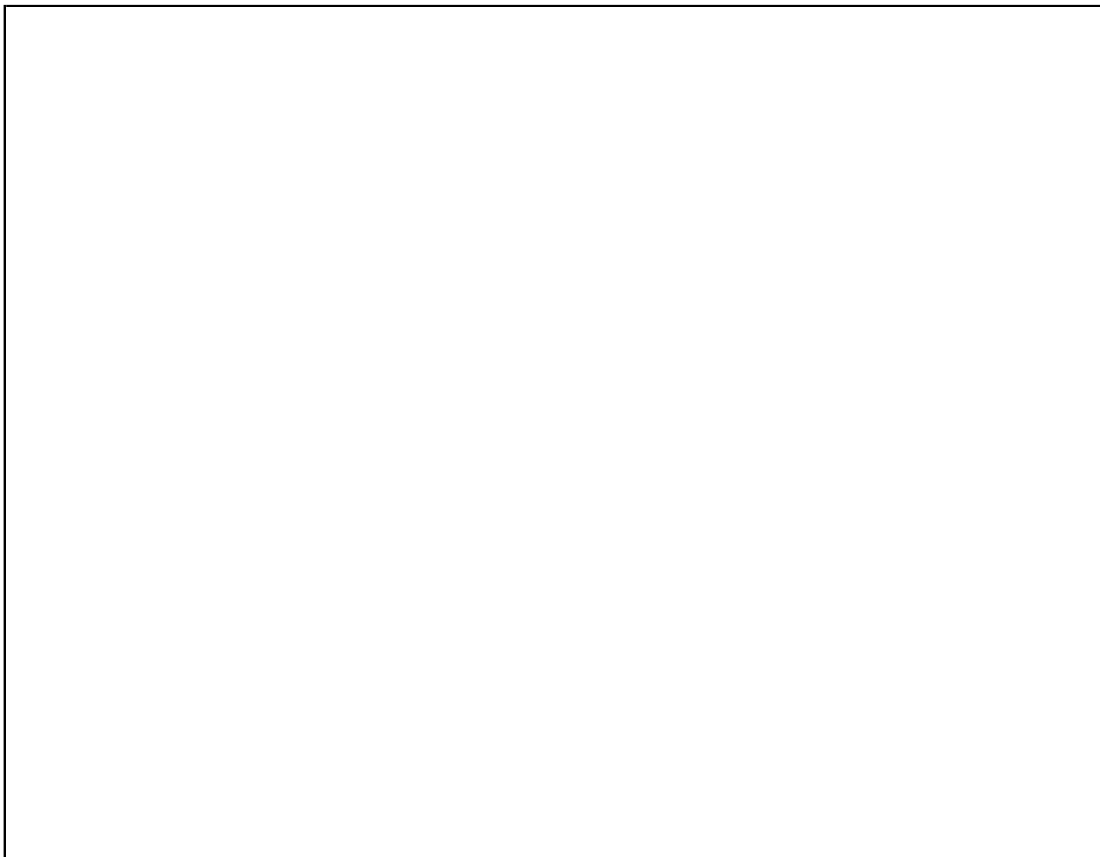
$$\begin{cases} \frac{d^2x}{dt^2} = -x, & t \in [0, +\infty[, \\ x(0) = x_0, \\ \frac{dx}{dt}(0) = x_1, \end{cases} \quad (1.3)$$

with initial data $x_0, x_1 \in \mathbb{R}$.

(i) Rewrite (1.3) in the form of a first order linear equation:

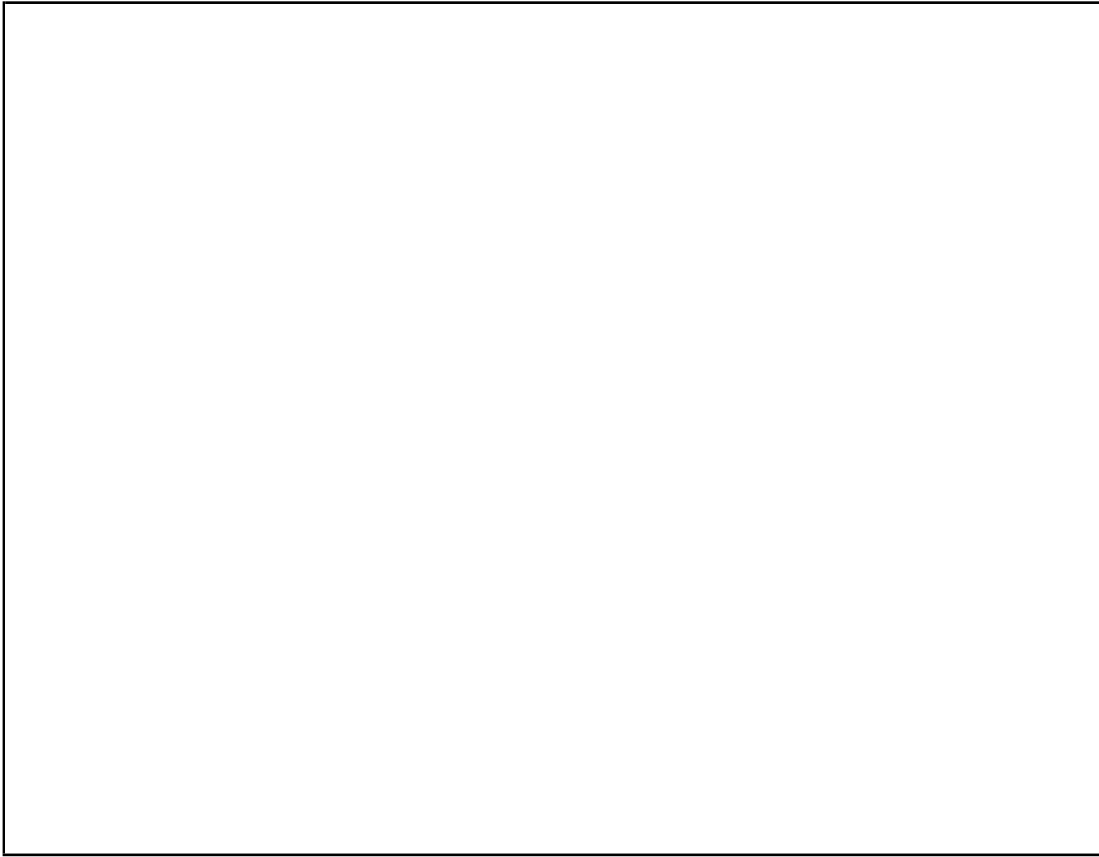
$$\frac{dX}{dt} = AX,$$

with A being a 2×2 matrix.



(ii) Assuming $\alpha + \beta = 1$, apply (1.2) to solve this first-order linear equation and find the 2×2 matrix $B(\Delta t)$ such that

$$X^{k+1} = B(\Delta t)X^k.$$



(iii) Discuss the behavior of X^k as $k \rightarrow +\infty$.



Problem 2**[20 Marks]**

Let k be a positive constant and consider the II order differential equation

$$\begin{cases} \frac{d^2x}{dt^2} = -k \sin x & \text{on } [0, T], \\ x(0) = x_0, \\ \frac{dx}{dt}(0) = x_1, \end{cases} \quad (2.1)$$

with initial data $x_0, x_1 \in \mathbb{R}$.

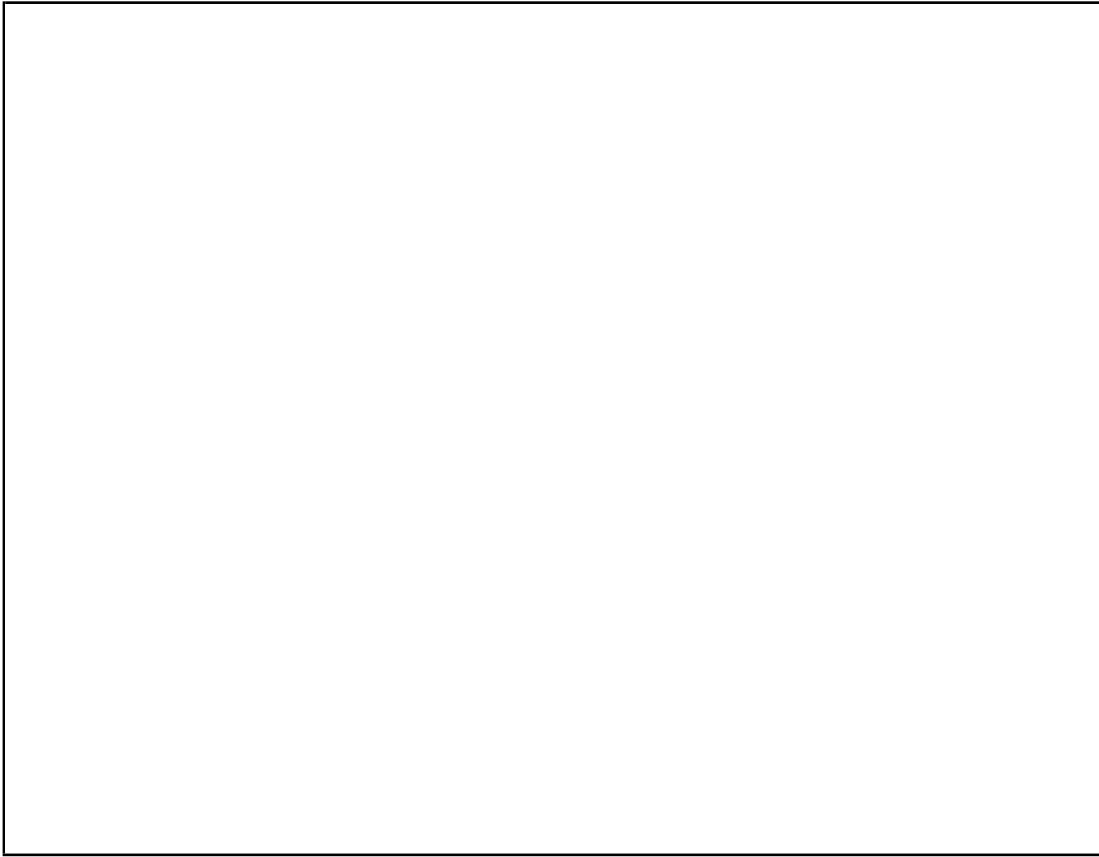
(2a)

(i) Set $p(t) = x(t)$, $q(t) = \frac{dx(t)}{dt}$ and $p_0 = x_0$, $q_0 = x_1$. Rewrite (2.1) in the following form

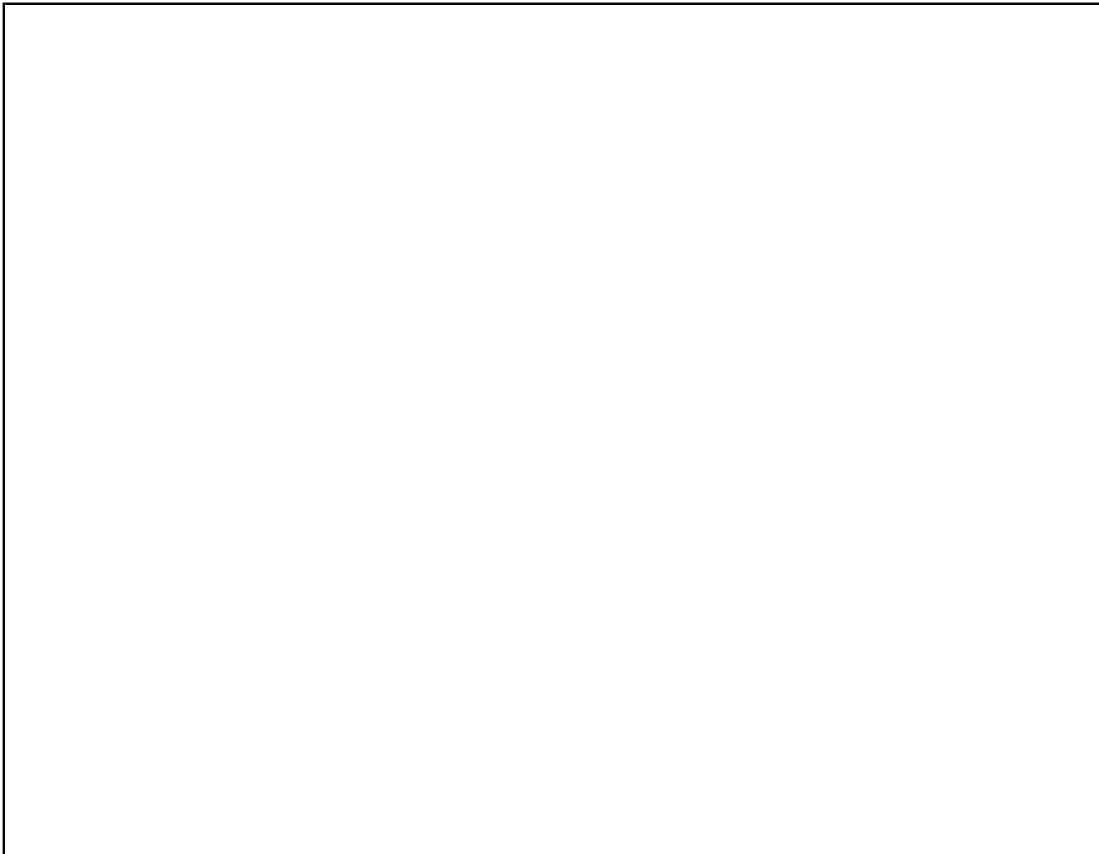
$$\begin{cases} \frac{dp}{dt} = q & \text{on } [0, T], \\ \frac{dq}{dt} = -k \sin p & \text{on } [0, T], \\ q(0) = q_0, \\ p(0) = p_0. \end{cases} \quad (2.2)$$



(ii) Is (2.2) a Hamiltonian system? You should justify your answer.



(iii) Find an invariant for (2.2), i.e., a function F such that $F(p(t), q(t))$ is constant for all $t \in [0, T]$.



(iv) Define the flow ϕ_t associated with (2.2) by $\phi_t(p_0, q_0) = (p(t), q(t))$.

Show that $\phi'_t(p_0, q_0)$ the jacobian of $\phi_t(p_0, q_0)$, i.e., $\phi'_t(p_0, q_0) = \frac{\partial \phi_t(p_0, q_0)}{\partial (p_0, q_0)}$, satisfies a linear differential equation and write it down.

(v) Is ϕ_t symplectic? Justify.

