Numerical Analysis II

Problem Sheet 4

Problem 4.1 Discrete Gronwall Lemma

Prove the discrete Gronwall Lemma for constant h:

If the sequence $(\xi_k)_{k\in\mathbb{N}_0}$, $\xi_k \geq 0$ satisfies the inequality

$$\xi_{k+1} \leq Ch^{p+1} + (1+Lh)\xi_k, \quad k \in \mathbb{N}_0, \quad C, h \geq 0,\ L > 0, p \in \mathbb{N}^*$$

then

$$\xi_k \leq Ch^p \frac{1}{L}\big(\mathrm{e}^{kLh} - 1\big) + \mathrm{e}^{kLh} \cdot \xi_0, \quad k \in \mathbb{N}_0.$$

HINT: Show, by induction, that

$$\xi_k \leq \frac{Ch^p}{L}\big[(1+Lh)^k - 1\big] + (1+Lh)^k \xi_0$$

and use the convexity of the exponential function.

Problem 4.2 Exponential of matrices

Let A, B be two $d \times d$ matrices ($d \geq 2$). Consider $x(t) \in \mathbb{R}^d$ solution to

$$\begin{cases} \dfrac{dx}{dt} = u(t)Ax(t) + (1-u(t))Bx(t), \\ x(0) = x_0, \end{cases} \tag{4.2.1}$$

where $u : t \mapsto u(t) \in [0,1]$ is a continuous function.

(4.2a) Prove using Cauchy-Lipschitz theorem that, for all u, there exists a unique solution x of Eqn. (4.2.1).

(4.2b) Verify that the solution of

$$\begin{cases} \dfrac{dx}{dt} = Ax(t), \\ x(0) = x_0, \end{cases}$$

is given by $x(t) = e^{tA}x_0$ where $e^{tA} := \sum_{n\geq 0} \frac{(tA)^n}{n!}$.

(4.2c) Suppose that $u(t) = \chi_E(t) \in \{0,1\}$ is the characteristic function of

$$E = \cup_{n\geq 0}[t_{2n}, t_{2n+1}] \subset [0,T]$$

where $(t_n)_{n\geq 0}$ is a strictly increasing sequence of real numbers in $[0,T]$ with $t_0 = 0$. Give an expression of $x(t)$ on each interval $[t_n, t_{n+1}]$, $n = 0, 1 \cdots$. $x(t)$ has to be continuous. Simplify this expression if $[A,B] = AB - BA = 0$ (i.e., A and B commute).

(4.2d) Let $d = 2$ and let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) Compute, for $s, t > 0$, e^{tA} and e^{sB}.

(ii) Do A and B commute?

(iii) Verify that $e^{tA}e^{sB} \neq e^{sB}e^{tA}$.

(iv) Verify that $e^{tA}e^{sB} \neq e^{tA+sB}$.

Problem 4.3 Linear System

Consider

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = A(\delta,\mu)x(t) \quad \text{on } [0,T] \\ x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases} \tag{4.3.1}$$

where

$$A(\delta,\mu) = \begin{pmatrix} -\delta & 1 \\ 0 & -\mu \end{pmatrix}$$

and δ, μ are positive parameters.

(4.3a) Solve this problem explicitly when $\mu = \delta$.

(4.3b) Solve it when $\mu \neq \delta$.

(4.3c) Show explicitly that, for any fixed $t > 0$, if we take the limit as $\mu \to \delta$, the two solutions become the same.

Problem 4.4 Second-order ODE

(4.4a) Consider the linear second-order ODE on $[1, 2]$ with parameter β:

$$\begin{cases} t^2 \dfrac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + t\dfrac{\mathrm{d}x(t)}{\mathrm{d}t} - \beta^2 x(t) = 0 \\ x(t=1) = 1 \\ \dfrac{\mathrm{d}x}{\mathrm{d}t}(t=1) = 0 \end{cases}$$

Verify that $x(t) = \cosh(\beta \log t)$ is the solution to the IVP. Is it a continuous function of β? Can it be differentiated with respect to β?

Problem 4.5 Exponential of Matrix

Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 3 \\ 1 & 4 \end{pmatrix}$$

(4.5a) Use Python to calculate $\exp(A)$, $\exp(B)$ and $\exp(A+B)$. Is $\exp(A)\exp(B) = \exp(A+B)$?

HINT: To calculate the exponential of matrix, use `expm` from the `scipy.linalg` package.

(4.5b) Now let

$$C = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix},$$

use Python to calculate $\exp(A)\exp(C)$ and $\exp(A+C)$. Is $\exp(A)\exp(C) = \exp(A+C)$? Can you briefly explain the reason?

(4.5c) Use Python to calculte the eigenvalues of $\exp(A)\exp(B)$ and $\exp(A+B)$. Do they have the same eigenvalues?

HINT: Use `eig` from the package `numpy.linalg` to calculate eigenvalues for matrices in Python.

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