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Numerical Analysis II

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D-MATH

Problem Sheet 7

Problem 7.1 Error estimate for the trapezium rule method

We consider the trapezium rule method

$$x_{n+1} = x_n + \frac{1}{2}h(f_{n+1} + f_n).$$

for the numerical solution of the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x),$$

where $x_0 = x(0)$ is given, $f_n = f(t_n, x_n)$ and $h = t_{n+1} - t_n$. Let us define the truncation error T_n as

$$T_n := \frac{x(t_{n+1}) - x(t_n)}{h} - \frac{1}{2}\Big(f(t_{n+1}, x(t_{n+1})) + f(t_n, x(t_n))\Big).$$

(7.1a) Show that

$$T_n = -\frac{1}{12}h^2 x'''(\xi_n),$$

for some ξ_n in the interval (t_n, t_{n+1}), where x is the solution of the initial value problem.

HINT: Apply integration by parts to the integral

$$\int_{t_n}^{t_{n+1}} (t - t_{n+1})(t - t_n)x'''(t)dt.$$

(7.1b) Suppose that f satisfies the Lipschitz condition

$$|f(t, x) - f(t, y)| \leq L|x - y|$$

for all real t, x, y, where L is a positive constant independent of t. Suppose also that there exists some constant M such that $|x'''(t)| \leq M$ for all t. Show that the global error $e_n = x(t_n) - x_n$ satisfies the inequality

$$|e_{n+1}| \leq |e_n| + \frac{1}{2}hL(|e_{n+1}| + |e_n|) + \frac{1}{12}h^3 M.$$

(7.1c) For a uniform step h satisfying $hL < 2$ deduce that, if $x_0 = x(t_0)$, then

$$|e_n| \leq \frac{h^2 M}{12L}\left[\left(\frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL}\right)^n - 1\right].$$

Problem 7.2 Truncation Error

Consider using a one-step method for the numerical solution of the initial value problem $x' = f(t,x)$, $x(t_0) = x_0$, $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. The method is given by

$$x_{n+1} = x_n + \frac{1}{2}h(k_1 + k_2),$$

where

$$k_1 = f(t_n, x_n), \quad \text{and} \quad k_2 = f(t_n + h, x_n + hk_1).$$

Show that the method is consistent and has truncation error

$$T_n = \frac{1}{6}h^2\left(f_x(f_t + f_x f) - \frac{1}{2}(f_{tt} + 2f_{tx}f + f_{xx}f^2)\right) + O(h^3)$$

Problem 7.3 Roundoff Error Effects

In practical situations, computers always round off real numbers. In numerical methods rounding erros become important when the step size Δt is comparable with the precision of the computations. Thus, if taking rounding error into consideration, the Explicit Euler method will become the following perturbed scheme:

$$x^{k+1} = x^k + \Delta t f(t_k, x^k) + (\Delta t)\mu^k + \rho^k,$$

where μ^k and ρ^k represent the errors in f and in the assembling, respectively. Assume that $|\mu^k| \le \mu$ and $|\rho^k| \le \rho$ for all $k = 0, 1, 2, \dots$ and $f \in C^1$. Let $e^k := x(t_k) - x^k$, and try to prove that

$$|e^{k+1}| \le (1 + \Delta t C)|e^k| + \Delta t \mu + \rho + \sup_{\xi \in [t_k, t_{k+1}]} |Df(\xi)| \frac{1}{2}(\Delta t)^2,$$

and hence

$$|e^k| \le e^{CT}|e^0| + \frac{\mu e^{CT}}{C} + \frac{\rho e^{CT}}{C\Delta t} + \frac{1}{2C} \sup_{\xi \in [0,T]} |Df(\xi)| e^{CT} \Delta t,$$

where C is the Lipschitz constant for f, and Df denotes the differentiation to f where $f(t, x(t))$ is regarded as a function with single parameter t.

Introduce

$$\phi(\Delta t) = \frac{\rho e^{CT}}{C\Delta t} + \frac{1}{2C} \sup_{\xi \in [0,T]} |Df(\xi)| e^{CT} \Delta t,$$

when does ϕ attain its minimum, and therefore what suggestion do you have for the minimal step size Δt? HINT: Use the the arithmetic mean–geometric mean inequality to find a bound for ϕ.

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