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Convex Optimization in Machine Learning and Computational Finance

Important:

- Put your student card on the table
- Begin each exercise on a new sheet of paper
- Write your name on each sheet
- Only pen, paper, and 10 single-sided A4 pages of your own notes are allowed. No books, slides, lecture notes, phones, laptops or tablets.

Please fill in the following table

Last name	
First name	
Student number (if available)	

Please do **not** fill in the following table

Question	Points	Control	Maximum
#1			10
#2			10
#3			10
#4			10
Total			40

Question 1 (10 Pts)

a) Let $Q_1, Q_2, \ldots$ be a sequence of convex subset of $\mathbb{R}^n$. Is the set

$$\liminf_{j\to\infty} Q_j := \bigcup_{j\geq 1} \bigcap_{k\geq j} Q_k$$

convex? If yes, prove it. Otherwise, give a counterexample. (2 Pts)

b) Let $Q_1, Q_2, \ldots$ be a sequence of convex subsets of $\mathbb{R}^n$. Is the set

$$\limsup_{j\to\infty} Q_j := \bigcap_{j\geq 1} \bigcup_{k\geq j} Q_k$$

convex? If yes, prove it. Otherwise, give a counterexample. (2 Pts)

c) What is a semidefinite representable set? (2 Pts)

d) Let $Q_1 \subseteq \mathbb{R}^m$ and $Q_2 \subseteq \mathbb{R}^n$ be semidefinite representable sets. Is $Q_1 \times Q_2 \subseteq \mathbb{R}^{m+n}$ semidefinite representable? If yes, prove it. Otherwise, give a counterexample. (2 Pts)

e) If $Q_1, Q_2 \subseteq \mathbb{R}^n$ are semidefinite representable, is the Minkowski sum $Q_1 + Q_2 \subseteq \mathbb{R}^n$ again semidefinite representable? If yes, prove it. Otherwise, give a counterexample. (2 Pts)

Question 2 (10 Pts)

a) Prove or disprove that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) := \sup_{-1\leq t\leq 1} \sum_{k=1}^{n} x_k t^k$$

is convex. (2 Pts)

b) Let $g\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be a lower semi-continuous convex function and Q a nonempty convex compact subset of $\mathbb{R}^n$. Consider the function $f\colon \mathbb{R}^m \to \mathbb{R}$ given by

$$f(x) := \inf_{y\in Q} g(x, y).$$

Can the "inf" be replaced by a "min"? Prove that f is convex. (3 Pts)

c) Let S be a nonempty bounded subset of $\mathbb{R}^n$. Prove that the support function σ_S of S has finite values. (2 Pts)

d) Compute the conjugate of σ_S. (3 Pts)

Question 3 (10 Pts)

In this question we study a fictive device to avoid crashes between satellites. Suppose that at a given time, two satellites are dangerously close to each other but are not touching yet. They are covered with electromagnets that can be activated where a collision is about to occur to repulse each other.

Assume that the satellites have the shapes of compact polyhedrons with nonempty interior of the form

$$Q_1 = \{x \in \mathbb{R}^3 : A_1 x \succeq_{\mathbb{R}^{m_1}_+} b_1\} \qquad \text{and} \qquad Q_2 = \{x \in \mathbb{R}^3 : A_2 x \succeq_{\mathbb{R}^{m_2}_+} b_2\}$$

for matrices $A_1 \in \mathbb{R}^{m_1 \times 3}$, $A_2 \in \mathbb{R}^{m_2 \times 3}$ and vectors $b_1 \in \mathbb{R}^{m_1}$, $b_2 \in \mathbb{R}^{m_2}$. We would like to determine as fast as possible which electromagnets have to be activated. This leads us to the optimization problem

$$\begin{aligned} p^* := \min \quad & \|x - y\|_2^2 \\ \text{s.t.} \quad & A_1 x \succeq_{\mathbb{R}^{m_1}_+} b_1 \\ & A_2 y \succeq_{\mathbb{R}^{m_2}_+} b_2 \\ & x, y \in \mathbb{R}^3. \end{aligned} \tag{P}$$

a) Show that problem (P) has a solution (x^*, y^*). Can it have more than one solution? Explain your answer. (2 Pts)

b) Unfortunately, due to the complex shape of the satellites, the number of constraints $m_1 + m_2$ is very large (more than 100). But we have a very fast optimization software for minimizing a convex quadratic objective function $f(u)$ subject to the non-negativity of the variables (that is, $u \succeq_{\mathbb{R}^m_+} 0$) and a small number of linear equality constraints (less than 10).

   Derive from (P) an optimization problem with the same optimal value p^* that our software can solve efficiently. Explain why the two problems have the same optimal value. (4 Pts)

c) Write down the KKT optimality conditions for (P) or the modified problem derived in b). (2 Pts)

d) If you have a solution $u^* \in \mathbb{R}^m_+$ to the modified problem, how can you derive a solution (x^*, y^*) to (P) without solving the full problem (P). (2 Pts)

Question 4 (10 Pts)

a) Describe Newton's method to minimize a function $f\colon \mathbb{R}^n \to \mathbb{R}$. (2 Pts)

b) Under which assumptions on f and the starting point does the theorem of Kantorovich guarantee that Newton's method converges? (2 Pts)

c) How can Newton's method be applied to convex minimization problems with linear equality constraints? (3 Pts)

d) How can Newton's method be applied to problems of the form $\min_{x \in Q} f(x)$, where $f\colon \mathbb{R}^n \to \mathbb{R}$ is convex and $Q \subseteq \mathbb{R}^n$ is a closed convex set with nonempty interior? (3 Pts)