ETH Zürich, HS 2018
D-MATH
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August 2019

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Please fill in the following table

Last name	
First name	
Programme of study	**MATH** ☐ **MScQF** ☐ **SAV** ☐ **Other** ☐
Matriculation number	

Leave blank

Question	**Maximum**	**Points**	**Check**
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration: 180 min.

Closed book examination: no notes, no books, no calculator, no mobile phones, etc. allowed.

Important:

- Please put your student card on the table.
- Only pen and paper are allowed on the table. Please **do not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- Take a new sheet for each question and write your name on every sheet.
- Except for Question 1, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas and results from the lecture or from the exercise classes without derivation.
- Simplify your results as much as possible.
- Most of the subquestions can be solved independently of each other.

⋆⋆⋆ Good luck! ⋆⋆⋆

Answer Sheet for Question 1

Please use this sheet to answer Question 1. Indicate the correct answer by ✗. If there is no cross or more than one cross in a line, this will be interpreted as "no answer".

	answer (1)	answer (2)	answer (3)
(a)			
(b)			
(c)			
(d)			
(e)			
(f)			
(g)			
(h)			

Do not fill in

correct	wrong	no answer

Do not fill in

	1st corr.	2nd corr.
correct		
wrong		
no answer		
Points		

Question 1 (8 **Points**)

For each of the following eight subquestions, there is **exactly one** correct answer. For each correct answer you get 1 point, for each wrong answer you get -0.5 point, and for no answer you get 0 points. You get at least 0 points for the whole exercise. **Please use the printed form for your answers.** It is enough to indicate your answer by a cross; you do not need to explain your choice.

Throughout subquestions (a) to (d), let $(\widetilde{S}^0, \widetilde{S})$ be an undiscounted financial market in discrete time on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with a finite time horizon $T \in \mathbb{N}$ and $\mathbb{F} := (\mathcal{F}_k)_{k=0,1,\dots,T}$ generated by $\widetilde{S}$. Let $\widetilde{S}^0_k := (1+r)^k$ for $k = 0, 1, \dots, T$ and constants $r > -1$ and $\widetilde{S}_0 := s_0 > 0$. The discounted market is denoted by (S^0, S).

(a) Let (S^0, S) admit an arbitrage. Then

(1) No payoff $H \in L^0_+(\mathcal{F}_T)$ is attainable.

(2) The set of EMMs for S is empty.

(3) There exists no probability measure $Q \approx P$.

(b) Let (S^0, S) be arbitrage-free and complete. Which of the following statements is **not** true?

(1) Every payoff $H \in L^0_+(\mathcal{F}_T)$ admits a unique replicating strategy.

(2) $(\widetilde{S}^0, \widetilde{S})$ is arbitrage-free.

(3) The set of ELMMs for (S^0, S) is a singleton.

(c) Let $Z = (Z_k)_{k=0,1,\dots,T}$ be the density process of some $Q \in \mathbb{P}_e(S)$ with respect to P. Then

(1) $\frac{1}{Z}$ is a $(P, \mathbb{F})$-martingale.

(2) ZS is a $(P, \mathbb{F})$-martingale.

(3) $P[A] = E_Q[Z_k \mathbb{1}_A]$ for any $A \in \mathcal{F}_k$.

(d) A trading strategy $\varphi = (\varphi^0, \vartheta)$ is self-financing if and only if

(1) $\varphi^0_{k-1} - \vartheta^{\mathrm{tr}}_{k-1} S_{k-1} = \varphi^0_k - \vartheta^{\mathrm{tr}}_k S_{k-1}$ for $k = 1, \dots, T$.

(2) $\varphi^0_k + \vartheta^{\mathrm{tr}}_k S_k = \varphi^0_{k-1} + \vartheta^{\mathrm{tr}}_{k-1} S_k$ for $k = 1, \dots, T$.

(3) $\vartheta^{\mathrm{tr}}_k S_{k-1} - \varphi^0_{k-1} = \vartheta^{\mathrm{tr}}_{k-1} S_{k-1} - \varphi^0_k$ for $k = 1, \dots, T$.

Throughout subquestions (e) to (h), W denotes a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions of P-completeness and right-continuity.

(e) Let X and Y be two local $(P, \mathbb{F})$-martingales. Which of the following statements is **not** true?

(1) $[aX] = a^2[X]$ for all $a \in \mathbb{R}$.

(2) $[X, Y] = 0$ if and only if $X = 0$ P-a.s. or $Y = 0$ P-a.s.

(3) $[X + Y] = [X] + 2[X, Y] + [Y]$ P-a.s.

(f) Let X be a general $(P, \mathbb{F})$-semimartingale and let L satisfy $dL = L_- dX$, $L_0 = 1$. Then

(1) L is a local $(P, \mathbb{F})$-martingale.

(2) L is a positive $(P, \mathbb{F})$-semimartingale if and only if X is continuous.

(3) L is a positive $(P, \mathbb{F})$-semimartingale if X is continuous.

(g) Let M be a right-continuous $(P, \mathbb{F})$-martingale. Which of the following statements about M is **not** true?

(1) M is predictable.

(2) If τ is an $\mathbb{F}$-stopping time, then M^τ is again a $(P, \mathbb{F})$-martingale.

(3) If $M_0 = 0$, then M is a local $(P, \mathbb{F})$-martingale.

(h) Which of the following statements about W is **not** true?

(1) $(W_{T+t} - W_T)_{t\geq 0}$ is a $(P, \mathbb{F})$-Brownian motion.

(2) W is a square-integrable local $(P, \mathbb{F})$-martingale in $\mathcal{M}_0^2$.

(3) $(W \bullet W)$ is a $(P, \mathbb{F})$-martingale.

Question 2 (8 **Points**)

Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ consisting of a bank account and one stock. The movements of the bank account $\widetilde{S}^0$ and the stock price $\widetilde{S}^1$ are described by the trees below, where the numbers beside the branches denote transition probabilities.

$$\begin{tikzcd}
\widetilde{S}^0 : & 1 \arrow[r, "1"] & 1+r \arrow[r, "1"] & (1+r)(1+2r) \\
 & & & (1+u)(1+2u) \\
 & & 1+u \arrow[ur, "\frac{1}{2}"] \arrow[dr, "\frac{1}{2}"'] & \\
 & & & (1+u)(1+2d) \\
\widetilde{S}^1 : & 1 \arrow[uur, "\frac{1}{2}"] \arrow[ddr, "\frac{1}{2}"'] & & \\
 & & & (1+d)(1+u) \\
 & & 1+d \arrow[ur, "\frac{1}{2}"] \arrow[dr, "\frac{1}{2}"'] & \\
 & & & (1+d)(1+d)
\end{tikzcd}$$

Note that the interest rate is $2r$ in the second period.

More precisely, let $(\Omega, \mathcal{F}, P)$ be the probability space given by $\Omega := \{-1, 1\}^2$, $\mathcal{F} := 2^\Omega$ and the probability measure P defined by $P\left[\{(x_1, x_2)\}\right] := p_{x_1} p_{x_1, x_2}$, where

$$p_1 = p_{-1} := \frac{1}{2} \quad \text{and} \quad p_{1,1} = p_{1,-1} = p_{-1,1} = p_{-1,-1} := \frac{1}{2}.$$

Next, let $u > d$ and $d, r > -0.5$ and consider the random variables Y_1 and Y_2 given by

$$\begin{aligned}
Y_1\big((1,1)\big) &:= 1+u, & Y_1\big((-1,1)\big) &:= 1+d, \\
Y_1\big((1,-1)\big) &:= 1+u, & Y_1\big((-1,-1)\big) &:= 1+d, \\
Y_2\big((1,1)\big) &:= 1+2u, & Y_2\big((-1,1)\big) &:= 1+u, \\
Y_2\big((1,-1)\big) &:= 1+2d, & Y_2\big((-1,-1)\big) &:= 1+d.
\end{aligned}$$

The bank account process $\widetilde{S}^0$ and the stock price process $\widetilde{S}^1$ are then given by $\widetilde{S}^0_k = \prod_{j=1}^k (1+jr)$ and $\widetilde{S}^1_k = \prod_{j=1}^k Y_j$ for $k \in \{0,1,2\}$, respectively. Finally, the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ is defined by $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_1 := \sigma(Y_1)$ and $\mathcal{F}_2 := \sigma(Y_1, Y_2) = 2^\Omega = \mathcal{F}$.

(a) Derive conditions on u, r and d under which the market (S^0, S^1) is arbitrage-free. (2 pt)

(b) Derive conditions on u, r and d under which the market (S^0, S^1) is not only arbitrage-free but also complete. (1 pt)

(c) Assume now that (S^0, S^1) is arbitrage-free and let $H \geq 0$, $H \in L^1(Q)$ for all $Q \in \mathbb{P}_e(S^1)$ be an attainable payoff with a replicating strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$. Argue why the unique arbitrage-free discounted price process $V^H = (V^H_k)_{k=0,1,2}$ of H is given by

$$V^H_k = E_Q\left[H \,|\, \mathcal{F}_k\right] \quad P\text{-a.s. for all } k \in \{0,1,2\},$$

where Q is any EMM in $\mathbb{P}_e(S^1)$. (3 pt)

(d) Assume now that (S^0, S^1) is arbitrage-free and complete and consider two payoffs $H, K \geq 0$, $H, K \in L^1(Q)$ for all $Q \in \mathbb{P}_e(S^1)$. Define $V^H = (V^H_k)_{k=0,1,2}$ and $V^K = (V^K_k)_{k=0,1,2}$ by

$$V^H_k = E_Q\left[H \,|\, \mathcal{F}_k\right] \quad \text{and} \quad V^K_k = E_Q\left[K \,|\, \mathcal{F}_k\right]$$

for some $Q \in \mathbb{P}_e(S^1)$. Argue why the market (S^0, S^1, V^H, V^K) is arbitrage-free. Is this market also complete? If yes, give the unique EMM for this market. (2 pt)

Question 3 (8 **Points)**

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ for some $T \in \mathbb{N}$ be a filtered probability space. Let $X = (X_k)_{k=0,1,\dots,T}$ be a local $(P, \mathbb{F})$-martingale.

(a) Show that if $|X_k| \le Y_k$ P-a.s. for all $k \in \{0, 1, \dots, T\}$, where $Y = (Y_k)_{k=0,1,\dots,T}$ is some integrable stochastic process, then X is a true $(P, \mathbb{F})$-martingale. (3 pt)

(b) Using the result given in (a), show that in finite discrete time, integrable local martingales are true martingales. (1 pt)

(c) Let $Y = (Y_n)_{n \in \mathbb{N}}$ be a sequence of *independent* random variables with

$$P[Y_n = k] = \begin{cases} \binom{n}{k} \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k} & k \in \{0, 1, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\tau := \inf\{n \in \mathbb{N} \,|\, Y_n > 1\}$ is a stopping time with respect the filtration generated by Y. (2 pt)

(d) Show that we have for the stopping time τ from (c) that $P[\tau = +\infty] = 0$, that is it is finite P-a.s. (2 pt)

Question 4 (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and $W = (W_t)_{t\geq 0}$ a $(P, \mathbb{F})$-Brownian motion.

(a) Consider a C^2 function $f : [0, \infty) \to \mathbb{R}$. Define the process $X = (X_t)_{t\geq 0}$ by

$$X_t = Z + f(t)W_t,$$

where $Z \sim \mathcal{N}(0, 1)$ is an $\mathcal{F}_0$-measurable random variable. Write down the distribution of X_t for a given $t \geq 0$ and compute the covariance $E[X_t X_s]$ for $t, s \geq 0$. (2 pt)

(b) State the continuous version of Itô's formula together with any assumptions. Using Itô's formula, find an expression for the process X from (a) involving the process $Y = (Y_t)_{t\geq 0}$ defined by

$$Y_t = \int_0^t f(s)dW_s,$$

and show that

$$\langle X \rangle_t = \langle Y \rangle_t = \int_0^t f^2(s)ds.$$

(2 pt)

(c) Fix $T > 0, u \in \mathbb{R}$ and assume that $E[\exp(uY_T)] < \infty$. Using the Markov property of Brownian motion, show that

$$E[\exp(uY_T) \mid \mathcal{F}_t] = E[\exp(uY_T) \mid Y_t]$$

and that the process $Z = (Z_t)_{t\in[0,T]}$ defined by

$$Z_t = E[\exp(uY_T) \mid Y_t] =: F(Y_t, t)$$

is a $(P, \mathbb{F})$-martingale. Deduce that if F is sufficiently smooth, F must satisfy the partial differential equation

$$\frac{\partial F}{\partial t}(Y_t, t) + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(Y_t, t)f^2(t) = 0 \quad P\text{-a.s. for all } t \in (0, T)$$

with terminal condition

$$F(y, T) = \exp(uy), \quad y \in \mathbb{R}.$$

(2 pt)

(d) Consider the partial differential equation

$$\frac{\partial F}{\partial t}(y, t) + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(y, t)f^2(t) = 0, \quad t \in (0, T),\ y \in \mathbb{R}.$$

Check directly that the function

$$F(y, t) = \exp\left(uy + \frac{u^2}{2}\int_t^T f^2(s)ds\right)$$

satisfies this equation. Assuming that this is the correct expression for F, compute $F(0, 0)$. Use this as well as the original definition of the function F to find the distribution of the random variable Y_T. (2 pt)

Question 5 (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions and $W = (W_t)_{t\in[0,T]}$ a $(P, \mathbb{F})$-Brownian motion. You can assume that $\mathbb{F} = \mathbb{F}^W = (\mathcal{F}_t^W)_{t\in[0,T]}$, the filtration generated by W, and that $\mathcal{F} = \mathcal{F}_T^W$. Consider the Black–Scholes model given by assets driven by the SDEs

$$\begin{aligned} d\widetilde{S}_t^0 &= \widetilde{S}_t^0 r dt, \\ d\widetilde{S}_t^1 &= \widetilde{S}_t^1 (\mu dt + \sigma dW_t), \end{aligned}$$

where $\widetilde{S}_0^0 = 1$, $\widetilde{S}_0^1 = x > 0$, $r > 0$, $\sigma > 0$ and $\mu \in \mathbb{R}$ are some fixed constants.

(a) Show that the solution to the second SDE $d\widetilde{S}_t^1 = \widetilde{S}_t^1(\mu dt + \sigma dW_t)$, $\widetilde{S}_0^1 = x$ is a positive $(P, \mathbb{F})$-semimartingale. (1 pt)

(b) Find the SDE satisfied by the process $\widehat{S}^0 := \widetilde{S}^0/\widetilde{S}^1$. (2 pt)

(c) Using Girsanov's theorem, find an EMM $\widehat{Q}$ for $\widehat{S}^0$. Make sure to argue that $\widehat{S}^0$ is indeed a $(\widehat{Q}, \mathbb{F})$-martingale. $\widehat{Q}$ is in fact the unique EMM for $\widehat{S}^0$. (3 pt)

(d) Consider a contingent claim whose payoff $\widetilde{H}$ in *undiscounted* terms is given by

$$\widetilde{H} = \widetilde{S}_T^1 \mathbb{1}_{\{\widetilde{S}_T^1 \leq K\}}$$

for some $K > 0$. Find the unique arbitrage-free *undiscounted* value process $\widetilde{V} = (\widetilde{V}_t)_{t\in[0,T]}$ of the payoff $\widetilde{H}$. (2 pt)

Hint: Note that there is no reason why any of the results we have seen in class should not hold when we use $\widetilde{S}^1$ as numéraire. Compute first the discounted value process corresponding to the payoff $\widetilde{H}$ discounted by $\widetilde{S}^1$.