ETH Zürich, HS 2018
D-MATH
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February 2019

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Please fill in the following table

Last name	
First name	
Programme of study	**MATH** □ **MScQF** □ **SAV** □ **Other** □
Matriculation number	

Leave blank

Question	Maximum	Points	Check
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration: 180 min.

Closed book examination: no notes, no books, no calculator, no mobile phones, etc. allowed.

Important:

- Please put your student card on the table.
- Only pen and paper are allowed on the table. Please **do not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- Take a new sheet for each question and write your name on every sheet.
- Except for Question 1, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas and results from the lecture or from the exercise classes without derivation.
- Simplify your results as much as possible.
- Most of the subquestions can be solved independently of each other.

⋆⋆⋆ Good luck! ⋆⋆⋆

Answer Sheet for Question 1

Please use this sheet to answer Question 1. Indicate the correct answer by ✗. If there is no cross or more than one cross in a line, this will be interpreted as "no answer".

	answer (1)	answer (2)	answer (3)
(a)			
(b)			
(c)			
(d)			
(e)			
(f)			
(g)			
(h)			

Do not fill in

correct	wrong	no answer

Do not fill in

	1st corr.	2nd corr.
correct		
wrong		
no answer		
Points		

Question 1 (8 **Points**)

For each of the following eight subquestions, there is **exactly one** correct answer. For each correct answer you get 1 point, for each wrong answer you get -0.5 point, and for no answer you get 0 points. You get at least 0 points for the whole exercise. **Please use the printed form for your answers.** It is enough to indicate your answer by a cross; you do not need to explain your choice.

Throughout subquestions (a) to (d), let $(\widetilde{S}^0, \widetilde{S}^1)$ be an undiscounted financial market in discrete time on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with a finite time horizon $T \in \mathbb{N}$ and $\mathbb{F} := (\mathcal{F}_k)_{k=0,1,\dots,T}$ generated by $\widetilde{S}^1$. Let $\widetilde{S}^0_k := (1+r)^k$ for $k = 0, 1, \dots, T$ and constants $r > -1$ and $\widetilde{S}^1_0 := s^1_0 > 0$. The discounted market is denoted by (S^0, S^1).

(a) Let $\varphi = (\varphi^0, \vartheta)$ be an arbitrage strategy for $(\widetilde{S}^0, \widetilde{S}^1)$. Then

(1) $G_k(\vartheta) \geq 0$ P-a.s. for all $k \in \{0, 1, \dots, T\}$.

(2) $V(\varphi) = \varphi^0_0 + G(\vartheta)$ P-a.s.

(3) $V_0(\varphi) > 0$ P-a.s.

(b) Which of the following conditions does **not** imply that $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free?

(1) The set of ELMMs for (S^0, S^1) is non-empty.

(2) S^1 is a $(Q, \mathbb{F})$-martingale for some probability measure $Q \approx P$.

(3) There exists an EMM for $(\widetilde{S}^0, \widetilde{S}^1)$.

(c) Which of the following statements is true?

(1) Let $D > 0$ be an integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$. Then $\frac{D}{E[D]}$ is the density (Radon–Nikodým derivative) of some probability measure $Q \approx P$.

(2) Every bounded $(P, \mathbb{F})$-martingale is P-a.s. constant.

(3) There exists no $(P, \mathbb{F})$-submartingale which is also a $(P, \mathbb{F})$-martingale.

(d) Let $(\widetilde{S}^0, \widetilde{S}^1)$ follow the binomial model. Then

(1) $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free and complete.

(2) $(\widetilde{S}^0, \widetilde{S}^1)$ is complete, provided that it is arbitrage-free.

(3) $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free.

Throughout subquestions (e) to (h), W denotes a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions of P-completeness and right-continuity.

(e) Let M be a local $(P, \mathbb{F})$-martingale and A be a process of finite variation. Then

(1) $[M, A] = 0$ P-a.s.

(2) $[M, A] = 0$ P-a.s. if and only if both M and A are continuous.

(3) $[M, A] = 0$ P-a.s. if A is continuous.

(f) Let $Z = (Z_t)_{t \in [0,T]}$ for some $T \in (0, \infty)$ be the density process of $Q \approx P$ on $\mathcal{F}_T$ with respect to P. Which of the following is true?

(1) $Z = Z_0 \mathcal{E}(L)$ for some continuous local $(P, \mathbb{F})$-martingale $L = (L_t)_{t \in [0,T]}$.

(2) $Z = Z_0 \mathcal{E}(L)$ for some local $(P, \mathbb{F})$-martingale $L = (L_t)_{t \in [0,T]}$.

(3) $Z = Z_0 \mathcal{E}(L)$ for some local $(Q, \mathbb{F})$-martingale $L = (L_t)_{t \in [0,T]}$.

(g) Which of the following statements about W is **not** true?

(1) P-a.a. paths of W have infinite 1-variation.

(2) W is a $(P, \mathbb{F})$-semimartingale.

(3) W is the unique continuous process with zero mean and normally distributed increments.

(h) Let X be a $(P, \mathbb{F})$-semimartingale and let $Q \overset{loc}{\approx} P$. Then

(1) $f(X)$ is a $(Q, \mathbb{F})$-semimartingale for any C^∞-function f.

(2) X is a $(Q, \mathbb{F})$-martingale.

(3) $f(X)$ is a $(P, \mathbb{F})$-semimartingale for any measurable function f.

Question 2 (8 **Points**)

Consider a **one-period** financial market on the probability space $(\Omega, \mathcal{F}, P)$ with a riskless asset $\widetilde{S}^0$ and two risky assets $\widetilde{S}^1$, $\widetilde{S}^2$. Assume that $(\widetilde{S}^0_0, \widetilde{S}^1_0, \widetilde{S}^2_0) = (1, 1, 1)$ and

$$\widetilde{S}^0_1 = (1 + r)S^0_0,$$
$$\widetilde{S}^i_1 = Y_i\widetilde{S}^i_0,$$

where $r > 0$, Y_1, Y_2 are two independent random variables such that $P[Y_i = 1 + u_i] = p_i$, $P[Y_i = 1 + d_i] = 1 - p_i$, where $u_1 > d_1 > -1, u_2 > d_2 > -1$ and $p_1, p_2 \in (0, 1)$. Assume that Ω is the canonical path space $\Omega = \{\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow\}$ and $\mathcal{F} = 2^\Omega$.

More precisely, on Ω defined as above we consider the probability measure defined by

$$P[\uparrow\uparrow] = p_1p_2, P[\uparrow\downarrow] = p_1(1 - p_2), P[\downarrow\uparrow] = (1 - p_1)p_2, P[\downarrow\downarrow] = (1 - p_1)(1 - p_2),$$

and the random variables Y_1, Y_2 are given explicitly by

$$Y_1(\uparrow\uparrow) = Y_1(\uparrow\downarrow) = 1 + u_1, Y_1(\downarrow\uparrow) = Y_1(\downarrow\downarrow) = 1 + d_1,$$
$$Y_2(\uparrow\uparrow) = Y_2(\downarrow\uparrow) = 1 + u_2, Y_2(\uparrow\downarrow) = Y_2(\downarrow\downarrow) = 1 + d_2.$$

The filtration is given by $\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_1 = 2^\Omega$.

(a) Let S^0, S^1, S^2 be the discounted prices with respect to $\widetilde{S}^0$. Prove that the conditions $u_1 > r > d_1$, $u_2 > r > d_2$ are necessary to ensure no-arbitrage, and prove (by writing it down explicitly) that the EMM Q for (S^0, S^1, S^2) is uniquely specified by $Q[\uparrow\uparrow]$, which must satisfy

$$Q[\uparrow\uparrow] \in \left(\max\left(0, \frac{r - d_2}{u_2 - d_2} + \frac{r - d_1}{u_1 - d_1} - 1\right), \min\left(\frac{r - d_1}{u_1 - d_1}, \frac{r - d_2}{u_2 - d_2}\right)\right).$$

Conversely, explain why any such $Q[\uparrow\uparrow]$ determines an equivalent martingale measure and why the conditions above are sufficient to ensure no-arbitrage. (3 pt)

Hint: Solve for $Q[\uparrow *] := Q[\uparrow\uparrow]+Q[\uparrow\downarrow]$, $Q[\downarrow *] := Q[\downarrow\downarrow]+Q[\downarrow\uparrow]$, $Q[* \uparrow] := Q[\uparrow\uparrow]+Q[\downarrow\uparrow]$ *and* $Q[* \downarrow] := Q[\downarrow\downarrow] + Q[\uparrow\downarrow]$ *first. An interval of the form* $(\max(a, b), \min(c, d))$ *is non-empty if and only if* $a < c, a < d, b < c, b < d$ *all hold.*

From now on assume that (r, u_1, u_2, d_1, d_2) satisfy the conditions in (a), such that the market is arbitrage-free.

(b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be two measurable functions. Briefly justify why the payoff

$$\widetilde{H} := f(\widetilde{S}^1_1) + g(\widetilde{S}^2_1)$$

is attainable for any such f, g. In the case where $f(x) = x^2$, $g(x) = (x-1)^-$, find a hedging strategy and then find the correct (undiscounted) price for $\widetilde{H}$ at time 0 given

$$r = 0, \quad u_1 = 0.2, \quad d_1 = -0.2, \quad u_2 = 0.6, \quad d_2 = -0.3, \quad p_1 = p_2 = 0.5.$$

(3 pt)

Hint: An arbitrage-free binomial model with only one risky asset is complete.

(c) Assume that (r, u_1, u_2, d_1, d_2) are as in (b). Prove that the market is incomplete by showing that the payoff

$$\widetilde{H} := \max(\widetilde{S}^1_1, \widetilde{S}^2_1)$$

is not attainable, and compute the set

$$\Pi(\widetilde{H}) := \left\{\widetilde{S}^0_0 E_Q\left[\frac{\widetilde{H}}{\widetilde{S}^0_1}\right] : Q \text{ is an EMM for } (\mathrm{S}^0, S^1, S^2)\right\}.$$

If you did not compute the set of EMMs in (a), describe how one can decide on attainability of a given payoff in general. (2 pt)

Question 3 (8 **Points**)

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ for some $T \in \mathbb{N}$ be a filtered probability space. Let $X = (X_k)_{k=0,1,\dots,T}$ be a $(P, \mathbb{F})$-local martingale and $\vartheta = (\vartheta_k)_{k=0,1,\dots,T}$ be an $\mathbb{F}$-predictable process.

(a) Let $Z = (Z_k)_{k=0,1,\dots,T}$ be the density process of $Q \approx P$ with respect to P. Show that $\frac{1}{Z}$ is a strictly positive Q-martingale with $E_Q\left[\frac{1}{Z_0}\right] = 1$ and that for every $k \in \{0, 1, \dots, T\}$ and every $A \in \mathcal{F}_k$ we have that

$$P[A] = E_Q\left[\frac{1}{Z_k}\mathbb{1}_A\right].$$

This shows that $\frac{1}{Z}$ is the density process of P with respect to Q. (2 pt)

(b) Let τ and σ be two $\mathbb{F}$-stopping times. Show that $\tau \wedge \sigma$ is an $\mathbb{F}$-stopping time. (2 pt)

(c) Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for X and define

$$\sigma_n := \inf\{k \in \mathbb{N}_0, \mid |\vartheta_{k+1}| > n\} \wedge T.$$

Show that $(\tau_n \wedge \sigma_n)_{n\in\mathbb{N}}$ is still a localizing sequence for X. (2 pt)

(d) Show that $(\tau_n \wedge \sigma_n)_{n\in\mathbb{N}}$ is in fact a localizing sequence for $\vartheta \bullet X$ and conclude that $\vartheta \bullet X$ is an $(P, \mathbb{F})$-local martingale. (2 pt)

Question 4 (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and $W = (W_t)_{t\geq 0}$, $B = (B_t)_{t\geq 0}$ two independent $(P, \mathbb{F})$-Brownian motions.

(a) Show that $BW = (B_t W_t)_{t\geq 0}$ is a $(P, \mathbb{F})$-martingale. (2 pt)

(b) Consider the process $X = (X_t)_{t\geq 0}$ defined by

$$X_t = \int_0^t s dW_s + B_t.$$

Write down the definition of the space $\mathcal{M}_0^2$. For a deterministic time $T > 0$, explain why $X^T = (X_{t\wedge T})_{t\geq 0} \in \mathcal{M}_0^2$. Prove that $X^2 = (X_t^2)_{t\geq 0}$ can be written in the form

$$X_t^2 = 2\int_0^t sX_s dW_s + 2\int_0^t X_s dB_s + \frac{t^3}{3} + t.$$

(2 pt)

Hint: You may use the fact that if $M_1, M_2 \in \mathcal{M}_0^2$ *then* $M_1 + M_2 \in \mathcal{M}_0^2$.

(c) Compute $Y_t = E[X_T^2 \mid \mathcal{F}_t]$ for $T > t \geq 0$, and express it as the sum of a constant and of stochastic integrals whose integrators are W and B. (1 pt)
Hint: You may use the fact that for $M \in \mathcal{M}_0^2$, $M^2 - [M]$ *is a martingale.*

(d) Consider the process $Z = \mathcal{E}(\alpha B + \beta W)$, which you may assume to be a $(P, \mathbb{F})$-martingale, for some $\alpha, \beta > 0$, and consider Q defined by $\frac{dQ}{dP} := Z_T$. Show that $Q \approx P$ on $\mathcal{F}_T$ and that the processes $\widetilde{B} = (\widetilde{B}_t)_{t\in[0,T]}$ and $\widetilde{W} = (\widetilde{W}_t)_{t\in[0,T]}$ given by $\widetilde{B}_t = B_t - \alpha t$ and $\widetilde{W}_t = W_t - \beta t$ are $(Q, \mathbb{F})$-Brownian motions on $[0, T]$. Rewrite X^2 in terms of $\widetilde{W}$ and $\widetilde{B}$. (2 pt)

(e) Does there exist an EMM for X^2? Justify your answer. (1 pt)

Question 5 (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions and $W = (W_t)_{t\in[0,T]}$ a $(P, \mathbb{F})$-Brownian motion. You can assume that $\mathbb{F} = \mathbb{F}^W = (\mathcal{F}_t^W)_{t\in[0,T]}$, the filtration generated by W, and that $\mathcal{F} = \mathcal{F}_T^W$. Consider the Black–Scholes model given by assets driven by the SDEs

$$d\widetilde{S}_t^0 = \widetilde{S}_t^0 r dt,$$
$$d\widetilde{S}_t^1 = \widetilde{S}_t^1(\mu dt + \sigma dW_t),$$

where $\widetilde{S}_0^0 = 1$, $\widetilde{S}_0^1 = x > 0$, $r > 0$, $\sigma > 0$ and $\mu \in \mathbb{R}$ are some fixed constants.

(a) Using Fubini's theorem, find the mean of $\int_0^t W_s ds$ and prove that its variance is $\frac{t^3}{3}$. Assuming that $\int_0^t W_s ds$ is normally distributed, use the Markov property of Brownian motion to find the conditional distribution of $\int_0^T W_s ds$ given $\mathcal{F}_t$. (3 pt)

Hint: You may want to rewrite $(\int_0^t W_s ds)^2 = \int_0^t \int_0^t W_s W_u ds du$.

Now we will price a so-called geometric Asian option. Assume for simplicity that $\mu = r = 0$ so that we have the explicit formulas

$$\widetilde{S}_t^0 = 1,$$
$$\widetilde{S}_t^1 = x \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)$$

and so that P is an EMM for S^1. Henceforth we write simply $S = S^1 = \widetilde{S}^1$.

(b) Consider the payoff $H = \mathbb{1}_{\{A_T > K\}}$, where $K > 0$ and $A_T = \exp\left(\frac{1}{T}\int_0^T \log(S_t)dt\right)$ is the geometric average over the interval $[0, T]$. Prove that the payoff's value process $V = (V_t)_{t\in[0,T]}$ is given by

$$V_t := E[H \mid \mathcal{F}_t] = 1 - \Phi(X_t)$$

where

$$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$$

is the cumulative distribution function of a standard normal random variable, and that the $(P, \mathbb{F})$-semimartingale $X = (X_t)_{t\in[0,T]}$ is given by

$$X_t = \frac{\sqrt{3}}{(T-t)^{\frac{3}{2}}} \left(\frac{T\log(K/x)}{\sigma} + \frac{T^2\sigma}{4} - \int_0^t W_s ds - (T-t)W_t \right).$$

(3 pt)

Hint: If you did not find the conditional distribution in (a), you can assume that $\int_0^T W_s ds$ *conditionally on* $\mathcal{F}_t$ *has* $\mathcal{N}(m_t, v_t^2)$ *distribution for some* $\mathbb{F}$*-adapted processes* m_t, v_t*, and compute* V_t *in this case.*

(c) Compute a hedging strategy $\varphi \widehat{=} (V_0, \theta_t)$ such that the value process is given by

$$V_t = V_0 + \int_0^t \theta_s dS_s$$

for all $t \in [0, T]$. If you are missing any ingredients from the previous subexercises to answer this question, write down the general form of the replicating strategy φ. (2 pt)