**ETH Zürich, HS 2019**
**D-MATH**
**Prof. E. W. Farkas** **January 2020**

# Examination

## Mathematical Foundations for Finance

**MATH, MScQF, SAV**

---

*Please fill in the following table*

| | |
|---|---|
| **Last name** | |
| **First name** | |
| **Programme of study** | **MATH** ☐ **MScQF** ☐ **SAV** ☐ **Other** ☐ |
| **Matriculation number** | |

*Leave blank*

| Question | Maximum | Points | Check |
|---|---|---|---|
| **1** | 8 | | |
| **2** | 8 | | |
| **3** | 8 | | |
| **4** | 8 | | |
| **5** | 8 | | |
| **Total** | **40** | | |

# Instructions

---

**Duration:** 180 min.

**Closed book examination:** no notes, no books, no calculator, no mobile phones, etc. allowed.

**Important:**

- ◇ Please put your student card on the table.
- ◇ Only pen and paper are allowed on the table. Please **do not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- ◇ Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- ◇ Take a new sheet for each question and write your name on every sheet.
- ◇ Except for Question 1, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas and results from the lecture or from the exercise classes without derivation.
- ◇ Simplify your results as much as possible.
- ◇ Most of the subquestions can be solved independently of each other.

⋆⋆⋆ Good luck! ⋆⋆⋆

# Answer Sheet for Question 1

Please use this sheet to answer Question 1. Indicate the correct answer by ✗. If there is no cross or more than one cross in a line, this will be interpreted as "no answer".

***Do not fill in***

| | answer (1) | answer (2) | answer (3) | correct | wrong | no answer |
|---|---|---|---|---|---|---|
| (a) | | | | | | |
| (b) | | | | | | |
| (c) | | | | | | |
| (d) | | | | | | |
| (e) | | | | | | |
| (f) | | | | | | |
| (g) | | | | | | |
| (h) | | | | | | |

***Do not fill in***

| | 1st corr. | 2nd corr. |
|---|---|---|
| correct | | |
| wrong | | |
| no answer | | |
| **Points** | | |

**Question 1** (8 **Points**)

For each of the following eight subquestions, there is **exactly one** correct answer. For each correct answer you get 1 point, for each wrong answer you get $-0.5$ point, and for no answer you get 0 points. You get at least 0 points for the whole exercise. **Please use the printed form for your answers.** It is enough to indicate your answer by a cross; you do not need to explain your choice.

Throughout subquestions (a) to (d), let $(\widetilde{S}^0, \widetilde{S}^1)$ be an undiscounted financial market in discrete time on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with a finite time horizon $T \in \mathbb{N}$ and $\mathbb{F} := (\mathcal{F}_k)_{k=0,1,\dots,T}$ generated by $\widetilde{S}^1$. Let $\widetilde{S}^0_k := (1+r)^k$ for $k = 0, 1, \dots, T$ and constants $r > -1$ and $\widetilde{S}^1_0 := s^1_0 > 0$. The discounted market is denoted by $(S^0, S^1)$.

(a) Let $\tau_1, \tau_2$ be two stopping times. Which of the following is a stopping time?

(1) $\lfloor \frac{\tau_1+\tau_2}{2} \rfloor$, where for any real number $x$, $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$.

(2) $\tau_1 \mathbb{1}_{\tau_1 \geq \tau_2}$.

(3) $\mathbb{1}_{\tau_1 > 0}$.

(b) Which of the following conditions does **not** imply that $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free?

(1) There exists a probability measure $Q \approx P$ such that $aS^0 + bS^1$ is a $(Q, \mathbb{F})$-martingale for any $a, b \in \mathbb{R}$.

(2) $\widetilde{S}^0 / \widetilde{S}^1$ is a positive martingale.

(3) There exists a positive martingale $Z$ such that $ZS^1$ is a martingale.

(c) Which of the following statements is **not** true about the binomial model?

(1) The market is complete if it is arbitrage-free.

(2) Every strategy is admissible.

(3) Every strategy is self-financing.

(d) Let $M$ be an adapted process. Which of the following does **not** imply that $M$ is a martingale?

(1) $M$ is a supermartingale such that $E[M_k]$ is an increasing sequence.

(2) For each $k = 1, \dots, T$, $E[M_k - M_{k-1} \mid \mathcal{F}_{k-1}] = 0$.

(3) $M$ is a bounded local martingale.

Throughout subquestions (e) to (h), $W$ denotes a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions of $P$-completeness and right-continuity.

(e) Let $M$ be a local $(P, \mathbb{F})$-martingale and $A$ be a process of finite variation. Then

(1) $H \bullet M$ is a martingale, if $H$ is bounded and predictable.

(2) $H \bullet A$ has finite variation, if $H$ is locally bounded and predictable.

(3) $H \bullet M$ is a martingale, if $M$ is a martingale and $H$ is locally bounded and predictable.

(f) Which of the following equations has a unique solution $Z = (Z_t)_{t\in[0,T]}$?

(1) $dZ_t = Z_t(3dt + dW_t)$

(2) $[Z]_t = t$, for $Z$ a continuous local martingale with $Z_0 = 0$.

(3) $Z_t = \int_0^t (2 + \cos(Z_s))dW_s$

(g) Suppose that $Q \approx P$ with density $\frac{dQ}{dP} = \exp(W_t - t/2)$. Which of the following is **not** true?

(1) $W_t^2 - t$ is a $Q$-martingale.

(2) $\exp(W_t - 3t/2)$ is a $Q$-martingale.

(3) $f(W_t)$ is a $(Q, \mathbb{F})$-semimartingale, for $f \in C^2(\mathbb{R})$.

(h) Consider the Black-Scholes model. Then:

(1) The price of a European put option increases, if the strike is decreased.

(2) The price of a European put option increases, if the interest rate is increased.

(3) None of the above.

**Question 2** (8 **Points**)

Let $\tilde{S} = (\tilde{S}^0_t, \tilde{S}^1_t)$ be a model of an arbitrage-free discrete time complete financial market with two assets and a finite time horizon $T$. Suppose that $\tilde{S}^0$ is a numéraire asset satisfying $\tilde{S}^0_{t+1} \geq \tilde{S}^0_t$ for all $t \geq 0$. Let $\tilde{C}(T, \tilde{K})$ be the initial undiscounted replication cost of a European Call option with strike $\tilde{K}$ and maturity $T$ written on the risky asset $\tilde{S}^1$. The goal of this exercise is to show that $T \to \tilde{C}(T, \tilde{K})$ is increasing and that $\tilde{K} \to \tilde{C}(T, \tilde{K})$ is decreasing and convex. *For simplicity you may assume $\mathcal{F}_0$ is trivial.*

(a) We define a *martingale deflator* to be an adapted process $Y$ such that $Y_t > 0$ for all $t \geq 0$ almost surely and such that the process $\tilde{S}Y = (\tilde{S}_t Y_t)_{t \geq 0}$ is a martingale (under the original measure $P$). Show that there is a one-to-one correspondence between martingale deflators and equivalent martingale measures (in finite time horizon models).

*Hint: Given a martingale deflator $Y$, consider the measure $Q$ defined by the Radon-Nykodym derivative*

$$\frac{dQ}{dP} = \frac{Y_T \tilde{S}^0_T}{E_P[Y_T \tilde{S}^0_T]}$$

*and show (using Bayes formula) that $Q$ defined this way is indeed an EMM. Conversely, given an EMM $Q$, consider the density process*

$$Z_t = E_P\left[\frac{dQ}{dP}|\mathcal{F}_t\right]$$

*and show that the process $Y$ defined by $Y_t = \frac{Z_t}{\tilde{S}^0_t}$ is a martingale deflator.*

Note that if $Y$ is a martingale deflator, then so is $cY$ for any $c > 0$. In what follows we will consider the unique martingale deflator such that $Y_0 = 1$.

(b) Let $Y$ be the unique martingale deflator such that $Y_0 = 1$. Show that $Y$ is a $P$-supermartingale. *Hint: for the integrability, you may use the fact that if the market model $\tilde{S}$ with $N$ assets is complete, then for each $t \geq 0$ the probability space $\Omega$ can be partitioned into no more than $N^t$ $\mathcal{F}_t$-measurable events of positive probability.*

(c) Show that the process defined by $Y_t(\tilde{S}^1_t - \tilde{K})^+ = (Y_t\tilde{S}^1_t - Y_t\tilde{K})^+$ is a $P$-submartingale.

(d) Write down the initial replication cost of a European Call option with strike $\tilde{K}$ and maturity $T$ in function of the martingale deflator $Y$.

(e) Conclude that $T \to \tilde{C}(T, \tilde{K})$ is increasing and that $K \to \tilde{C}(T, \tilde{K})$ is decreasing and convex.

**Question 3** (8 **Points**)

Let $(\Omega, \mathcal{F}, \mathbb{F}, P, \widetilde{S}^0, \widetilde{S}^1)$ be the canonical setup of a two asset, $T$-period trinomial model. More precisely, let the undiscounted price processes of the assets in our market be defined by

$$\tilde{S}^0_k = (1+r)^k \quad \text{for } k = 0, 1, \dots, T$$
$$\frac{\tilde{S}^1_{k+1}}{\tilde{S}^1_k} = Y_{k+1} \quad \text{for } k = 0, 1, \dots, T-1$$

where the $Y_k$ are i.i.d. random variables describing the returns of the risky asset. For simplicity assume that the initial undiscounted prices are given by $\tilde{S}^0_0 = \tilde{S}^1_0 = 1$. Suppose that the filtration $\mathbb{F}$ is given by the canonical filtration of $\tilde{S}^1$. Suppose also that the distribution of $Y_k$ under $P$ is given by

$$Y_k = \begin{cases} 1+u & \text{with probability } p_u > 0 \\ 1+m & \text{with probability } p_m > 0 \\ 1+d & \text{with probability } p_d > 0 \end{cases}$$

where $p_u$, $p_m$, $p_d > 0$, $p_u + p_m + p_d = 1$ , $-1 < d < m < u$ and $u > r > d$.

(a) Find the set of all EMMs for the discounted price process $S^1 = \frac{\tilde{S}^1}{\tilde{S}^0}$. Give your answer for a general $T \geq 1$.

From now on, we suppose that there is only one trading period, i.e. $T = 1$. Also recall that we have assumed $\tilde{S}^0_0 = \tilde{S}^1_0 = 1$ for simplicity.

(b) Find the set of all payoffs $\tilde{H}$ that can be replicated by trading in the risky underlying asset and the bank account.

(c) When one cannot perfectly replicate a payoff, one can try to at least approximately replicate the contingent claim. Mean-variance hedging is the problem of approximating, with minimal mean squared error, a given payoff by the final value of a self-financing trading strategy in a financial market. We thus consider the problem

$$\min_{\vartheta \in \mathcal{A}} E\left[(\tilde{H} - \tilde{c} - (\vartheta \bullet \tilde{S}^1)_T)^2\right].$$

where the expectation is taken under the original measure $\mathbb{P}$ and $\tilde{c}$ is supposed to be known and corresponds to the undiscounted initial investment, and the optimisation is over suitably integrable predictable processes:

$$\mathcal{A} = \{\text{all predictable processes } \vartheta = (\vartheta_k)_{k=1,\dots,T} : (\vartheta \bullet \tilde{S}^1)_k \in L^2 \text{ for } k = 1, \dots, T\}$$

For simplicity, assume that we are only interested in strategies with zero initial wealth, i.e $\tilde{c} = 0$ and suppose that the interest rate is $r = 0$. Recall that we have assumed $T = 1$ and $\tilde{S}^0_0 = \tilde{S}^1_0 = 1$ to simplify the computations. Let $p_u = \frac{2}{3}$ and $p_m = p_d = \frac{1}{6}$ and $d = -0.5$, $m = 0$ and $u = 0.5$. Consider a European call option on the risky asset $\tilde{S}^1$ with undiscounted strike $\tilde{K} = 1$. Find the hedging strategy for a European call option that minimizes the quadratic hedging error.

**Question 4** (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and $\hat{W} = (\hat{W}_t)_{t\geq 0}$ a Brownian motion adapted to $\mathbb{F}$ with respect to a fixed measure $Q$ that is equivalent to $P$. In this question we model stochastic interest rate processes and derive an arbitrage-free price for the zero-coupon bond with maturity $T$. For simplicity, suppose that there is no risky asset in the market and the undiscounted bank account price process $\widetilde{S}^0$ satisfies the following SDE:

$$d\widetilde{S}_t^0 = \widetilde{S}_t^0 r_t dt \quad \widetilde{S}_0^0 = 1$$

where $(r_t)_t$ is itself a stochastic process. In 1977, Vasicek proposed the following model (under a fixed EMM $Q$) for the stochastic short rate process $r$:

$$dr_t = \lambda(\bar{r} - r_t)dt + \sigma d\hat{W}_t$$

with a certain initial condition $r_0$. The parameters $\bar{r}$, $\lambda$ and $\sigma$ are given and assumed to be strictly positive.

(a) Give a financial interpretation of the parameters $\bar{r} > 0$, $\lambda > 0$ and $\sigma > 0$

(b) Derive a closed form solution for $\tilde{S}_t^0$

(c) Let $\tilde{P}_t^{(T)}$ denote the undiscounted time $t$ price of a zero-coupon bond of maturity $T$ and face value 1. This is a financial instrument that makes no periodic interest payment until it's maturity where it pays it's face value. In particular one must have $\tilde{P}_T^{(T)} = 1$ for a zero coupon bond with maturity $T$ and face value 1.

Express $\tilde{P}_t^{(T)}$ in terms of a conditional expectation of a process involving $r$ under the fixed measure $Q$.

Using the results from the previous question, one could actually compute $\tilde{P}_t^{(T)}$ explicitly since under the Vasicek model, one can solve the SDE for $r$ using an integrating factors technique. It turns out that $r_t$ is Gaussian and so is $\int_t^T r_s ds$ for all $t$ and hence the expectation obtained in the previous question can be evaluated using the moment generating function of a Gaussian random variable. In this exercise, we however take another approach and derive a PDE for the price process. You can take for granted that $\tilde{P}_t^{(T)}$ can be expressed as a function of current time $t$ and current spot rate $r_t$. To fix the notations, let $\tilde{V}(t, r_t) = \tilde{P}_t^{(T)}$.

(d) Using Ito's lemma, show that $\tilde{V}$ satisfies the following PDE:

$$\frac{\partial \tilde{V}}{\partial t}(t, r_t) + \lambda(\bar{r} - r_t)\frac{\partial \tilde{V}}{\partial r_t}(t, r_t) + \frac{1}{2}\sigma^2 \frac{\partial^2 \tilde{V}}{\partial r_t^2}(t, r_t) = r_t \tilde{V} \tag{1}$$

on $(0, \infty) \times (-\infty, \infty)$ with terminal condition $\tilde{V}(T, r_T) = 1$.

(e) ***Bonus question:*** *you can gain* ***additional 3 points*** *for solving this question*

To solve the above PDE (1), we make the ansatz (guess) that there exists unknown functions $R$ and $Q$ that depend only on time (not the interest rate) such that the solution of the PDE (1) is given by

$$\tilde{V}(t, r_t) = \exp\left(r_t R(T - t) + Q(T - t)\right)$$

Derive a system of ODEs for $R$ and $Q$. Solve this system of ODEs and conclude that the time $t$ undiscounted price $\tilde{V}(t, r_t) = \tilde{P}_t^{(T)}$ of a zero-coupon bond with maturity $T$ and face value 1 is given by

$$\begin{aligned}\tilde{V}(t, r_t) &= \tilde{P}_t^{(T)} \\ &= \exp\left(r_t \frac{e^{-\lambda(T-t)} - 1}{\lambda} + \bar{r}\frac{1 - e^{-\lambda(T-t)} - \lambda(T-t)}{\lambda} + \frac{\sigma^2\left(4e^{-\lambda(T-t)} - e^{-2\lambda(T-t)} + 2\lambda(T-t) - 3\right)}{4\lambda^3}\right)\end{aligned}$$

**Question 5** (8 **Points**)

Let $T \in (0, \infty)$ be a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by $W$ and augmented by the $P$-nullsets in $\sigma(W_s; 0 \le s \le T)$. Consider the Black–Scholes model with two assets, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ satisfy

$$\begin{aligned} d\widetilde{S}^0_t &= \widetilde{S}^0_t r dt, & \widetilde{S}^0_0 &= 1, \\ d\widetilde{S}^1_t &= \widetilde{S}^1_t \left(\mu dt + \sigma dW_t\right), & \widetilde{S}^1_0 &= S^1_0, \end{aligned}$$

with constants $\mu, r \in \mathbb{R}$, $\sigma > 0$, and $\widetilde{S}^1_0 = S^1_0 > 0$.

Find a replicating strategy and the associated wealth process for a claim with the following payouts. You may choose to handle **2 out of the following 4** payoffs.

(a) $\tilde{H} = (\tilde{S}^1_T)^p$ for some $p \in \mathbb{R}$

(b) $\tilde{H} = \left(\log \tilde{S}^1_T\right)^2$

(c) $\tilde{H} = \max(\tilde{S}^1_T - \tilde{K}, 0)$ for some strike $\tilde{K} \in \mathbb{R}$

*Hint 1: start by showing that if $X \sim \mathcal{N}(0,1)$ is a standard normal random variable, and $c$ and $m$ are positive constants, then the expectation $E\left[\left(e^{-v/2+\sqrt{v}X} - m\right)^+\right]$ can be expressed in terms of $\Phi$, the cumulative distribution function of $X$ as follows:*

$$E\left[\left(e^{-c/2+\sqrt{c}X} - m\right)^+\right] = \Phi\left(-\frac{\log m}{\sqrt{c}} + \frac{\sqrt{c}}{2}\right) - m\Phi\left(-\frac{\log m}{\sqrt{c}} - \frac{\sqrt{c}}{2}\right)$$

*Hint 2: You can assume without proof that the partial derivative of the function*

$$v(t, S^1_t) = S^1_t \Phi\left(\frac{\log\left(\frac{S^1_t}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) - K\Phi\left(\frac{\log\left(\frac{S^1_t}{K}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)$$

*with respect to its spatial variable is*

$$\frac{\partial}{\partial x} v(t, S^1_t) = \Phi\left(\frac{\log\left(\frac{S^1_t}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)$$

(d) $\tilde{H} = \max((\tilde{S}^1_T)^2 - \tilde{K}, 0)$ for some strike $\tilde{K} \in \mathbb{R}$

*Hint: What process does $(\tilde{S}^1_t)^2$ follow. How could you use the result for the standard European Call option to solve d)?*