Question 1

The correct answers are:

(a) (2)

(b) (1)

(c) (2)

(d) (3)

(e) (2)

(f) (1)

(g) (2)

(h) (1)

Question 2

(a) Any probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_2$ can be described by

$$\mathbb{Q}[\{(x_1, x_2)\}] := q_{x_1} q_{x_1,x_2}, \tag{1}$$

where q_{x_1}, q_{x_1,x_2} are in $(0,1)$ and satisfy $\sum_{x_1 \in \{1,2\}} q_{x_1} = 1$, $\sum_{x_2 \in \{1,2,3\}} q_{1,x_2} = 1$ and $\sum_{x_2 \in \{1,2\}} q_{2,x_2} = 1$. Next, since $\mathcal{F}_0$ is trivial, $\mathcal{F}_1 = \sigma(S_1^1)$ and S_1^1 only takes two values, S^1 is a $\mathbb{Q}$-martingale if and only if

$$\mathbb{E}_{\mathbb{Q}}[S_1^1] = 100, \quad \mathbb{E}_{\mathbb{Q}}[S_2^1 \mid S_1^1 = 200] = 200 \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}[S_2^1 \mid S_1^1 = 50] = 50.$$

Thus, $q_1, q_2, q_{1,1}, q_{1,2}, q_{1,3}, q_{2,1}, q_{2,2} \in (0,1)$ define an equivalent martingale measure for S^1 if and only if they satisfy the three systems of equations

$$\begin{cases} q_1 + q_2 &= 1, \\ 50q_1 + 200q_2 &= 100; \end{cases} \tag{I}$$

$$\begin{cases} q_{2,1} + q_{2,2} &= 1, \\ 100q_{2,1} + 300q_{2,2} &= 200; \end{cases} \tag{II}$$

$$\begin{cases} q_{1,1} + q_{1,2} + q_{1,3} &= 1, \\ 30q_{1,1} + 50q_{1,2} + 70q_{1,3} &= 50. \end{cases} \tag{III}$$

It is straightforward to check that the solution to (I) and (II) is given by

$$q_1 = \frac{2}{3}, \;\; q_2 = \frac{1}{3} \quad \text{and} \quad q_{2,1} = \frac{1}{2}, \;\; q_{2,2} = \frac{1}{2}. \tag{2}$$

Moreover, (III) is equivalent to

$$\begin{cases} q_{1,1} + q_{1,2} + q_{1,3} &= 1, \\ -q_{1,1} + q_{1,3} &= 0. \end{cases} \tag{III'}$$

Recalling that $q_{1,1}, q_{1,2}, q_{1,3} \in (0,1)$ shows that the solution to (III') is given by

$$q_{1,1} = \rho, \;\; q_{1,2} = 1 - 2\rho, \;\; q_{1,3} = \rho, \quad \rho \in (0, 1/2). \tag{3}$$

Thus, $\mathbb{P}_e(S^1) = \{\mathbb{Q}^\rho : \rho \in (0,1/2)\}$, where $\mathbb{Q}^\rho[\{(x_1,x_2)\}] = q^\rho_{x_1} q^\rho_{x_1,x_2}$ with

$$q^\rho_1 = \frac{2}{3}, \;\; q^\rho_2 = \frac{1}{3}, \quad q^\rho_{1,1} = \rho, \;\; q^\rho_{1,2} = 1 - 2\rho, \;\; q^\rho_{1,3} = \rho \quad \text{and} \quad q^\rho_{2,1} = \frac{1}{2}, \;\; q^\rho_{2,2} = \frac{1}{2}. \tag{4}$$

Because $\mathbb{P}_e(S^1) \neq \emptyset$, we conclude that the market is free of arbitrage.

(b) Since the strike price K is greater than or equal to 70 and less than 300, the payoff from the call option is not zero if and only if the price of S^1 has increased in the first step, i.e., on the set $\{S_1^1 = 200\}$.

Working backwards through the tree, i.e. starting from $k = 2$, we obtain the values of the call option for $k = 1$ and $k = 0$ as

$$\begin{tikzcd}
 & & 300-K \\
 & \frac{1}{2}(300-K) + \frac{1}{2}(100-K)^+ \arrow[ur] \arrow[r] & (100-K)^+ \\
V^{C^K} : \frac{1}{6}(300 - K + (100-K)^+) \arrow[ur] \arrow[dr] & & 0 \\
 & 0 \arrow[ur] \arrow[r] \arrow[dr] & 0 \\
 & & 0
\end{tikzcd}$$

To calculate the replication strategy ϑ_k, $k = 1, 2$, we use Δ-hedging $\Delta V_k^{C^K} = \vartheta_k \Delta S_k^1$, which gives

$$\vartheta_k = \frac{V_k^{C^K} - V_{k-1}^{C^K}}{S_k^1 - S_{k-1}^1}.$$

Hence, we get that the initial capital is $v_0 = \frac{1}{6}(300 - K + (100 - K)^+)$, $\vartheta_0 = 0$,

$$\begin{aligned} \vartheta_1 &= \frac{\left(\frac{1}{2} - \frac{1}{6}\right)(300 - K + (100 - K)^+)}{200 - 100} \\ &= \frac{300 - K + (100 - K)^+}{300} \\ &= \frac{200 - K}{150} 1_{\{70 \le K \le 100\}} + \frac{300 - K}{300} 1_{\{K > 100\}}, \end{aligned}$$

and

$$\begin{aligned} \vartheta_2 &= \frac{300 - K - (100 - K)^+}{200} 1_{\{S_1^2 = 200\}} \\ &= 1_{\{S_1^2 = 200,\ 70 \le K \le 100\}} + \frac{300 - K}{200} 1_{\{S_1^2 = 200,\ K > 100\}}, \end{aligned}$$

and the holdings on the bank account are determined by the relation $\varphi_k^0 = V_k^{C^K} - \vartheta_k S_k^1$, $k = 1, 2$.

(c) The call option is **not** attainable for $50 \le K < 70$. Indeed, fix $0 < \rho < 1/2$ in the parametrization of the EMM $\mathbb{Q}^\rho$ in (3). Then, similarly as in (b), we solve

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & 300 - K \\
 & 200 - K \arrow[ur] \arrow[r] & 100 - K \\
V^{C^K} : \frac{200-K}{3} + \rho\frac{140-2K}{3} \arrow[ur] \arrow[dr] & & 70 - K \\
 & \rho(70 - K) \arrow[ur] \arrow[r] \arrow[dr] & 0 \\
 & & 0
\end{tikzcd}$$

and obtain

$$\mathbb{E}_{\mathbb{Q}^\rho}[C^K] = V_0^{C^K, \mathbb{Q}^\rho} = \frac{200 - K}{3} + \rho \frac{140 - 2K}{3}, \quad K \in [50, 70). \tag{5}$$

The mapping $\rho \mapsto \mathbb{E}_{\mathbb{Q}^\rho}[C^K]$, $0 < \rho < 1/2$, is non-constant. This implies that the payoff C^K is not attainable for the strike price $50 \le K < 70$ (Theorem 1.2.3 on p. 49 in the lecture notes).

(d) By put-call parity,

$$S_2^1 - K = (S_2^1 - K)^+ - (K - S_2^1)^+,$$

the put option P^K is attainable precisely for those values of the strike price $50 \le K < 300$ for which the call C^K is attainable. So, the put option is attainable for $70 \le K < 300$ and not attainable for $50 \le K < 70$.

Question 3

(a) It clearly suffices to show that for all $k = 1, \ldots, T-1$, we have

$$E_Q\left[\frac{\widetilde{C}^{Eu}_{k+1}}{\widetilde{S}^0_{k+1}}\right] \geq E_Q\left[\frac{\widetilde{C}^{Eu}_{k}}{\widetilde{S}^0_{k}}\right]. \tag{6}$$

Fix $k \in \{1, \ldots, T-1\}$. Using the *tower property* of conditional expectations, *Jensen's inequality* for conditional expectations (for the convex function $x \mapsto x^+$), the fact that S^1 is a Q-martingale and $r \geq 0$, we get

$$\begin{aligned}
E_Q\left[\frac{\widetilde{C}^{Eu}_{k+1}}{\widetilde{S}^0_{k+1}}\right] &= E_Q\left[\left(S^1_{k+1} - \frac{\widetilde{K}}{(1+r)^{k+1}}\right)^+\right] \\
&= E_Q\left[E_Q\left[\left(S^1_{k+1} - \frac{\widetilde{K}}{(1+r)^{k+1}}\right)^+ \middle| \mathcal{F}_k\right]\right] \\
&\geq E_Q\left[\left(E_Q\left[S^1_{k+1} - \frac{\widetilde{K}}{(1+r)^{k+1}} \middle| \mathcal{F}_k\right]\right)^+\right] \\
&= E_Q\left[\left(S^1_{k} - \frac{\widetilde{K}}{(1+r)^{k+1}}\right)^+\right] \\
&\geq E_Q\left[\left(S^1_{k} - \frac{\widetilde{K}}{(1+r)^{k}}\right)^+\right] \\
&= E_Q\left[\frac{\widetilde{C}^{Eu}_{k}}{\widetilde{S}^0_{k}}\right].
\end{aligned}$$

(b) Since the function $x \mapsto x^+$ is convex, we have for $k = 1, \ldots, T$ that

$$\begin{aligned}
\widetilde{C}^{As}_k &= \left(\frac{1}{k}\sum_{j=1}^k \widetilde{S}^1_j - \widetilde{K}\right)^+ = \left(\sum_{j=1}^k \frac{1}{k}\left(\widetilde{S}^1_j - \widetilde{K}\right)\right)^+ \\
&\leq \sum_{j=1}^k \frac{1}{k}\left(\widetilde{S}^1_j - \widetilde{K}\right)^+ = \frac{1}{k}\sum_{j=1}^k \widetilde{C}^{Eu}_j.
\end{aligned} \tag{7}$$

By linearity and monotonicity of expectation and since $r \geq 0$, we get

$$\begin{aligned}
E_Q\left[\frac{\widetilde{C}^{As}_k}{\widetilde{S}^0_k}\right] &= E_Q\left[\frac{\widetilde{C}^{As}_k}{(1+r)^k}\right] \leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{\widetilde{C}^{Eu}_j}{(1+r)^k}\right] \\
&\leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{\widetilde{C}^{Eu}_j}{(1+r)^j}\right] = \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{\widetilde{C}^{Eu}_j}{\widetilde{S}^0_j}\right].
\end{aligned} \tag{8}$$

(c) Putting the results of (a) and (b) together yields for $k = 1, \ldots, T$ that

$$E_Q\left[\frac{\widetilde{C}^{As}_k}{\widetilde{S}^0_k}\right] \leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{\widetilde{C}^{Eu}_j}{\widetilde{S}^0_j}\right] \leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{\widetilde{C}^{Eu}_k}{\widetilde{S}^0_k}\right] = E_Q\left[\frac{\widetilde{C}^{Eu}_k}{\widetilde{S}^0_k}\right]. \tag{9}$$

(d) We have

$$
\begin{aligned}
E_Q\left[\frac{\widetilde{C}^{lb}_{k+1}}{\widetilde{S}^0_{k+1}}\middle|\mathcal{F}_k\right] &= E_Q\left[\frac{\left(\max_{j\le k+1}\widetilde{S}^1_j-\widetilde{K}\right)^+}{(1+r)^{k+1}}\middle|\mathcal{F}_k\right]\\
&\ge \frac{\left(E_Q\left[\max_{j\le k+1}\widetilde{S}^1_j\middle|\mathcal{F}_k\right]-\widetilde{K}\right)^+}{(1+r)^{k+1}}\\
&\ge \frac{\left(\max_{j\le k}\widetilde{S}^1_j-\widetilde{K}\right)^+}{(1+r)^{k+1}}\\
&\ge \frac{\left(\max_{j\le k}\widetilde{S}^1_j-\widetilde{K}\right)^+}{(1+r)^{k}}\\
&= \frac{\widetilde{C}^{lb}_k}{\widetilde{S}^0_k},
\end{aligned}
$$

where the first inequality is Jensen's inequality for the conditional expectation, the second follows from the fact that $\max_{j\le k+1}\widetilde{S}^1_j=\widetilde{S}^1_{k+1}\vee\max_{j\le k}\widetilde{S}^1_j$ and the last from the fact that $r\ge 0$. So, $\left(\widetilde{C}^{lb}_k/\widetilde{S}^0_k\right)_{k=1,\dots,T}$ is a Q-submartingale.

(e) Let us denote the process $\widetilde{C}^{Eu}_k=(\widetilde{S}^1_k-\widetilde{K})^+$, $k=1,\dots,T$, by $X=(X_k)_{k=1,\dots,T}$ and the process $\widetilde{C}^{lb}_k=(\max_{j\le k}\widetilde{S}^1_j-\widetilde{K})^+=\max_{j\le k}(\widetilde{S}^1_j-\widetilde{K})^+=\max_{j\le k}X_k$, $k=1,\dots,T$, by $X^*=(X^*_k)_{k=1,\dots,T}$. Repeating the argument for (a) with conditional expectations given $\mathcal{F}_k$ shows that X is a non-negative Q-submartingale, and repeating the argument in (d) with $r=0$ shows that X^* is a non-negative Q-submartingale. The stopping time τ can now be written as

$$\tau=\inf\{k\in\{1,\dots,T\}:X^*_k\ge M\}\wedge T.$$

Moreover, we have

$$X_\tau\ge M$$

on $A:=\{X^*_T\ge M\}$ and

$$\tau=T$$

on $\Omega\setminus A=\{X^*_T<M\}$. Since $\tau\le T$, by the (optional) stopping/sampling theorem, we get

$$E_Q\left[X_T\mid\mathcal{F}_\tau\right]\ge X_\tau=X_\tau 1_A+X_\tau 1_{\Omega\setminus A}\ge M1_A.$$

Taking Q-expectations on both sides, we get

$$Q[A]\le\frac{1}{M}E_Q[X_T],$$

i.e.,

$$Q\left[\widetilde{C}^{lb}_T\ge M\right]\le\frac{1}{M}E_Q[\widetilde{C}^{Eu}_T]$$

as claimed.

(f) We have

$$H^2(\omega)=H(\omega)\ \forall\omega\in\Omega$$

if and only if

$$H(\omega)(1-H(\omega))=0\ \forall\omega\in\Omega.$$

So, H takes only the values 0 and 1 and is $\mathcal{F}_T$-measurable, so

$$H = 1_F =: H^F$$

for some $F \in \mathcal{F}_T$. There are exactly 2^N such options and since $r = 0$, their price under Q is equal to the Q-expectation

$$E_Q\left[H^F\right] = Q[F].$$

Question 4

(a) By Itô's formula,

$$h(W_t) = h(W_0) + \int_0^t h'(W_s)dW_s + \frac{1}{2}\int_0^t h''(W_s)ds,$$

i.e.,

$$\int_0^t h'(W_s)dW_s = h(W_t) - h(W_0) - \frac{1}{2}\int_0^t h''(W_s)ds$$

for any C^2-function $h : \mathbb{R} \to \mathbb{R}$. We want to find a function h whose derivative $h'(x)$ is xe^x, so we pick a candidate $h(x) = xe^x - e^x + c$, where c is a constant. For this particular choice of h, we have $h(W_0) = h(0) = -1 + c$, $h'(x) = xe^x$ and $h''(x) = e^x + xe^x = e^x(1 + x)$, so we pick $f(x) = h(x) - h(0) = xe^x - e^x + 1$ and $g(x) = -\frac{1}{2}h''(x) = -\frac{1}{2}e^x(1+x)$. We return to Itô's formula to verify that

$$\begin{aligned}\int_0^t W_s e^{W_s} dW_s &= W_t e^{W_t} - e^{W_t} + 1 - \frac{1}{2}\int_0^t e^{W_s}(1 + W_s)\,ds \\ &= f(W_t) + \int_0^t g(W_s)ds.\end{aligned}$$

(b) We have $S = \mathcal{E}(\sigma W + \mu t)$, so $S > 0$ and $|S|^3 = S^3$. Since $x \mapsto x^3$ is in C^2, we may compute, by Itô's formula,

$$dY_t = dS_t^3 = 3S_t^2 dS_t + \frac{1}{2}6S_t d\langle S\rangle_t,$$

where

$$d\langle S\rangle_t = \sigma^2 S_t^2 d\langle W\rangle_t = \sigma^2 S_t^2 dt$$

so that

$$\begin{aligned}dY_t &= 3\mu S_t^3 dt + 3\sigma S_t^3 dW_t + 3\sigma^2 S_t^3 dt \\ &= 3\sigma Y_t dW_t + 3(\mu + \sigma^2)Y_t dt \\ &= Y_t\left(3\sigma dW_t + 3(\mu + \sigma^2)dt\right),\end{aligned}$$

i.e., $Y = \mathcal{E}(3\sigma W + 3(\mu + \sigma^2)t)$.

(c) Let us try to find a measure Q which admits a continuous density process $Z = (Z_t)_{t\in[0,T]}$ of the form

$$Z_t = \mathcal{E}\left(-\int \nu_s dW_s\right)_t$$

for ν in $L^2_{loc}(W)$. Then, by Girsanov's theorem (lecture notes Theorem 6.2.3), given a P-Brownian motion W, the process $\widetilde{W}$ given as

$$\widetilde{W}_t = W_t - \left\langle \int \nu_s dW_s, W\right\rangle_t = W_t - \int_0^t \nu_s ds,\ t \in [0, T],$$

is a Q-Brownian motion. We want

$$X_t = \widetilde{W}_t$$

for all $t \in [0, T]$, and this we have for ν for which

$$\int_0^t \nu_s ds = t^3 - t,\ t \in [0, T],$$

i.e.,

$$\nu_t = 3t^2 - 1,\ t \in [0,T],$$

which is apparently in $L^2_{loc}(W)$. We may now explicitly compute the density process $Z = (Z_t)_{t\in[0,T]}$ as

$$\begin{aligned} Z_t &= \mathcal{E}\left(-\int \nu_s dW_s\right)_t \\ &= \exp\left(-\int_0^t \nu_s dW_s - \frac{1}{2}\int_0^t \nu_s^2 ds\right) \\ &= \exp\left(-\int_0^t (3s^2-1) dW_s - \frac{1}{2}\int_0^t (3s^2-1)^2 ds\right) \\ &= \exp\left(-3\int_0^t s^2 dW_s + W_t - \frac{9}{10}t^5 + t^3 - \frac{1}{2}t\right) \end{aligned}$$

and we see that the process Z is indeed continuous. The Radon–Nikodým derivative $\frac{dQ}{dP}$ is then obtained as

$$\frac{dQ}{dP} = Z_T = \exp\left(-3\int_0^T s^2 dW_s + W_T - \frac{9}{10}T^5 + T^3 - \frac{1}{2}T\right),$$

which uniquely characterizes the measure Q as

$$Q[F] = \int_F Z_T dP,\ F \in \mathcal{F}_T.$$

Question 5

(a) Let W^Q denote the Q-Brownian motion given by

$$W_t^Q = W_t + \frac{\mu - r}{\sigma} t, \; t \in [0, T].$$

For the *discounted* stock price process

$$S_t = S_0 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right) t\right) = S_0 \exp\left(\sigma W_t^Q - \frac{1}{2}\sigma^2 t\right),$$

we obtain by Itô's formula the dynamics under measure P as

$$dS_t = S_t\left((\mu - r)dt + \sigma dW_t\right) = \sigma S_t \left(\frac{\mu - r}{\sigma} dt + dW_t\right).$$

So, under the measure Q, we have

$$dS_t = \sigma S_t dW_t^Q,$$

i.e.,

$$S_t = S_0 \exp\left(\sigma W_t^Q - \frac{1}{2}\sigma^2 t\right).$$

So,

$$\begin{aligned} 1/S_t &= \frac{1}{S_0} \exp\left(-\sigma W_t^Q + \frac{1}{2}\sigma^2 t\right) \\ &= \frac{1}{S_0} \exp\left(-\sigma W_t^Q - \frac{1}{2}\sigma^2 t + \sigma^2 t\right) \end{aligned}$$

so that $1/S = \frac{1}{S_0}\mathcal{E}\left(-\sigma W^Q + \sigma^2 t\right)$ satisfies

$$d\left(\frac{1}{S_t}\right) = \frac{1}{S_t}\left(\sigma^2 dt - \sigma dW_t^Q\right).$$

(b) We have $\log \frac{S_t}{S_0} \sim \mathcal{N}\left((\mu - r - \frac{1}{2}\sigma)t, \sigma^2 t\right)$, so by the fact that $\log \frac{S_t}{S_0} = -\log \frac{S_0}{S_t}$, we have $\log \frac{S_0}{S_t} \sim \mathcal{N}\left((\frac{1}{2}\sigma - \mu + r)t, \sigma^2 t\right)$. In particular, we conclude that the adapted process $1/S$ is integrable. It is apparent that $1/S > 0$, and because the function $1/x$ is convex for $x > 0$, we get by Jensen's inequality that

$$E_Q\left[1/S_t \mid \mathcal{F}_s\right] \geq 1/E_Q\left[S_t \mid \mathcal{F}_s\right] = 1/S_s,$$

where the equality on the right follows by the Q-martingale property of S. Indeed, the process S is a Q-martingale (see Proposition 4.2.2. in the lecture notes). Thus, we have shown that $1/S$ is a Q-submartingale.

(c) We note that $g(S_T) = \sigma W_T^Q$. Indeed,

$$\log \frac{S_T}{S_0} + \frac{1}{2}\sigma^2 T = \sigma W_T^Q.$$

We have

$$\sigma W_T^Q = \sigma W_0^Q + \int_0^T \sigma dW_u^Q = 0 + \int_0^T S_u^{-1} dS_u,$$

so that the self-financing strategy whose initial capital is $V_0 = 0$ and which at $0 \leq t \leq T$ holds $\vartheta_t = S_t^{-1}$ shares of stock and $\varphi_t^0 = V_t - \vartheta_t S_t = V_t - 1$ units of cash on the bank account replicates the payoff $g(S_T)$. Here,

$$V_t = V_0 + \int_0^t \vartheta_u dS_u = \int_0^t S_u^{-1} dS_u = \sigma W_t^Q = \log \frac{S_t}{S_0} + \frac{1}{2}\sigma^2 t.$$