ETH Zürich, HS 2020
D-MATH
Prof. Dr. M. Schweizer **August 2021**

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Important!

Enter the first two letters of your surname and your first name, as well as the last six digits of your matriculation number in the boxes below. Write the same information and **only this information** clearly at the top of each additional sheet of paper that you submit.

Surname	First name	Matriculation (legi) number	Exam number
first *two letters of each*		***last*** *six digits only*	*leave blank*

Question	Maximum	Points	Check
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration: 180 min.

Closed book examination: no notes, no books, no calculator, no mobile phones, etc. allowed.

Important:

- Please put your student card on the table.
- Only pen and paper are allowed on the table. Please **do not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question, but try to solve as many questions as possible.
- Take a new sheet for each question.
- Except for Question 1, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas and results from the lecture or from the exercise classes without derivation.
- Simplify your results as much as possible.
- Most of the subquestions can be solved independently of each other.

⋆ ⋆ ⋆ Good luck! ⋆ ⋆ ⋆

Answer Sheet for Question 1

Please use this sheet to answer Question 1. Indicate the correct answer by ✗. If there is no ✗ or more than one ✗ in a line, this will be interpreted as "no answer".

	answer (1)	answer (2)	answer (3)
(a)			
(b)			
(c)			
(d)			
(e)			
(f)			
(g)			
(h)			

Leave blank

correct	wrong	no answer

Leave blank

	1st corr.	2nd corr.
correct		
wrong		
no answer		
Points		

Question 1 (8 **Points)**

For each of the following eight subquestions, there is **exactly one** correct answer. For each correct answer, you get 1 point; for each wrong answer, you get -0.5 point; and for no answer, you get 0 points. You get at least 0 points overall for the question. **Please use the printed form for your answers.** It is enough to indicate your answer by a cross; you need not explain your choice.

Throughout subquestions (a) to (d), let $(\Omega, \mathcal{F}, \mathbb{F}, P, \widetilde{S}^0, \widetilde{S}^1)$ or shortly $(\widetilde{S}^0, \widetilde{S}^1)$ be an undiscounted financial market in discrete time with a finite time horizon $T \in \mathbb{N}$ and with the filtration $\mathbb{F} := (\mathcal{F}_k)_{k=0,1,\dots,T}$ generated by $\widetilde{S}^1$. Furthermore, $\widetilde{S}^0_k := (1+r)^k$ for $k = 0, 1, \dots, T$ and constants $r > -1$ and $\widetilde{S}^1_0 := s^1_0 > 0$. The prices discounted by $\widetilde{S}^0$ are denoted by S^0 and S^1, respectively.

(a) Which of the following statements is correct?

(1) Every model for $(\widetilde{S}^0, \widetilde{S}^1)$ can be expressed as a multinomial model.

(2) The market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free if and only if there exists a probability measure $Q \approx P$ such that S^1 is a $(Q, \mathbb{F})$-local martingale.

(3) The market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free if and only if $\widetilde{S}^1_k / \widetilde{S}^1_{k-1}$, $k = 1, \dots, T$, are independent and identically distributed random variables.

(b) Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a binomial model with $u > d$. Which of the following statements is true?

(1) There exists a unique EMM for S^1.

(2) The set Ω contains at least 2^T elements.

(3) The probability measure P does not have any atoms in $\mathcal{F}$.

(c) Let $\varphi = (\varphi^0, \vartheta)$ be an arbitrage strategy for $(\widetilde{S}^0, \widetilde{S}^1)$. Then:

(1) $V_0(\varphi) = 0$ P-a.s.

(2) $G_k(\vartheta) \geq 0$ P-a.s. for all $k \in \{0, 1, \dots, T\}$.

(3) φ is not necessarily admissible since we are in finite discrete time.

(d) Which of the following statements about a local martingale $X = (X_k)_{k=0,1,\dots,T}$, $T \in \mathbb{N}$, on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is **not** true?

(1) If X is bounded, then X is a true $(P, \mathbb{F})$-martingale.

(2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a concave function. Then $f(X)$ is an $\mathbb{F}$-adapted stochastic process.

(3) Let τ be an $\mathbb{F}$-stopping time. Then the stopped process X^τ is not a $(P, \mathbb{F})$-local martingale.

Throughout subquestions (e) to (h), W denotes a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions of P-completeness and right-continuity.

(e) Let B be another Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Which of the following statements is true?

(1) (B, W) is a two-dimensional Brownian motion.

(2) If B and W are independent, then BW is a Brownian motion.

(3) $W^2 - B^2$ is a $(P, \mathbb{F})$-martingale.

(f) Which of the following statements about the optional quadratic variation $[M] = ([M]_t)_{t\geq 0}$ of a local martingale $M = (M_t)_{t\geq 0}$ is **not** true?

(1) $[M]$ is P-a.s. increasing, right-continuous and has left limits.

(2) There exists an increasing function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $[M]_t = g(t)$ for all $t \geq 0$ P-a.s. As a special case of this, we have $[W]_t = t$ for all $t \geq 0$ P-a.s.

(3) The sample paths of $[M]$ are of finite variation P-a.s.

(g) Let $X = (X_t)_{t\geq 0}$ be defined by $X_t := \mathcal{E}(W)_t$. We then have for $X^2 = (X_t^2)_{t\geq 0}$ that

(1) $X_t^2 = e^t \mathcal{E}(2W)_t$.

(2) $X_t^2 = e^{-2t} \mathcal{E}(2W)_t$.

(3) $X_t^2 = e^{2t} \mathcal{E}(2W)_t$.

(h) Let $N = (N_t)_{t\geq 0}$ be a $(P, \mathbb{F})$-Poisson process with parameter $\lambda > 0$ and define the process $X = (X_t)_{t\geq 0}$ by $X_t := 2 + \int_0^t \sqrt{s} dW_s + N_t$. Then we have P-a.s. that

(1) $[X]_t = \frac{1}{2}t^2 + N_t + 2\int_0^t \sqrt{s} dN_t$.

(2) $[X]_t = \frac{1}{2}t^2 + \lambda t$.

(3) $[X]_t = \frac{1}{2}t^2 + N_t$.

Question 2 (8 **Points**)

Consider a one-period model for a financial market with one bank account and one stock. The bank account has zero interest rate and hence a price process given by $S_0^0 = S_1^0 = 1$. The stock has an initial price of $S_0^1 = 100$. At time 1, its price either doubles, stays the same or halves, all with positive probability. We realise this model on a probability space $(\Omega, \mathcal{F}, P)$ with $|\Omega| = 3$ and $\mathcal{F} = 2^\Omega$ with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1}$, where $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_1 := \mathcal{F}$.

(a) Find two different equivalent martingale measures for S^1.

(b) For any payoff $H \geq 0$ at time 1, a *superreplicating strategy* is a self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ with $V_1(\varphi) \geq H$ P-a.s. Show that for any payoff $H \geq 0$, there exist infinitely many superreplicating strategy.

(c) Now let H be the payoff of a call option on S^1 with (discounted) strike K so that we have $H = \left(S_1^1 - K\right)^+$. For $K = 90$, find the value at time 0 of the cheapest superreplicating strategy for H, i.e.

$$\pi(H) = \inf\{V_0 \,:\, \exists \varphi \mathrel{\widehat{=}} (V_0, \vartheta) \text{ with } V_1(\varphi) \geq H\}.$$

(d) Show that there exists a martingale measure Q^* for S^1 such that $\pi(H) = E_{Q^*}[H]$. Is Q^* equivalent to P?

Question 3 (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

(a) Let $X = (X_k)_{k\in\mathbb{N}_0}$ be a local $(P, \mathbb{F})$-martingale, and let the process $Y = (Y_k)_{k\in\mathbb{N}_0}$ be such that $|X_j| \le Y_k$ P-a.s. for all $0 \le j \le k$ and $Y_k \in L^1(P)$ for all $k \in \mathbb{N}_0$. Show that X is a true $(P, \mathbb{F})$-martingale.

(b) Conclude that any integrable local $(P, \mathbb{F})$-martingale X is a true $(P, \mathbb{F})$-martingale.
Remark: This result is only true in discrete time.

(c) Show that any local $(P, \mathbb{F})$-martingale $X \ge 0$ P-a.s. with $X_0 \in L^1(P)$ is a true $(P, \mathbb{F})$-martingale.

(d) Let $H = (H_k)_{k\in\mathbb{N}_0}$ be an $\mathbb{F}$-predictable process. Show that H is locally bounded.

Question 4 (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Let $W = (W_t)_{t\geq 0}$ be a $(P, \mathbb{F})$-Brownian motion, $N = (N_t)_{t\geq 0}$ a $(P, \mathbb{F})$-Poisson process with parameter $\lambda > 0$ and define $\widetilde{N} = (\widetilde{N}_t)_{t\geq 0}$ by $\widetilde{N}_t := N_t - \lambda t$, $t \geq 0$. Let $T > 0$ be a deterministic time.

(a) Show that $\widetilde{N}_- \in L^2(W^T)$, where we note for clarity that W^T denotes the process W stopped at time T.
Hint: You can use that N_- is locally bounded.

(b) Using your result from (a), compute

$$\mathrm{Var}\left[\int_0^T \widetilde{N}_{t-} dW_t\right].$$

(c) Show that the process $M = (M_t)_{t\in[0,T]}$ defined by

$$M_t := \int_0^t W_s d\widetilde{N}_s$$

is a $(P, \mathbb{F})$-martingale.
Hint: You can use the fact that all moments of $\sup_{t\in[0,T]} |W_t|$ *are finite.*

(d) Define the process $S = (S_t)_{t\geq 0}$ by $S_t := \sum_{k=0}^{N_t} W_k$. This is a special case of a so-called *compound Poisson process*. Derive the quadratic variation of S.

Question 5 (8 **Points**)

Let $T \in (0, \infty)$ be a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s, 0 \leq s \leq T)$. Assume that the undiscounted price process of the bank account $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ satisfy

$$
\begin{aligned}
d\widetilde{S}^0_t &= \widetilde{S}^0_t r dt, & \widetilde{S}^0_0 &= 1,\\
d\widetilde{S}^1_t &= \widetilde{S}^1_t \left(\mu dt + \sigma dW_t\right), & \widetilde{S}^1_0 &= S^1_0,
\end{aligned}
$$

with constants $\mu, r \in \mathbb{R}, \sigma > 0$ and $S^1_0 > 0$. As usual, let $S^0 = 1$ and $S^1 = (S^1_t)_{t\in[0,T]}$, where $S^i = \widetilde{S}^i / \widetilde{S}^0$ denote the discounted price processes.

(a) Find an EMM $\widehat{Q}$ for $\widetilde{S}^0/\widetilde{S}^1$, i.e. for $1/S^1$.

(b) What is the distribution of the random variable $\log \widetilde{S}^1_T$ under the measure $\widehat{Q}$?
Hint: First work out the dynamics of $\widetilde{S}^1$ under the measure $\widehat{Q}$.

(c) Show the change of numéraire formula, i.e. that for any $H \in L^0_+(\mathcal{F}_T)$, one has

$$
E_Q[H] = \widetilde{S}^0_T S^1_0 E_{\widehat{Q}}\left[\frac{H}{\widetilde{S}^1_T}\right],
$$

where Q is the unique EMM for S^1.

(d) Find the arbitrage-free value of the undiscounted payoff $\widetilde{H} = \widetilde{S}^1_T \log \widetilde{S}^1_T$.

(e) *In this question, you need not provide a rigorous mathematical argument.*
Consider the Black–Scholes market with two risky assets $\widetilde{S}^i = (\widetilde{S}^i_t)_{t\in[0,T]}$ satisfying

$$
d\widetilde{S}^i_t = \widetilde{S}^i_t\big(\mu_i dt + \sigma dW^i_t\big), \quad \widetilde{S}^i_0 = s,
$$

for $i = 1, 2$ with constants $\sigma, s > 0$, $\mu_1 \neq \mu_2$ and W^1 and W^2 two independent Brownian motions. In addition, there is also a bank account with constant interest rate $r \in \mathbb{R}$. How will the prices of European call options with the same strike price and maturity on the two assets compare?