ETH Zürich HS 2015
D-MATH
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January 2016

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Please fill in the following table

Last name	
First name	
Programme of study	**MATH** □ **MScQF** □ **SAV** □ **Other** □
Matriculation number	

Leave blank

Question	**Maximum**	**Points**	**Check**
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration of exam: 180 min.

Closed book examination: no notes, no books, no calculator, no mobile phones, etc. allowed.

Important:

- ◇ Please put your student card on the table.
- ◇ Only pen and paper are allowed on the table. Please do **not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- ◇ Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- ◇ Take a new sheet for each question and write your name on every sheet.
- ◇ Except for question 1, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas and results from the lecture or from the exercise classes without derivation.
- ◇ Simplify your results as far as possible.
- ◇ Most of the subquestions can be solved independently of each other.

$\star\star\star$ Good luck! $\star\star\star$

Answers Sheet for Question 1

Please use this sheet to solve Question 1. Indicate the correct answer by a **cross ✗**. If there is no cross or more than one cross in a line, this will be interpreted as "no answer".

	answer (1)	answer (2)	answer (3)
(a)			
(b)			
(c)			
(d)			
(e)			
(f)			
(g)			
(h)			

Do not fill in

correct	wrong	no answer

Do not fill in

	1st corr.	2nd corr.
correct		
wrong		
no answer		
Points		

Question 1 (8 Points)

For each of the following 8 subquestions, there is exactly one correct answer. For each correct answer you get 1 point, for each wrong answer you get $-1/2$ points, and for no answer you get 0 points. You get at least zero points for the whole exercise. **Please use the printed form for your answers.** It is enough to put a cross; you need not explain your choice.

Throughout subquestions (a) to (d), let $(\widetilde{S}^0, \widetilde{S}^1)$ be an undiscounted financial market in discrete time with time horizon $T \in \mathbb{N}$, where $\widetilde{S}^0_k := (1+r)^k$ for $k = 0, \dots, T$ and some $r > -1$. Assume that $\widetilde{S}^1_0 := s^1_0 > 0$. Finally, the discounted market is denoted by (S^0, S^1) and the filtration $\mathbb{F} := (\mathcal{F}_k)_{k=0,\dots,T}$ is always generated by $\widetilde{S}^1$.

(a) Consider the binomial model with a time horizon $T < \infty$ on the canonical probability space. Suppose that $u > r > d$ and let $\varphi \mathrel{\widehat{=}} (1, \vartheta)$ be a self-financing trading strategy. Then

(1) the cost process $C(\varphi)$ satisfies $C(\varphi) \equiv 0$.

(2) φ is admissible.

(3) for any choice of ϑ, the value $P[G_T(\vartheta) > 0]$ cannot be strictly larger than 0.

(b) Let σ, τ be two stopping times with respect to $\mathbb{F}$. Then it is always true that

(1) $(\tau - \sigma)^+$ is a stopping time.

(2) $\sigma \mathbb{1}_{\{\sigma \text{ is even}\}} + 1$ is a stopping time.

(3) $P[\tau = 0] \in \{0, 1\}$.

(c) Suppose that $S^1_T = s_T \geq 0$. Then

(1) the market is arbitrage free if and only if $s^1_0 = s_T$.

(2) if the market is arbitrage free, then $V(\varphi) \equiv 0$ for every admissible self-financing trading strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$.

(3) there is no arbitrage free market with this property.

(d) For $k = 0, \dots, T$, define $Z_k := E\left[\frac{\mathrm{d}Q^*}{\mathrm{d}P} \middle| \mathcal{F}_k\right]$, where Q^* is an equivalent martingale measure for S^1. Then it is always true that

(1) $Z_k > 0$ P-a.s. for all $k = 0, \dots, T$.

(2) ZS^1 is a $(Q^*, \mathbb{F})$-martingale.

(3) $E[S^1_T] \neq s^1_0$.

Throughout subquestions (e) to (h), W denotes a Brownian motion and N a Poisson process with parameter $\lambda > 0$, relative to the same filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$.

(e) Which of the following assertions is true?

(1) the process $W^{2016} - 2016 \int 2015 W_s^{2014}\ \mathrm{d}s$ is a $(P, \mathbb{F})$-martingale.

(2) the process W^{2016} is a $(P, \mathbb{F})$-submartingale.

(3) None of the previous answers is correct.

(f) Let B be another $\mathbb{F}$-Brownian motion on $(\Omega, \mathcal{F}, P)$ such that W and B are independent. Then

(1) $\mathcal{E}(WB) = 0$.

(2) $\mathcal{E}(WB)$ is a $(P, \mathbb{F})$-local martingale.

(3) $\mathcal{E}(WB) = \left(e^{B_t W_t} e^{-t}\right)_{t\geq 0}$.

(g) Which of the following processes is **not** a $(P, \mathbb{F})$-martingale?

(1) $\left(N_t^2 - (2\lambda t + 1)N_t + (\lambda t)^2\right)_{t\geq 0}$.

(2) $\left(N_t^2 - 2\lambda t N_t + (\lambda t)(\lambda t - 1)\right)_{t\geq 0}$.

(3) $\left(N_t^2 - 2\lambda t N_t + (\lambda t)^2\right)_{t\geq 0}$.

(h) Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a C^2 function. Then the process $\left(f(t, W_t)\right)_{t\geq 0}$ is a $(P, \mathbb{F})$-local martingale if and only if

(1) $\frac{\partial^2 f}{\partial w^2} = 0$.

(2) $\frac{1}{2}\frac{\partial^2 f}{\partial w^2} + \frac{\partial f}{\partial t} = 0$.

(3) $\frac{\partial f}{\partial w} = 0$.

Question 2 (8 **Points**)

Consider the one-period trinomial model on the probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{\omega_d, \omega_m, \omega_u\}$, $\mathcal{F} = 2^\Omega$, and P is the probability measure given by

$$p_d := P[\{\omega_d\}] = \frac{1}{3}, \qquad p_m := P[\{\omega_m\}] = \frac{1}{3}, \qquad p_u := P[\{\omega_u\}] = \frac{1}{3}.$$

Consider the filtration $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1)$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 = \mathcal{F}$. Recall that this market consists of a bank account whose price process is denoted by $\widetilde{S}^0 := \big(\widetilde{S}^0_k\big)_{k=0,1}$ and a risky asset whose price process is denoted by $\widetilde{S}^1 := \big(\widetilde{S}^1_k\big)_{k=0,1}$, where

$$\begin{aligned} \widetilde{S}^0_1 &= (1+r)\widetilde{S}^0_0 \qquad \text{with} \qquad \widetilde{S}^0_0 = 1, \\ \widetilde{S}^1_1 &= Y\widetilde{S}^1_0 \qquad \text{with} \qquad \widetilde{S}^1_0 = 101, \end{aligned}$$

for some $r > -1$ and a random variable Y which takes positive values $1+d$, $1+m$ and $1+u$ on ω_d, ω_m and ω_u, respectively. Finally denote by $S^i := \widetilde{S}^i/\widetilde{S}^0$, $i = 0, 1$, the discounted price processes.
Suppose that $u = 0.05$, $m = 0.02$, and $r = d = 0.01$.

(a) Show that this market is not free of arbitrage by explicitly constructing an arbitrage opportunity. By which values can you replace the value of d in order to make the market free of arbitrage?

For the remaining part of the exercise, set $d = -0.01$.

(b) Show that the discounted payoff $H^{Put} := (102 - S^1_1)^+$ of a put option with (discounted) strike 102 is not attainable, by computing its expectation under all equivalent martingale measures (EMMs) Q for S^1.

(c) Consider the equivalent martingale measure Q^* for S^1 given by

$$q^*_d := Q^*[\{\omega_d\}] = \frac{1}{2}, \qquad q^*_m := Q^*[\{\omega_m\}] = \frac{1}{3}, \qquad q^*_u := Q^*[\{\omega_u\}] = \frac{1}{6}.$$

Consider the enlargement of the market given by $\big(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2\big)$, where

$$\widetilde{S}^2_0 := E_{Q^*}\big[H^{Put}\big] \qquad \text{and} \qquad \widetilde{S}^2_1 := (1+r)H^{Put}.$$

(i) Is this enlarged market free of arbitrage?

(ii) Is it complete?

(d) Consider again the enlargement of the market of point (c) and let $\widetilde{H}^{Call} := \big(\widetilde{S}^1_1 - 102.01\big)^+$ be the undiscounted payoff of a call option on $\widetilde{S}^1$ with undiscounted strike 102.01. Compute its replication strategy.

Hint: Note that $102.01 = 101 \cdot 1.01$.

Question 3 (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T \in \mathbb{N}$ the time horizon. Consider a collection $\big(Y_j\big)_{j=1,\dots,T}$ of i.i.d., $\mathcal{U}\big((0,3/2)\big)$-distributed random variables.
Define the discounted financial market (S^0, S^1) by

$$S^0 \equiv 1, \qquad S_k^1 := S_0^1 \prod_{j=1}^{k} Y_j \quad \text{for } k = 1, \dots, T, \qquad \text{with} \qquad S_0^1 := 1.$$

Consider the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,\dots,T}$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(Y_1, \dots, Y_k)$ for each $k = 1, \dots, T$.

Hint: $X \sim \mathcal{U}\big((0,3/2)\big)$ means that X has a uniform distribution on $(0,3/2)$; so its density function is $\frac{2}{3}\mathbb{1}_{(0,3/2)}$, and hence $P[X \in A] = \frac{2}{3}\int_0^{3/2} \mathbb{1}_A(y)\,\mathrm{d}y$, for all Borel sets $A \subseteq (0,3/2)$. For this distribution, $E[X] = E[X^2] = 3/4$.

(a) Consider the probability measure Q^* given by $\frac{\mathrm{d}Q^*}{\mathrm{d}P} := \left(\frac{4}{3}\right)^T S_T^1$. Compute its density process and prove that it is an equivalent martingale measure (EMM) for S^1.

(b) Consider the random variable

$$\tau := \inf\{k = 1, \dots, T : Y_k > 1\} \wedge T,$$

where we agree that $\inf \emptyset = \infty$. Show that τ is a stopping time and prove that the stochastic process $\varphi = (\varphi^0, \vartheta)$, where $\varphi^0 := (\varphi_k^0)_{k=0,\dots,T}$ and $\vartheta := (\vartheta_k)_{k=0,\dots,T}$ with

$$\begin{aligned} \varphi_k^0 &:= \mathbb{1}_{\{k \le \tau\}} \quad \text{for } k = 1, \dots, T, \qquad \text{and} \qquad \varphi_0^0 = 0, \\ \vartheta_k &:= -\mathbb{1}_{\{k \le \tau\}} \quad \text{for } k = 1, \dots, T, \qquad \text{and} \qquad \vartheta_0 = 0, \end{aligned}$$

is a trading strategy.

(c) The trading strategy of point (b) is not self-financing. Find the process $\overline{\varphi}^0 := (\overline{\varphi}_k^0)_{k=0,\dots,T}$ with $\overline{\varphi}_0^0 = 0$ making $\overline{\varphi} := (\overline{\varphi}^0, \vartheta)$ self-financing, and compute the discounted value process $V(\overline{\varphi})$ of $\overline{\varphi}$. Is the strategy $\overline{\varphi}$ admissible?

(d) Show that $V(\overline{\varphi})$ is a $(Q^*, \mathbb{F})$-martingale. Is it true that $V(\varphi)$ is a $(Q^*, \mathbb{F})$-martingale for every admissible self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$?

Question 4 (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, and let $W = (W_t)_{t\geq 0}$ be a Brownian motion with respect to P and $\mathbb{F}$.

(a) Fix $T \in (0, \infty)$ and define the processes $I = (I_t)_{t\in[0,T]}$ and $X = (X_t)_{t\in[0,T]}$ by

$$I_t = \int_0^t \frac{1}{2T - u} \, \mathrm{d}W_u \quad \text{and} \quad X_t = (2T - t)I_t.$$

Calculate the quadratic variation $[X]_t$, $t \in [0, T]$. Is X a Brownian motion (on $[0, T]$) with respect to P and $\mathbb{F}$?

(b) Determine all $a, b \in \mathbb{R}$ such that the process $M = (M_t)_{t\geq 0}$ given by

$$M_t = atW_t + bW_t^3, \quad t \geq 0,$$

is a martingale with respect to P and $\mathbb{F}$.

(c) Let $\alpha \in \mathbb{R}$ and set $Z_t := \exp(-(W_t + \alpha t)^2)$, $t \geq 0$. Determine $\liminf_{t\to\infty} Z_t$ and $\limsup_{t\to\infty} Z_t$.

Question 5 (8 **Points**)

Let $T \in (0, \infty)$ be a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; 0 \leq s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ satisfy the SDEs

$$\begin{aligned} \mathrm{d}\widetilde{S}^0_t &= \widetilde{S}^0_t r \,\mathrm{d}t, & \widetilde{S}^0_0 &= 1, \\ \mathrm{d}\widetilde{S}^1_t &= \widetilde{S}^1_t \left(\mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t\right), & \widetilde{S}^1_0 &= S^1_0, \end{aligned}$$

with constants $\mu, r \in \mathbb{R}$, $\sigma > 0$, and $S^1_0 > 0$. Let Q^* denote the unique equivalent martingale measure for the discounted stock price process $S^1 := \frac{\widetilde{S}^1}{\widetilde{S}^0}$.

(a) Define $\widehat{Q}$ by $\frac{\mathrm{d}\widehat{Q}}{\mathrm{d}Q^*} = \frac{S^1_T}{S^1_0}$. Show that $\widehat{Q}$ is a probability measure equivalent to Q^* on $\mathcal{F}_T$ and compute the density process $Z = (Z_t)_{t\in[0,T]}$ of $\widehat{Q}$ with respect to Q^*. Moreover, show that for all $\widetilde{H} \in L^0_+(\mathcal{F}_T)$ and $t \in [0, T]$,

$$\widetilde{S}^0_t E_{Q^*}\left[\frac{\widetilde{H}}{\widetilde{S}^0_T}\,\middle|\, \mathcal{F}_t\right] = \widetilde{S}^1_t E_{\widehat{Q}}\left[\frac{\widetilde{H}}{\widetilde{S}^1_T}\,\middle|\, \mathcal{F}_t\right] \qquad P\text{-a.s.}$$

(b) Construct a $\widehat{Q}$-Brownian motion $\widehat{W}$ such that the process $\widehat{S}^0 := \frac{\widetilde{S}^0}{\widetilde{S}^1}$ satisfies the SDE

$$\mathrm{d}\widehat{S}^0_t = \widehat{S}^0_t \sigma \,\mathrm{d}\widehat{W}_t. \tag{$*$}$$

(c) Show that the undiscounted stock price process $\widetilde{S}^1$ satisfies the SDE

$$\mathrm{d}\widetilde{S}^1_t = \widetilde{S}^1_t \left((r + \sigma^2)\,\mathrm{d}t - \sigma \,\mathrm{d}\widehat{W}_t\right).$$

Remark: You have to use $(*)$ from point (b).

(d) Construct a replicating strategy for the *asset-or-nothing call* with undiscounted payoff $\widetilde{S}^1_T \mathbb{1}_{\{\widetilde{S}^1_T \geq \widetilde{K}\}}$ for some constant $\widetilde{K} > 0$, i.e., find explicitly a number $V_0 \in \mathbb{R}$ and a predictable, locally bounded process ϑ such that

$$V_0 + \int_0^T \vartheta_t \,\mathrm{d}S^1_t = \frac{\widetilde{S}^1_T}{\widetilde{S}^0_T} \mathbb{1}_{\{\widetilde{S}^1_T \geq \widetilde{K}\}} \qquad P\text{-a.s.}$$

and the stochastic integral process $\int \vartheta \,\mathrm{d}S^1$ is a $(Q^*, \mathbb{F})$-martingale.

Remark: In your expressions for V_0 and ϑ, you can use the standard symbols φ and Φ for the density and the cumulative distribution function of the standard normal distribution.