ETH Zürich HS 2016
D-MATH
Prof. M. Schweizer
Prof. E.W. Farkas

February 2017

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Please fill in the following table

Last name	
First name	
Programme of study	**MATH** □ **MScQF** □ **SAV** □ **Other** □
Matriculation number	

Leave blank

Question	**Maximum**	**Points**	**Check**
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration of exam: 180 min.

Closed book examination: no notes, no books, no calculator, no mobile phones, etc. allowed.

Important:

- ◇ Please put your student card on the table.
- ◇ Only pen and paper are allowed on the table. Please do **not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- ◇ Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- ◇ Take a new sheet for each question and write your name on every sheet.
- ◇ Except for question 1, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas and results from the lecture or from the exercise classes without derivation.
- ◇ Simplify your results as far as possible.
- ◇ Most of the subquestions can be solved independently of each other.

⋆⋆⋆ Good luck! ⋆⋆⋆

Answers Sheet for Question 1

Please use this sheet to solve Question 1. Indicate the correct answer by a **cross ✗**. If there is no cross or more than one cross in a line, this will be interpreted as "no answer".

	answer (1)	answer (2)	answer (3)
(a)			
(b)			
(c)			
(d)			
(e)			
(f)			
(g)			
(h)			

Do not fill in

correct	wrong	no answer

Do not fill in

	1st corr.	2nd corr.
correct		
wrong		
no answer		
Points		

Question 1 (**8 points**)

For each of the following 8 subquestions, there is exactly one correct answer. For each correct answer you get 1 point, for each wrong answer you get $-1/2$ points, and for no answer you get 0 points. You get at least zero points for the whole question 1. **Please use the printed form for your answers.** It is enough to put a cross; you need not explain your choice.

Throughout subquestions (a) to (d), let $(\widetilde{S}^0, \widetilde{S}^1)$ be an undiscounted financial market in discrete time with time horizon $T \in \mathbb{N}$, where $\widetilde{S}^0_k := (1+r)^k$ for $k = 0, \dots, T$ and some $r > -1$. Assume that $\widetilde{S}^1_0 := s^1_0 > 0$. Finally, the discounted market is denoted by (S^0, S^1) and the filtration $\mathbb{F} := (\mathcal{F}_k)_{k=0,\dots,T}$ is always generated by $\widetilde{S}^1$.

(a) Assume that there exists an arbitrage opportunity. Then

(1) every contingent claim is attainable.

(2) there exists an admissible arbitrage opportunity.

(3) the set of equivalent martingale measures is non-empty.

(b) Let σ and τ be two $\mathbb{F}$-stopping times satisfying $\sigma \le \tau$. Then it is always true that

(1) a random time $\widetilde{\sigma}$ satisfying $\sigma \le \widetilde{\sigma} \le \tau$ is an $\mathbb{F}$-stopping time.

(2) $\tau - \sigma$ is an $\mathbb{F}$-stopping time.

(3) $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

(c) Suppose that P is a probability measure and $\widetilde{S}^1$ is a local $(P, \mathbb{F})$-submartingale. Then

(1) the stopped process $(\widetilde{S}^1_{k\wedge\tau})_{k=0}^T$ is a $(P, \mathbb{F})$-submartingale for every $\mathbb{F}$-stopping time τ.

(2) the stochastic integral process $\varphi \cdot \widetilde{S}^1$ is a local $(P, \mathbb{F})$-submartingale for every bounded predictable process φ.

(3) the process $\widetilde{S}^1$ is a $(P, \mathbb{F})$-submartingale if it is bounded from above.

(d) Let Q be a martingale measure for S^1 and equivalent to a probability measure P. Then

(1) the process S^1 is an $\mathbb{F}$-martingale under P.

(2) there exists an $\mathcal{F}_T$-measurable random variable Z such that $Q[A] = \int_\Omega 1_A Z dP$ for every $A \in \mathcal{F}_T$ and $\frac{1}{Z}$ is Q-integrable.

(3) we have $P[A] = Q[A]$ for every $A \in \mathcal{F}_T$.

Throughout subquestions (e) to (h), W denotes a Brownian motion relative to a fixed filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, P)$.

(e) Which of the following processes is a Brownian motion under P?

(1) $(cW_{t/c^2})_{t\geq 0}$ for a fixed $c > 0$.

(2) $(\int_0^t W_u dW_u)_{t\geq 0}$.

(3) $\big(\exp(cW_t - \frac{1}{2}c^2 t)\big)_{t\geq 0}$ for a fixed $c > 0$.

(f) Which of the following processes is a $(P, \mathbb{F})$-martingale?

(1) $(4W_t^3 - tW_t)_{t\geq 0}$.

(2) $(W_t^3 - 3t^2 W_t)_{t\geq 0}$.

(3) $(W_t^3 - 3tW_t)_{t\geq 0}$.

(g) Define the stopping time $\tau = \inf\{t \geq 0 : W_t = 1\}$ and the process Y as $Y_t = W_t^\tau$ for $t \geq 0$. Then

(1) Y is a local martingale, but not a martingale.

(2) Y is a martingale and a local martingale.

(3) Y is neither a martingale nor a local martingale.

(h) Let B be another $(P, \mathbb{F})$-Brownian motion which is independent of W and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a C^2-function such that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \geq 0$. Then

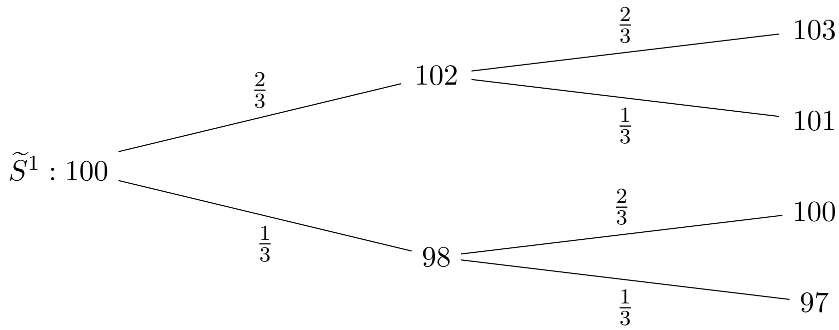
(1) the process $\Big(f(W_t, B_t)\Big)_{t\geq 0}$ is a local $(P, \mathbb{F})$-submartingale.

(2) the process $\Big(f(W_t, B_t)\Big)_{t\geq 0}$ is increasing.

(3) neither of the previous two answers is correct.

Question 2 (8 **points)**

Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ consisting of a bank account and one stock. The movements of the undiscounted stock price $\widetilde{S}^1$ are described by the following tree, where the numbers beside the branches denote transition probabilities.



More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space with $\Omega := \{u, d\}^2$, $\mathcal{F} := 2^\Omega$ and the probability measure $\mathbb{P}$ defined by $\mathbb{P}[\{(x_1, x_2)\}] := p_{x_1} p_{x_2}$, where $p_u := \frac{2}{3}$ and $p_d := \frac{1}{3}$. Next, consider Y_1 and Y_2 given by

$$Y_1((u,u)) := Y_1((u,d)) := \frac{102}{100}, \quad Y_1((d,u)) := Y_1((d,d)) := \frac{98}{100}$$

and

$$Y_2((u,u)) := \frac{103}{102}, \, Y_2((u,d)) := \frac{101}{102}, \, Y_2((d,u)) := \frac{100}{98}, \, Y_2((d,d)) := \frac{97}{98}.$$

The bank account $\widetilde{S}^0$ and the stock $\widetilde{S}^1$ are then given by $\widetilde{S}^0_k = (1+r)^k$ and $\widetilde{S}^1_k = 100 \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$, respectively. Finally, the filtration $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$ is defined by $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_1 := \sigma(Y_1)$ and $\mathcal{F}_2 := 2^\Omega = \mathcal{F}$.

- Assume in (a) and (b) that the risk-free rate is $r = 0$.

(a) Construct (i.e. write explicitly) an equivalent martingale measure $\mathbb{Q}$ for the discounted stock price S^1.

(b) Determine the "one-step conditional densities" of $\mathbb{Q}$, i.e., the random variables D_1 and D_2 such that the process $Z = (Z_k)_{k=0,1,2}$ defined by $Z_0 := 1$ and $Z_k := \prod_{j=1}^k D_j$ for $k = 1, 2$ is the density process of $\mathbb{Q}$ with respect to $\mathbb{P}$. Are D_1 and D_2 independent under $\mathbb{P}$?

(c) Give the definition of an arbitrage-free market in the current setup and determine in both cases, (c1) and (c2), if the market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free.

(c1) The risk-free rate is $r = -0.02$.

(c2) The risk-free rate is $r = 0.01$.

Question 3 (8 **points**)

Let $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ be a filtration on the probability space $(\Omega, \mathcal{F}, Q)$ and $X = (X_k)_{k=0,1,2}$ an adapted Q-integrable process. Define the process $Y = (Y_k)_{k=0,1,2}$ recursively by

$$Y_k := \begin{cases} X_k & \text{if } k = 2, \\ \max\{X_k, E_Q[Y_{k+1} \mid \mathcal{F}_k]\} & \text{if } k = 0, 1. \end{cases}$$

(a) Show that Y is the smallest Q-supermartingale dominating X, i.e., that Y is a Q-supermartingale satisfying $Y_k \geq X_k$ Q-a.s. for $k = 0, 1, 2$, and if Z is another Q-supermartingale with $Z_k \geq X_k$ Q-a.s. for $k = 0, 1, 2$, then $Z_k \geq Y_k$ Q-a.s. for $k = 0, 1, 2$.

(b) Show that $\tau := \inf\{k \in \{0, 1, 2\} : Y_k = X_k\}$ is a $\{0, 1, 2\}$-valued $\mathbb{F}$-stopping time.

(c) Show that the stopped process $Y^\tau = (Y_{\tau \wedge k})_{k=0,1,2}$ is a Q-martingale by using the identity

$$Y^\tau_{k+1} - Y^\tau_k = 1_{\{k+1 \leq \tau\}}(Y_{k+1} - Y_k). \tag{1}$$

You do not have to prove that the identity (1) is true.

(d) Show that $Y_0 = E_Q[X_\tau]$.

(e) Show that if S is a Q-martingale and $f : \mathbb{R} \to \mathbb{R}$ is a convex function such that $f(S)$ is Q-integrable, then $f(S)$ is a Q-submartingale.

(f) Assume now that $X = f(S)$, where $\mathbb{R} \ni x \mapsto f(x) = (x - K)^+$ for some $K \in \mathbb{R}$ and S is a Q-martingale. Denote

$$V^{C,Am} := \sup_{\sigma \in \mathcal{T}} E_Q[f(S_\sigma)],$$

where σ ranges over the family $\mathcal{T}$ of all $\{0, 1, 2\}$-valued $\mathbb{F}$-stopping times. Show that $V^{C,Am} = V^{C,Eu}$, where $V^{C,Eu} := E_Q[f(S_2)]$.

Question 4 (8 points)

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, and let $W = (W_t)_{t\geq 0}$ be a Brownian motion with respect to P and $\mathbb{F}$. Fix $T \in (0, \infty)$.

(a) Compute $E[W_T^4]$ via Itô's formula.

(b) Compute the process $\langle W^2, W^2 \rangle$.

(c) Let $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ be such that

$$\frac{\partial f}{\partial t} = -\frac{1}{2}\frac{\partial^2 f}{\partial x^2}.$$

Put $X_t = f(t, W_t)$ for $t \geq 0$. Show that the process X is a local martingale, and if in addition

$$E\left[\int_0^T \left(\frac{\partial f}{\partial x}(t, W_t)\right)^2 dt\right] < \infty,$$

then X is a martingale on $[0, T]$.

Question 5 (8 **points)**

Let $T \in (0, \infty)$ be a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; 0 \leq s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{B} = (\widetilde{B}_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S} = (\widetilde{S}_t)_{t\in[0,T]}$ satisfy the SDEs

$$\begin{aligned} \mathrm{d}\widetilde{B}_t &= \widetilde{B}_t r \,\mathrm{d}t, & \widetilde{B}_0 &= 1, \\ \mathrm{d}\widetilde{S}_t &= \widetilde{S}_t \left(\mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t\right), & \widetilde{S}_0 &= S_0, \end{aligned}$$

with constants $\mu, r \in \mathbb{R}$, $\sigma > 0$, and $\widetilde{S}_0 = S_0 > 0$.

(a) Find a measure Q' (by giving its Radon–Nikodým derivative) such that the process

$$W_t^{Q'} := W_t + \frac{\mu + r}{\sigma} t, \ t \in [0, T],$$

is a Q'-Brownian motion. (Be careful with the signs.)

(b) Consider a so-called *power option* with the undiscounted payoff $h(\widetilde{S}_T) = \widetilde{S}_T^p$ at time T, for some $p \in \mathbb{R}$. Compute the undiscounted value $\widetilde{V}_t$ for this payoff for each time $t \in [0, T]$.

(c) Compute the hedging strategy $\varphi = (\eta, \vartheta) = (\eta_t, \vartheta_t)_{t\in[0,T]}$ for the power option.