D–MATH

Exam Probability Theory

401-3601-00L

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Please do not turn the page yet!

Please take note of the information on the answer-booklet.

All random variables are defined on an a fixed implicit probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

For a real random variable X, we write $\mathrm{E}(X)$ for its expectation, and ϕ_X for its characteristic function, defined by

$$\forall t \in \mathbb{R} \quad \phi_X(t) = \mathrm{E}(e^{itX}).$$

$\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$ of non-negative integers.

The last page contains some additional results from the lectures that you may use without proof.

Exercise 1

Let $(X_n)_{n\geq 1}$ be iid real random variables such that

$$\forall t \in \mathbb{R} \quad \mathrm{P}(X_1 > t) = \begin{cases} \frac{1}{1+t^2} & \text{if } t \geq 0, \\ 1 & \text{if } t < 0. \end{cases}$$

1.1 [1 Point] Show that, almost surely, we have $X_n > \sqrt{n}$ for infinitely many $n \geq 1$.

1.2 [3 Points] Show that $\sqrt{n}\min(X_1, \ldots, X_n)$ converges in distribution as $n \to \infty$.

1.3 [2 Points] Show that

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow[n\to\infty]{a.s.} \frac{\pi}{2}.$$

1.4 [2 Points] Show that $(X_1 + \cdots + X_n)^{1/2}\min(X_1, \ldots, X_n)$ converges in distribution as $n \to \infty$.

Exercise 2

Let (X,Y) be a random vector taking values in $\mathbb{R}^2$ that has joint density given by

$$f(x,y) = \begin{cases} \frac{1}{4}\left[1 + xy(x^2 - y^2)\right] & \text{if } |x| \leq 1 \text{ and } |y| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

2.1 [2 Points] Show that both X and Y are uniform random variables in $[-1,1]$.

2.2 [1 Point] Check that

$$\int_{-1}^{1}\int_{-1}^{1} e^{it(x+y)} xy(x^2 - y^2)\, dxdy = 0$$

for all $t \in \mathbb{R}$.

2.3 [2 Points] Show that

$$\forall t \in \mathbb{R} \quad \varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t).$$

2.4 [2 Points] Are X and Y independent? Justify your answer.

Exercise 3

Let $(Z_n)_{n\geq 1}$ be iid random variables such that

$$\mathrm{P}(Z_1 = 1) = \mathrm{P}(Z_1 = -1) = 1/2.$$

Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $X_0 = 0$. For $n \geq 1$, let

$$\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n) \quad \text{and} \quad X_n = Z_1 + \cdots + Z_n.$$

3.1 [**2 Points**] Show that (X_n) is a $(\mathcal{F}_n)$-martingale.

3.2 [**2 Points**] Show that $(X_n^2 - n)$ is a $(\mathcal{F}_n)$-martingale.

Let $T : \Omega \to \mathbb{N}$ be a $(\mathcal{F}_n)$-stopping time.

3.3 [**2 Points**] Let $t > 0$. Prove that

$$\mathrm{E}\Big(X^2_{\lfloor t^2 \rfloor \wedge T}\Big) \leq \mathrm{E}(t^2 \wedge T).$$

3.4 [**2 Points**] Prove that

$$\mathrm{P}\left(\sup_{n\geq 0} |X_{n\wedge T}| \geq t\right) \leq \mathrm{P}\Big(T \geq t^2\Big) + \mathrm{P}\left(\sup_{n\leq \lfloor t^2 \rfloor} |X_{n\wedge T}| \geq t\right).$$

3.5 [**2 Points**] Use the previous two parts to deduce that

$$\mathrm{P}\left(\sup_{n\geq 0} |X_{n\wedge T}| \geq t\right) \leq \mathrm{P}\Big(\sqrt{T} \geq t\Big) + \mathrm{E}\left(\frac{T\wedge t^2}{t^2}\right).$$

3.6 [**2 Points**] Show that

$$\int_0^\infty \frac{\mathrm{E}(T\wedge t^2)}{t^2}\, dt = 2\mathrm{E}\Big(\sqrt{T}\Big).$$

3.7 [**1 Point**] Conclude that

$$\mathrm{E}\left(\sup_{n\geq 0} |X_{n\wedge T}|\right) \leq 3\mathrm{E}\Big(\sqrt{T}\Big).$$

3.8 [**3 Points**] Let $S = \min\{n \geq 0 : X_n = -1\}$. Show that

$$\mathrm{E}\Big(\sqrt{S}\Big) = \infty.$$

Exercise 4

4.MC1 [3 Points] Let X, Y be two independent uniform random variable in $[0, 1]$. Write f_X and f_Y for the densities of X and Y, respectively. Which of the following are true?

(A) The random variable $X + Y$ admits a density f_{X+Y} given by

$$\forall x \in \mathbb{R} \quad f_{X+Y}(x) = f_X(x) f_Y(x).$$

(B) $\mathrm{E}((X + Y)^2) = \mathrm{E}(X^2) + \mathrm{E}(Y^2)$

(C) $\mathrm{Var}(X + Y) = 2\mathrm{Var(X)}$

(D) For every $a, b \in \mathbb{R}$ we have $\mathrm{E}((X - a)(X - b)) = \mathrm{E}(X - a)\mathrm{E}(X - b)$.

(E) There exists $a, b \in \mathbb{R}$ such that $\mathrm{E}((X - a)(X - b)) = \mathrm{E}(X - a)\mathrm{E}(X - b)$.

4.MC2 [3 Points] Let $(X_n)_{n\geq 1}$ be an iid sequence of $\mathcal{N}(0, 1)$ random variables. Which of the following sequences converge in distribution to a $\mathcal{N}(0, 1)$ random variable as $n \to \infty$?

(A) $\frac{X_1+\cdots+X_n}{\sqrt{n}}$

(B) $\frac{X_1+\cdots+X_n}{n}$

(C) $\frac{X_1^2+\cdots+X_n^2}{\sqrt{n}}$

(D) $\frac{2}{n}\sum_{k=1}^{n} \sqrt{k}\, X_k$

(E) $\frac{X_1^2+\cdots+X_n^2-n}{\sqrt{n}}$

4.MC3 [3 Points] Let X, Y, Z be iid $\mathcal{N}(0, 1)$ random variables. Which of the following are true?

(A) $\mathrm{E}(X + Y + Z|X) = X$ a.s.

(B) $\mathrm{E}(X + Y + Z|X + Y) = X + Y$ a.s.

(C) $\mathrm{E}(X + Y + Z|\sigma(X, Y)) = X + Y$ a.s.

(D) $\mathrm{E}(X^2 + Y^2 + Z^2|Z) = Z^2$ a.s.

(E) $\mathrm{E}(X^2 + Y^2 + Z^2|Z) = Z^2 + 2$ a.s.

Results from lectures

A formula for $\mathrm{E}(X)$.

Let X be a non-negative real random variable. Then

$$\mathrm{E}(X) = \int_0^\infty \mathrm{P}(X \geq t)\, dt.$$

Doob's maximal inequality.

Let $(X_n)_{n\geq 0}$ be a sub-martingale. Then for every $n \geq 0$ and for every $a > 0$, we have

$$\mathrm{P}\Big(\max_{0\leq k\leq n} X_k \geq a\Big) \leq \frac{\mathrm{E}\Big(\max(X_n, 0)\Big)}{a}.$$