D–MATH

# Exam Probability Theory

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*Please take note of the information on the answer-booklet.*

All random variables are defined on an a fixed implicit probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

For a real random variable $X$, we write $\mathrm{E}(X)$ for its expectation, and $\phi_X$ for its characteristic function, defined by

$$\forall t \in \mathbb{R} \quad \phi_X(t) = \mathrm{E}(e^{itX}).$$

$\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$ of non-negative integers. For every $x > 0$, we write $\log(x)$ for the logarithm in base $e$ (that is, $\exp(\log(x)) = x$).

# Exercise 1

Let $(X_n)_{n\geq 1}$ be independent real random variables such that for each $n \geq 1$, $X_n$ has density given by

$$f_n(x) = \frac{n+1}{x^{n+2}} 1_{x>1}.$$

**1.1 [1 Point]** Check that for all $n \geq 1$, $f_n$ defines a density.

**1.2 [2 Points]** For $n \geq 1$, compute $\mathrm{E}(X_n)$.

**1.3 [2 Points]** Let $\varepsilon > 0$. Show that for all $n \geq 1$,

$$\mathrm{P}(|X_n - 1| > \varepsilon) = \frac{1}{(1+\varepsilon)^{n+1}}.$$

**1.4 [2 Points]** Does $(X_n)$ converge almost surely? Does $(X_n)$ converge in probability? Justify your answers.

**1.5 [1 Point]** Does $(X_n)$ converge in $L^1$? Justify your answer.

Let $(Y_n)_{n\geq 1}$ be independent real random variables such that for each $n \geq 1$, $Y_n$ has density given by

$$g_n(x) = \frac{1+\log(n+1)}{x^{2+\log(n+1)}} 1_{x>1}.$$

**1.6** **[1 Point]** Show that for all $n \geq 1$ and $a \geq 1$,

$$\mathrm{P}(Y_n \geq a) = \frac{1}{a^{1+\log(n+1)}}.$$

**1.7** **[1 Point]** Show that $Y_n \xrightarrow[n\to\infty]{\mathrm{P}} 1$.

**1.8** **[2 Points]** Show that almost surely, we have $Y_n \geq e$ for infinitely many $n \geq 1$.

**1.9** **[1 Point]** Does $(Y_n)$ converge almost surely as $n \to \infty$?

**1.10** [**2 Points**] What is $\liminf_{n\to\infty} Y_n$ almost surely?

**1.11** [**2 Points**] What is $\limsup_{n\to\infty} Y_n$ almost surely?

# Exercise 2

Let $(X_{m,n})_{m,n\geq 1}$ be independent random variables such that

$$\forall m,n \geq 1 \quad \mathrm{P}(X_{m,n}=1) = 1-\mathrm{P}(X_{m,n}=0) = \frac{1}{n}.$$

For $n \geq 1$, define

$$S_n = X_{1,n} + \cdots + X_{n,n}.$$

**2.1 [3 Points]** Let $n \geq 1$. Let $\phi_{S_n}$ be the characteristic function of $S_n$. For $t \in \mathbb{R}$, compute $\phi_{S_n}(t)$.

**2.2 [2 Points]** Let $Z \sim \mathrm{Poi}(1)$, that is,

$$\forall k \in \mathbb{N} \quad \mathrm{P}(Z=k) = \frac{e^{-1}}{k!}.$$

Let $\phi_Z$ be the characteristic function of $Z$. For $t \in \mathbb{R}$, compute $\phi_Z(t)$.

**2.3 [3 Points]** Deduce that $S_n \xrightarrow[n\to\infty]{(d)} Z$.

# Exercise 3

Let $(Z_n)_{n\geq 1}$ be iid random variables such that

$$\mathrm{P}(Z_1 = 1) = \mathrm{P}(Z_1 = -1) = 1/2.$$

Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(Z_1, \dots, Z_n)$ for $n \geq 1$. Let $X_0 = 0$ and $X_n = Z_1 + \cdots + Z_n$ for $n \geq 1$.

**3.1 [2 Points]** Show that $(X_n)_{n\geq 0}$ is a $(\mathcal{F}_n)$-martingale.

**3.2 [2 Points]** Show that $(X_n^2 - n)_{n\geq 0}$ is a $(\mathcal{F}_n)$-martingale.

Let $T : \Omega \to \mathbb{N} \cup \{+\infty\}$ be a stopping time.

**3.3 [2 Points]** Show that

$$\forall n \geq 0 \quad \mathrm{E}\Big(X_{n\wedge T}^2\Big) \leq \mathrm{E}(T).$$

**3.4 [2 Points]** Deduce that

$$\forall n \geq 0 \quad \mathrm{E}\left(\sup_{n\geq 0} X_{n\wedge T}^2\right) \leq 4\mathrm{E}(T).$$

**3.5 [2 Points]** Assume that $\mathrm{E}(T) < \infty$. Prove that $\mathrm{E}(X_T) = 0$.

**3.6** **[2 Points]** Let $S = \min\{n \geq 1 : X_n = -1\}$ (by convention, $\min \emptyset = +\infty$).
Show that $\mathrm{E}(S) = +\infty$.

**3.7** **[2 Points]** Let $S' = \min\{n \geq 1 : X_n = 0\}$. What is $\mathrm{E}(S')$?

# Exercise 4

In the following questions, mark all statements that are true (several statements can be true).

**4.MC1 [3 Points]** Let $(Z_n)_{n\geq 1}$ be iid random variables such that

$$\mathrm{P}(Z_1 = 1) = \mathrm{P}(Z_1 = -1) = 1/2.$$

Let $X \sim \mathcal{N}(0,1)$ be independent of $(Z_n)_{n\geq 1}$. Which of the following is/are true?

(A) $\frac{Z_1+\cdots+Z_n}{\sqrt{n}} \xrightarrow[n\to\infty]{(d)} X$.

(B) $\frac{Z_1+\cdots+Z_n}{\sqrt{n}} \xrightarrow[n\to\infty]{L^1} X$.

(C) $\forall t \in \mathbb{R} \quad \phi_{\frac{Z_1+\cdots+Z_n}{n}}(t) \xrightarrow[n\to\infty]{} \phi_X(t)$.

(D) $\forall t \in \mathbb{R} \quad \left(\phi_{\frac{Z_1}{\sqrt{n}}}(t)\right)^n \xrightarrow[n\to\infty]{} \phi_X(t)$.

(E) $\forall t \in \mathbb{R} \quad \phi_{\frac{Z_1+\cdots+Z_n}{\sqrt{n}}}(t) \xrightarrow[n\to\infty]{} \phi_X(t)$.

**4.MC2 [3 Points]** Let $(Z_n)_{n\geq 1}$ be iid random variables such that

$$\mathrm{P}(Z_1 = 1) = \mathrm{P}(Z_1 = -1) = 1/2.$$

Let $X_0 = 1$ and for $n \geq 0$, let

$$X_{n+1} = \begin{cases} X_n + 1 & \text{if } Z_{n+1} = +1, \\ \lfloor X_n/2 \rfloor & \text{if } Z_{n+1} = -1, \end{cases}$$

where we write $\lfloor x \rfloor$ for the integer part of $x$ (that is, the greatest integer less than or equal to $x$). Let $(\mathcal{F}_n)_{n\geq 1}$ be the filtration generated by $(Z_n)_{n\geq 1}$. Which of the following is/are true?

(A) $X_n$ is a $(\mathcal{F}_n)$-submartingale.

(B) $X_n$ is a $(\mathcal{F}_n)$-supermartingale.

(C) $X_n$ is a $(\mathcal{F}_n)$-martingale.

(D) $\limsup\limits_{n\to\infty} X_n = +\infty$ almost surely.

(E) $X_n \xrightarrow[n\to\infty]{a.s.} 0$.

**4.MC3 [3 Points]** Let $(X_n)_{n\geq 1}$ be iid $\mathcal{U}[0,1]$ random variables. Which of the following is/are true?

(A) $\lim\limits_{n\to\infty}(X_1^2+\cdots+X_n^2)/n > 1/2$ almost surely.

(B) $(X_1+\cdots+X_n)/n \xrightarrow[n\to\infty]{a.s.} 1/2$.

(C) $X_1\cdots X_n \xrightarrow[n\to\infty]{a.s.} 1/2$.

(D) There exists $r\in\mathbb{R}$ such that $(X_1\cdots X_n)^{1/n} \xrightarrow[n\to\infty]{a.s.} r$.

(E) $\left((X_1\cdots X_n)^{1/n^2}\right)_{n\geq 1}$ does not converge a.s. as $n\to\infty$.