Prof. P. Cheridito August 13, 2019

Quantitative Risk Management

Important:

- Put your student card on the table
- Begin each exercise on a new sheet of paper, and write your name on each sheet
- Only pen, paper and ten A4 sides of summary are allowed

Please fill in the following table.

Last name	
First name	
Student number (if available)	

Please do not fill in the following table.

Question	**Points**	**Control**	**Maximum**
#1			11
#2			10
#3			8
#4			11
#5			10
Total			50

Question 1 (11 Pts)

a) Let X be a random variable with cdf

$$F(x) = \frac{1}{1 + e^{-\frac{x-\mu}{\sigma}}}, \quad x \in \mathbb{R},$$

for parameters $\mu \in \mathbb{R}$ and $\sigma > 0$. Calculate $\mathrm{VaR}_\alpha(X)$ and $\mathrm{ES}_\alpha(X)$ for $\alpha \in (0,1)$. (3 Pts)

b) Denote by $L^2(\Omega, \mathcal{F}, \mathbb{P})$ the space of all square integrable random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Which axioms of coherence does the mapping $\rho \colon L^2(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, given by

$$\rho(X) := \mathrm{sd}(X) = \sqrt{\mathbb{E}\left[(X - \mathbb{E}X)^2\right]},$$

satisfy? Please, prove your statements. (4 Pts)

c) Construct a two-dimensional random vector (X_1, X_2) such that

(i) $X_i \sim \mathrm{Exp}(\lambda_i)$ for $\lambda_i > 0$, $i = 1, 2$, and

(ii) $\mathrm{VaR}_\alpha(X_1 + X_2) = \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)$ for all $\alpha \in (0,1)$. (4 Pts)

Question 2 (10 Pts)

a) Let X be a d-dimensional random vector with a $t_d(\nu, 0, \Sigma)$-distribution for $d \geq 2$, $\nu > 0$ and a positive definite $d \times d$-matrix Σ. Are the components of X exchangeable? (3 Pts)

b) Let $X \sim S_d(\psi)$ for some $d \geq 2$. Show that all univariate marginal distributions of X are equal. (3 Pts)

c) Denote by $\mathbb{S}^2_+$ the set of all positive semidefinite symmetric 2×2-matrices, and let X be a two-dimensional random vector with a $N_2(\mu, \Sigma)$-distribution for a fixed mean vector $\mu \in \mathbb{R}^2$ and a covariance matrix Σ in the set

$$S = \left\{ \Sigma \in \mathbb{S}^2_+ : \Sigma_{ii} = \sigma_i^2 \text{ for } i = 1, 2, \text{ and } \underline{\rho} \leq \frac{\Sigma_{12}}{\sigma_1 \sigma_2} \leq \overline{\rho} \right\},$$

where $\sigma_i > 0$, $i = 1, 2$, and $-1 \leq \underline{\rho} \leq \overline{\rho} \leq 1$ are given constants. The set S models correlation uncertainty between the components of X. Consider a vector $w \in \mathbb{R}^2_+$ and a probability level $\alpha \in (1/2, 1)$. Compute the worst-case value-at-risk

$$\sup_{\Sigma \in S} \mathrm{VaR}_\alpha\left(-\sum_{i=1}^{2} w_i X_i\right)$$

of the portfolio loss $-\sum_{i=1}^{2} w_i X_i$. (4 Pts)

Question 3 (8 Pts)

a) Compute the lower tail dependence coefficient λ_l of the two-dimensional copulas $W(u,v) = (u + v - 1)^+$ and $M(u,v) = \min\{u, v\}$. (3 Pts)

b) Let (X_1, X_2) be a two-dimensional random vector with joint distribution given by the cdf

$$F(x_1, x_2) = \exp\left(-(-x_1 - x_2)^{1/\beta}\right) \quad \text{for } x_1, x_2 \leq 0 \quad \text{and} \quad \beta \geq 1.$$

Calculate the marginal distributions and the copula of (X_1, X_2). (5 Pts)

Question 4 (11 points)

Let X be a random variable with cdf

$$F(x) = 1 - x^{-\alpha}, \quad x \geq 1,$$

for a parameter $\alpha > 0$.

a) Does X have a density? If yes, derive it. (1 Pts)

b) Find all $k = 1, 2, \ldots$ such that $\mathbb{E}\left[|X|^k\right] < \infty$. (2 Pts)

c) Does F belong to $\mathrm{MDA}(H_\xi)$ for a generalized extreme value distribution H_ξ? If yes, what is ξ and what are the normalizing sequences? (3 Pts)

d) Calculate the excess distribution function $F_u(x) = \mathbb{P}\left[X - u \leq x \mid X > u\right]$, $x \geq 0$. (2 Pts)

e) Does there exist a parameter $\xi \in \mathbb{R}$ and a positive measurable function β such that

$$\lim_{u \to \infty} \sup_{x > 0} \left| F_u(x) - G_{\xi, \beta(u)}(x) \right| = 0,$$

where $G_{\xi,\beta}$ is a generalized Pareto distribution? If yes, what are ξ and β? (3 Pts)

Question 5 (10 points)

a) Assuming that you can only simulate $U \sim \mathrm{Unif}(0,1)$ and $Z \sim N_d(0, I_d)$, where I_d denotes the d-dimensional identity matrix, describe an algorithm for simulating $X \sim M_d(\mu, \Sigma, \hat{F}_W)$. (4 Pts)

b) Name three methods for estimating a copula C from data. (3 Pts)

c) Explain what principal component analysis is. (3 Pts)