D–MATH

Exam Quantitative Risk Management

401-3629-00L

Last Name | *First Name* | *Legi-Nr.*

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Please take note of the information on the answer-booklet.

Question 1

(a) [**4 Points**] Consider a random variable X with distribution function F given by

$$F(x) \quad = \quad \begin{cases} 0 & ; \quad x < 0 \\ 0.5 & ; \quad 0 \leq x < 3 \\ 1 - \frac{1}{x^2}; & \quad x \geq 3. \end{cases}$$

Compute $\mathrm{VaR}_\alpha(X)$, $\mathrm{AVaR}_\alpha(X)$ and $\mathrm{ES}_\alpha(X)$ at level $\alpha = 0.8$.

(b) For a random variable L with distribution function F_L, we can define

$$\mathrm{VaR}_1(L) := \sup\{x \in \mathbb{R} : F_L(x) < 1\} \in \mathbb{R} \cup \{\infty\}.$$

That means $\mathrm{VaR}_1(L)$ corresponds to the right endpoint of F_L.

In this exercise, we are going to show that VaR_1 is a coherent risk measure on the space $\mathcal{L}$ of essentially bounded random variables, that is, $\mathcal{L} = \{L \mid \mathrm{VaR}_1(L) < \infty\}$.

(i) [**2 Points**] First show that for any random variable $L \in \mathcal{L}$ it holds that

$$\mathrm{VaR}_1(L) = \sup_{\alpha \in (0,1)} \mathrm{AVaR}_\alpha(L). \tag{1}$$

Hint: You may use without proof that for any $L \in \mathcal{L}$, the quantile function $(0, 1] \ni \alpha \to \mathrm{VaR}_\alpha(L)$ is left-continuous and increasing.

(ii) [**4 Points**] Use the representation (247) to prove that VaR_1 is a coherent risk measure on $\mathcal{L}$.

Hint: You may use any properties of AVaR_α established in the lecture.

(iii) [**1 Point**] Briefly discuss the adequacy of VaR_1 as a risk measure in quantitative risk management and regulation.

Question 2

(a) (i) **[1 Point]** Provide the definition of an ARCH(p) process.

(ii) **[2 Points]** Is an ARCH(p) process a white noise process? Justify your answer.

(iii) **[1 Point]** Is an ARCH(p) process a *strict* white noise process? Justify your answer.

(iv) **[1 Point]** Discuss the adequacy of a strict white noise process to model financial log-returns, referring to the relevant stylized facts of financial log-returns.

(b) Suppose you have a portfolio with d assets $X_1, \dots, X_d$, which all have variance 1.

(i) **[1 Point]** What does it mean that the vector $(X_1, \dots, X_d)$ has an exchangeable distribution?

(ii) **[1 Point]** Show that exchangeability implies that all pairs (X_i, X_j), $1 \le i < j \le d$, have the same Pearson correlation $\rho \in [-1, 1]$.

(iii) **[2 Points]** Show that necessarily $\rho \ge -1/(d-1)$.

(iv) **[3 Points]** Show that for $\rho = -1/(d-1)$, the distribution of $(X_1, \dots, X_d)$ cannot have a joint density.

Hint: Start by calculating the variance $\mathrm{Var}(X_1 + X_2 + \cdots + X_d)$.

Question 3

(a) **[3 Points]** Let $X_1, X_2, \dots$ be iid random variables with distribution F. Let $M_n = \max\{X_1, \dots, X_n\}$ and suppose that $x_F = \sup\{x \in \mathbb{R} : F(x) < 1\} < \infty$.

Show that M_n converges in distribution to x_F.

(b) Let F be the distribution function of a uniform distribution on $[0, 1]$.

(i) **[2 Points]** Does F belong to the maximum domain of attraction, $\mathrm{MDA}(H_\xi)$, of a standard GEV distribution H_ξ? If yes, determine the parameter ξ.

(ii) **[1 Point]** Calculate the excess distribution function $F_u(x) = \mathbb{P}[X - u \leq x \mid X > u]$, $0 \leq u < x_F$, $x \in [0, x_F - u)$.

(iii) **[2 Points]** Does there exist a parameter $\xi \in \mathbb{R}$ and a function $\beta\colon \mathbb{R} \to (0, \infty)$ such that

$$\lim_{u \uparrow x_F} \sup_{x > 0} |F_u(x) - G_{\xi, \beta(u)}(x)| = 0,$$

where $G_{\xi,\beta}$ denotes the cumulative distribution function of a GPD? If yes, for which ξ and β does this hold?

Question 4

(a) Consider the random vector $(Z, |Z|)$, where Z follows a standard normal distribution.

(i) **[2 Points]** Sketch a scatterplot of a random sample from the distribution of $(Z, |Z|)$ and also a scatterplot of a random sample from the underlying copula of $(Z, |Z|)$.

(ii) **[2 Points]** Compute the Pearson correlation of Z and $|Z|$. Are Z and $|Z|$ independent? Provide an argument for your claim.

(iii) **[1 Point]** How would the scatterplot of a random sample from the copula change if we considered (Z, Z^2)? Provide a brief argument for your claim.

(b) Suppose you have iid observations $(X_1, Y_1), \ldots, (X_n, Y_n)$ of a two-dimensional random vector (X, Y).

(i) **[1 Point]** Provide a formula for the sample version of Kendall's tau, $r_\tau(n)$.

(ii) **[1 Point]** Suppose that $n = 4$ and your random sample is

$$\{(-1, 3), (0, 0), (2, 2), (3, 1)\}. \tag{2}$$

Compute $r_\tau(n)$ on this sample.

(iii) **[2 Points]** Suppose you have a new data point (X_{n+1}, Y_{n+1}). Describe a condition under which this new data point influences the sample version of Kendall's tau maximally. Provide an appropriate example for the concrete sample in (248).

(iv) **[1 Point]** What is the maximal influence of a new data point (X_{n+1}, Y_{n+1}) on the sample version of Kendall's tau? That is, what is an upper bound for the difference

$$|r_\tau(n) - r_\tau(n+1)|,$$

which holds for any data set $(X_1, Y_1), \ldots, (X_n, Y_n), (X_{n+1}, Y_{n+1})$?
Decide whether it is either

$$2 \quad \text{or} \quad \frac{4}{n+1} \quad \text{or} \quad \frac{4}{(n+1)^2}.$$

Write down the solution on your submission sheet (answers on this sheet will not be counted). You do not need to provide an explanation.

(v) **[1 Point]** Compare and possibly contrast your previous result to Pearson's correlation coefficient.

Question 5

In this exercise, any scoring function $S(x, y)$, $x, y \in \mathbb{R}$, has the interpretation that $x \in \mathbb{R}$ is a forecast and $y \in \mathbb{R}$ an observation.

(a) [**1 Point**] In a comparative backtest you consider $\mathrm{VaR}_{0.95}$-forecasts from two different models, M1 and M2. You compute average scores on a validation window for the following scoring functions:

$$
\begin{aligned}
S_1(x, y) &= |x - y| \\
S_2(x, y) &= (x - y)^2 \\
S_3(x, y) &= (1_{\{y \le x\}} - 0.95)(x - y)
\end{aligned}
$$

The results are summarized in the following table.

	M1	M2
S_1	3.88	3.23
S_2	7.49	6.91
S_3	2.54	3.02

Which model would you prefer and why?

(b) [**1 Point**] Why is it important to evaluate forecasts for a distribution-based risk measure using a strictly consistent scoring function for this risk measure?

(c) [**2 Points**] Show that for any strictly convex function $\phi\colon \mathbb{R} \to \mathbb{R}$ with strictly positive second derivative, the score

$$S(x, y) = -\phi(x) + \phi'(x)(x - y), \quad x, y \in \mathbb{R},$$

is strictly consistent for the mean functional on the class of distributions with a finite mean.

(d) [**4 Points**] Consider an AR(1) process $(Y_t)_{t \in \mathbb{N}}$ of the form

$$Y_0 = 0 \quad \text{and} \quad Y_t = \theta Y_{t-1} + u_t, \quad t \ge 1,$$

where $|\theta| < 1$ and $\mathbb{E}[u_t \,|\, Y_{t-1}] = 0$.

Two competing forecasts are available, $X_t = 0$ and $X_t^* = \theta Y_{t-1}$, where $t \ge 1$.

If $S\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a strictly consistent scoring function for the mean functional, is it possible to establish an inequality between the expected scores $\mathbb{E}[S(X_t, Y_t)]$ and $\mathbb{E}[S(X_t^*, Y_t)]$? Explain your answer.

Hint: Start by computing $\mathbb{E}[Y_t]$ and $\mathbb{E}[Y_t \,|\, Y_{t-1}]$.