D–MATH

Exam Quantitative Risk Management

401-3629-00L

Last Name

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Please take note of the information on the answer-booklet.

Question 1

Consider a risk measure $\rho\colon \mathcal{L} \to \mathbb{R}$ on a vector space $\mathcal{L}$ of random variables.
Show the following:

(a) [**3 Points**] If ρ is positively homogeneous, then it is subadditive if and only if it is convex.

(b) [**2 Points**] If ρ is subadditive, then $\rho(0) \geq 0$.

(c) [**3 Points**] If ρ is subadditive and $\rho(X) \leq 0$ for all $X \in \mathcal{L}$ such that $X \leq 0$ a.s., then ρ is monotone.

(d) [**3 Points**] Let $X \sim \mathrm{Par}(\theta, \kappa)$ with cumulative distribution function

$$F(x) = 1 - \left(\frac{\kappa}{\kappa + x}\right)^{\theta}, \quad x \geq 0,$$

for parameters $\kappa > 0$ and $\theta > 1$. Calculate $\mathrm{VaR}_{\alpha}(X)$ and $\mathrm{AVaR}_{\alpha}(X)$.

Question 2

(a) [**3 Points**] Define the notion of radial symmetry of a d-dimensional random vector X. Show that if X is spherical, then X is radially symmetric around some $\mu \in \mathbb{R}^d$.

(b) [**4 Points**] Show that two d-dimensional random vectors X, Y are equal in distribution if and only if $a^T X \overset{(d)}{=} a^T Y$ for all $a \in \mathbb{R}^d$.

(c) [**3 Points**] Let a correlation matrix $C \in \mathbb{R}^{d \times d}$ be an "equicorrelation" matrix, i.e. all off-diagonal elements are equal to the same ρ. Show that $-\frac{1}{d-1} \leq \rho \leq 1$.

Question 3

(a) **[2 Points]** Let C be the copula of (X_1, X_2, X_3), where X_1 is a *positive* random variable with a continuous cumulative distribution function, $X_2 = 1/X_1$ and $X_3 = \exp(-X_1)$.

Explain why C is also the copula of $(X_1, -X_1, -X_1)$.

(b) **[4 Points]** Derive the form of copula C from 3.(a).

(c) **[4 Points]** Let (X, Y) be a two-dimensional random vector with a cumulative distribution function

$$F(x,y) = \frac{2x(1 - e^{-2y^2})}{\sqrt{4x^2(1 - e^{-2y^2}) + 1}}, \ x, y \geq 0.$$

Compute the copula of (X, Y).

Question 4

Let X be a random variable with a cumulative distribution function

$$F(x) = \begin{cases} 1 - \dfrac{1}{(3x+4)^{3/4}} & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0. \end{cases}$$

(a) [**1 Point**] Does X have a density? If yes, can you derive it?

(b) [**1 Point**] Find all $k \in \mathbb{N} = \{1, 2, \dots\}$ such that $\mathbb{E}[|X|^k] < \infty$.

(c) [**3 Points**] Does F belong to $\mathrm{MDA}(H_\xi)$ for a standard generalized extreme value distribution H_ξ? If yes, what is ξ, and what are the normalizing sequences?

(d) [**2 Points**] Calculate the excess distribution function $F_u(x) = \mathbb{P}[X - u \leq x \mid X > u]$, $x \geq 0$.

(e) [**3 Points**] Does there exist a parameter $\xi \in \mathbb{R}$ and a function β such that

$$\lim_{u \to \infty} \sup_{x > 0} |F_u(x) - G_{\xi, \beta(u)}(x)| = 0,$$

where $G_{\xi,\beta}$ denotes the cumulative distribution function of a generalized Pareto distribution? If yes, for which ξ and β does this hold?

Question 5

(a) **[2 Points]** Name advantages of Spearman's ρ compared to linear correlation.

(b) **[3 Points]** Name advantages and disadvantages of using risk measures based on loss distributions for quantitative risk management.

(c) **[4 Points]** Denote by Φ the cumulative distribution function of $N(0,1)$. Assuming that you can generate independent realizations of Unif$(0,1)$-distributed random variables, describe an algorithm for generating realizations of a bivariate random vector $X = (X_1, X_2)$ whose distribution can be described as follows:

- X_1 and X_2 have the same distribution.
- The cdf F of X_1 is given by

$$F(x) = \lambda \Phi\left(\frac{x-\mu_1}{\sigma_1}\right) + (1-\lambda)\Phi\left(\frac{x-\mu_2}{\sigma_2}\right), \quad x \in \mathbb{R}$$

  for some $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 > 0$, and $\lambda \in (0,1)$.
- The copula of X is given by a Gaussian copula $C_P \colon \mathbb{R}^2 \to [0,1]$ corresponding to a given correlation matrix $P \in \mathbb{R}^{2\times 2}$.