Prof. Dr. P. Nolin ETH Zürich Summer 2013

Probability Theory and Statistics (BSc D-ITET)

1. (10 Points) Every Saturday Thomas goes to Bellevue to take an ice–cream, either on a cone or in a cup. In his opinion, there are only two places where it is worth buying it: Mövenpick or Globus. We denote by M the event <Thomas goes to Mövenpick>, by G the event <Thomas goes to Globus>, and by ∇ the event <Thomas orders a cone>. It is known that

- the probability that Thomas goes to Globus is equal to 0.3.
- the probability Thomas goes to Mövenpick *and* orders a cone is equal to 0.5,
- the probability Thomas orders a cup, *knowing that* he went to Globus is equal to 0.2.

a) Using the above notation, answer the following questions:

i) What is the probability that Thomas orders a cone, knowing that he goes to Mövenpick?

ii) What is the probability that Thomas goes to Globus and orders a cone?

b) You meet Thomas in front of Opernhaus with his ice–cream on a cone. What is the probability that he went to Mövenpick?

In Mövenpick, they ask people to stand in two different lines: one for the fruity sorbets, one for the creamy ice–creams. It is known that, at precisely 3 PM the number of people in the line for the fruity sorbets is Poisson distributed with parameter $\lambda_s = 1$, while in the creamy ice–cream line it is Poisson distributed with parameter $\lambda_c = 3$. Under the assumption of independence between the distributions of the number of people in the two lines, the total number of people queuing in the shop is Poisson distributed with parameter $\lambda = \lambda_s + \lambda_c = 4$ (this result was proven in class and does *not* need to be proven here).

c) Thomas wants a strawberry sorbet at 3 PM and he counts 20 people in the shop (not counting himself).

i) Show that the number of people (not counting himself) in the sorbet line is Binomially distributed with parameters $n = 20$ and $p = 0.25$.

ii) What is the expected number of people queuing in the line for the sorbets before him?

Please turn the page!

2. (10 Points) Let X be a $\Gamma(3,1)$ distributed random variable, i.e. its density is given by

$$f_X(x) = \begin{cases} \frac{1}{2}x^2e^{-x}, & x > 0, \\ 0 & x \leq 0. \end{cases}$$

and let Y have a uniform distribution on the random interval $[0, X]$. This implies that the joint density $f_{X,Y}$ of (X, Y) is given by

$$f_{X,Y}(x, y) = \frac{1}{2}xe^{-x}\,1(0 < y < x).$$

This result does *not* need to be proven here.

a) Calculate the marginal density f_Y of Y.

b) Compute $E[Y]$.

HINT: You may use, without proving it, that $\int_0^\infty x^k e^{-x}dx = k!$.

c) Compute $\mathrm{Cov}(X, Y)$.

d) Determine whether X, Y are independent random variables. Justify your answer.

3. (10 Points) As a casino controller, it is your job to verify that the dice used in a casino are fair. When inspecting a casino, you select a die at random on the casino floor and check whether the face marked with "1" has probability $\frac{1}{6}$ of appearing. After randomly selecting one die, we denote by $D \in \{1, \ldots, 6\}$ the random variable that gives the result of a throw of the selected die. Additionally, we denote by p the probability of the selected die showing the face marked with "1", i.e. $p = P(D = 1) \in (0, 1)$. We repeatedly throw the selected die and we denote by D_i, $i = 1, 2, \ldots$ the result of the i-th throw. We assume the random variables D_i are independent of each other and have the same distribution as D.

a) Let T be the random variable that counts the number of throws necessary to make the face marked with "1" appear. Calculate $P(T = n)$ for all $n = 1, 2, \ldots$ (*justify your calculations*). What is the name of the distribution of T?

b) Prove that $E[T] = \frac{1}{p}$.

You repeat the experiment of part **a)** k times, and hence obtain a sample T_i for $i = 1, \ldots, k$ of independent random variables that have the same distribution as T. We want to use this sample to estimate p.

c) Propose a "natural" estimator for p.

d) Calculate the maximum-likelihood estimator of p.