Probability and Statistics (English) FS 2020 Session Exam

13.08.2020 **Time Limit: 180 Minutes**

Surname	**First name**	**Legi Number**	**Examnumber**
the first two letters of each		last 6 digits	leave blank

Enter **the first two letters of your surname and first name now**, as well as **the last six digits of your Legi number**. If you are attaching separate sheets, write the same information and **only this information clearly at the top of each sheet**.

This exam contains 10 problems and in addition statistical tables.

Grading Table (for grading use only, please leave empty)

Question:	1	2	3	4	5	6	7	8	9	10	Total
Maximal:	10	10	10	10	10	10	10	10	10	10	100
Points:											
Checkup:											

Instructions

The exam is a closed-book exam (no pocket calculators or mobile phones either).

Before the exam:

- ◇ Put your student card (Legi) on the table.
- ◇ Fill out the cover sheet as appropriate. Do not open the exam.

During the exam:

- ◇ Read the questions carefully, and try to answer as many as you can. Start each problem on a new sheet, and write the information above on each sheet. Do not use red or green ink, nor pencil.
- ◇ You may use results from the lecture or from the formulae sheet without proof, unless you are explicitly asked to give a proof. But to get full points, it must be clear from your solutions how you found your answer. A correct result alone will not always give all points.
- ◇ Try to simplify your answers as much as possible. If a numerical calculation requires a calculator, you may give your answer as a numerical expression — e.g. $\frac{\sqrt{3}}{7^2}$ can be a valid solution (but for instance $\int_0^1 \frac{\sqrt{3}}{7^2}dx$ or $\frac{\sqrt{300}}{490}$ would not give full points).

After the exam:

- ◇ Order your answer sheets in a reasonable way, and place everything inside the envelope you are given.
- ◇ After your exam sheets have been collected by a supervisor, remain seated and follow instructions.

1. (10 points) An urn contains 5 red, 5 black and 5 white balls. We randomly draw 3 balls, replacing each one directly after noting its colour.

   (a) (3 points) Describe this situation by an appropriate probability space.

   (b) (3 points) What is the probability that the 3 drawn balls are of exactly two different colours?

   (c) (4 points) What is the probability of the event in (b) if we do not replace the balls after drawing?

Solution:

(a) By using a Laplace model, $\Omega := \{r, b, w\}^3$, $\mathcal{F} := 2^\Omega$ and $\mathbb{P} : \mathcal{F} \to [0,1], A \mapsto \mathbb{P}[A] = \frac{|A|}{|\Omega|} = \frac{|A|}{27}$ define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(b) There are at least two methods to solve this question.

Method 1: We want to compute the probability p of the event

$$\begin{aligned} A &= \{2 \text{ red and } 1 \text{ white}\} \cup \{2 \text{ red and } 1 \text{ black}\} \cup \{2 \text{ white and } 1 \text{ red}\} \cup \\ &\quad \{2 \text{ white and } 1 \text{ black}\} \cup \{2 \text{ black and } 1 \text{ white}\} \cup \{2 \text{ black and } 1 \text{ red}\}. \end{aligned}$$

Since $|\{2 \text{ red and } 1 \text{ white}\}| = 3$ and $|A| = 6 \times 3$ by symmetry, we get

$$p = \frac{6 \times 3}{27} = \frac{2}{3} \approx 0.66666666.$$

Method 2: Let $p := \mathbb{P}[A]$ be the probability of the event

$$A := \{\text{the 3 balls are of exactly 2 different colours}\}.$$

Then

$$\begin{aligned} p &= 1 - \mathbb{P}[\text{the 3 balls have the same colour or all have different colours}] \\ &= 1 - \mathbb{P}[S] - \mathbb{P}[D], \end{aligned}$$

where $S = \{\text{same colour}\}$ and $D = \{\text{all different colours}\}$. We compute

$$\begin{aligned} \mathbb{P}[S] &= \mathbb{P}[\text{all red}] + \mathbb{P}[\text{all black}] + \mathbb{P}[\text{all white}] \\ &= \frac{1}{27} + \frac{1}{27} + \frac{1}{27} \\ &= \frac{3}{27} = \frac{1}{9} \approx 0.11111111; \end{aligned}$$

$$\begin{aligned} \mathbb{P}[D] &= \mathbb{P}[\{\text{one ball white, one ball red, one ball black}\}] \\ &= \frac{3!}{27} \\ &= \frac{6}{27} = \frac{2}{9} \approx 0.22222222. \end{aligned}$$

In conclusion,

$$p = 1 - \frac{1+2}{9} = \frac{2}{3} \approx 0.66666666.$$

(c) We need a new probability space for this part. First we define the set of all 15 balls as $M := \{\underbrace{1,2,3,4,5}_{\text{5 red balls}}, \underbrace{6,7,8,9,10}_{\text{5 black balls}}, \underbrace{11,12,13,14,15}_{\text{5 white balls}}\}$. Then we define $\Omega := \{W \subseteq M : |W| = 3\}$ as the set of all three-element subsets of M. Then we take $\mathcal{F} = 2^{\Omega}$ and $\mathbb{P} : \mathcal{F} \to [0,1], A \mapsto \mathbb{P}[A] = \frac{|A|}{|\Omega|} = \frac{|A|}{\binom{15}{3}}$. There are at least two methods to solve this question.

Method 1: We want to compute the probability p of the event

$$\begin{aligned} A \ &= \{\text{2 red and 1 white}\} \cup \{\text{2 red and 1 black}\} \cup \{\text{2 white and 1 red}\} \cup \\ &\quad \{\text{2 white and 1 black}\} \cup \{\text{2 black and 1 white}\} \cup \{\text{2 black and 1 red}\}. \end{aligned}$$

Since $|A| = 6 \times \binom{5}{2} \times \binom{5}{1}$ by symmetry,

$$p = \frac{6 \times \binom{5}{2} \times \binom{5}{1}}{\binom{15}{3}} = \frac{6 \times \frac{5\times 4}{2} \times 5}{\frac{15\times 14\times 13}{6}} = \frac{60}{91} \approx 0.6593.$$

Method 2: Let $p := \mathbb{P}[A]$ be the probability of the event

$$A := \{\text{the 3 balls are of exactly 2 different colours}\}.$$

Then

$$\begin{aligned} p &= 1 - \mathbb{P}[\text{the 3 balls have the same colour or all have different colours}] \\ &= 1 - \mathbb{P}[S] - \mathbb{P}[D], \end{aligned}$$

where $S = \{\text{same colour}\}$ and $D = \{\text{all different colours}\}$. Using the notation $\binom{n}{k}$ $\left(= C_n^k\right)$ for binomial coefficients, we compute

$$\begin{aligned} \mathbb{P}[S] &= \mathbb{P}[\text{all red}] + \mathbb{P}[\text{all black}] + \mathbb{P}[\text{all white}] \\ &= \frac{\binom{5}{3}}{\binom{15}{3}} + \frac{\binom{5}{3}}{\binom{15}{3}} + \frac{\binom{5}{3}}{\binom{15}{3}} \\ &= \frac{3 \times \frac{5\times 4\times 3}{3\times 2}}{\frac{15\times 14\times 13}{3\times 2}} \\ &= \frac{3 \times 4}{14 \times 13} \\ &= \frac{3 \times 2}{7 \times 13} \\ &= \frac{6}{91}; \end{aligned}$$

$$
\begin{aligned}
\mathbb{P}[D] &= \mathbb{P}[\{\text{one ball white, one ball red, one ball black}\}] \\
&= \frac{\binom{5}{1} \times \binom{5}{1} \times \binom{5}{1}}{\binom{15}{3}} \\
&= \frac{125 \times 6}{15 \times 14 \times 13} \\
&= \frac{25 \times 2}{14 \times 13} \\
&= \frac{25}{7 \times 13} \\
&= \frac{25}{91}.
\end{aligned}
$$

In conclusion,

$$p = 1 - \frac{6 + 25}{91} = \frac{60}{91} \approx 0.6593.$$

2. (10 points) An octaeder is a geometric body with 8 equal sides. It can be used as a die with 8 possible outcomes. We have two fair octaeder dice. One has on its sides the odd numbers $1, 3, 5, \dots, 15$, the other has the even numbers $2, 4, 6, \dots, 16$. We flip a coin to decide which die is rolled. If the coin shows heads, we roll the odd-numbered die, and otherwise the even-numbered one.

   (a) (3 points)
      (i) Describe this experiment by a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$, using a Laplace model.
      (ii) What is the cardinality $|\mathcal{F}|$ of $\mathcal{F}$? Give examples of events E_1, E_2, E_3, E_4 in $\mathcal{F}$ such that $\mathbb{P}[E_i] \neq \mathbb{P}[E_j]$ for all $i \neq j$.

   (b) (2 points) What is the probability that rolling the die produces a number which is divisible by 3?

   (c) (5 points)
      (i) Define random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ such that X and Y represent the outcomes of flipping the coin and of rolling the die, respectively.
      (ii) Are the random variables X and Y independent? Prove your answer.

Solution:

(a) (i) We mentally label the sides of each die by $1, 2, \dots, 8$ in increasing order of the number on each side. Then we can take $\Omega := \{0, 1\} \times \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\mathcal{F} := 2^\Omega$, $\mathbb{P} : \mathcal{F} \to [0, 1], A \mapsto \mathbb{P}[A] = \frac{|A|}{|\Omega|} = \frac{|A|}{16}$.
Alternatively one could define

$$\Omega' := \{ (0,1), (0,3), (0,5), (0,7), (0,9), (0,11), (0,13), (0,15), (1,2), (1,4), (1,6), (1,8), (1,10), (1,12), (1,14), (1,16) \}.$$

Another possibility would be to define

$$\Omega'' := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}.$$

(ii) $|\mathcal{F}| = 2^{16} = 65536$, so there are many different possibilities to choose the examples—e.g. $E_1 = \varnothing$, $E_2 = \{(0, 1)\}$, $E_3 = \{(0, 1), (0, 3), (1, 2)\}$, $E_4 = \Omega$ with $\mathbb{P}[E_1] = 0$, $\mathbb{P}[E_2] = \frac{1}{16}$, $\mathbb{P}[E_3] = \frac{3}{16}$, $\mathbb{P}[E_4] = 1$.

(b) The numbers divisible by 3 are $3, 6, 9, 12, 15$. This corresponds to $A = \{(0, 2), (1, 3), (0, 5), (1, 6), (0, 8)\}$ or $A' = \{(0, 3), (1, 6), (0, 9), (1, 12), (0, 15)\}$ or $A'' = \{3, 6, 9, 12, 15\}$. In every case, the event has cardinality 5 so that the desired probability is $\frac{5}{16}$.

(c) (i) $X : \Omega \to \mathbb{R}, \omega \mapsto X(\omega) := \omega_1$ and

$$Y : \Omega \to \mathbb{R}, \omega \mapsto Y(\omega) := \begin{cases} 2\omega_2 - 1, & \omega_1 = 0, \\ 2\omega_2, & \omega_1 = 1. \end{cases}$$

An alternative way to write this is $Y(\omega) := (2\omega_2 - 1) I_{\{\omega_1 = 0\}} + (2\omega_2) I_{\{\omega_1 = 1\}}$ or $Y(\omega) := 2\omega_2 - 1 + \omega_1$.

On Ω', Y can be defined more simply as $Y(\omega') := \omega'_2$.
On Ω'', Y can be defined more simply as $Y(\omega'') := \omega''$, but X would need to be defined via a case analysis.

(ii) X and Y are dependent. To show this, it is sufficient to find one example of $x, y \in \mathbb{R}$ such that $\mathbb{P}[X = x, Y = y] \neq \mathbb{P}[X = x] \mathbb{P}[Y = y]$. A simple choice is $x = 0$ and $y = 2$. First we compute the left-hand side as

$$\mathbb{P}[X = 0, Y = 2] = \frac{|\{\omega \in \Omega : X(\omega) = 0 \text{ and } Y(\omega) = 2\}|}{16} = 0.$$

Now, we consider the right-hand side and first calculate the probability

$$\mathbb{P}[X = 0] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = 0\}] = \frac{|\{\omega \in \Omega : X(\omega) = 0\}|}{16} = \frac{8}{16} = \frac{1}{2}.$$

Analogously, we get

$$\mathbb{P}[Y = 2] = \mathbb{P}[\{\omega \in \Omega : Y(\omega) = 2\}] = \frac{|\{\omega \in \Omega : Y(\omega) = 2\}|}{16} = \frac{1}{16}.$$

So we can conclude that

$$\mathbb{P}[X = 0, Y = 2] = 0 \neq \frac{1}{32} = \mathbb{P}[X = 0] \mathbb{P}[Y = 2].$$

3. (10 points) A car towing company services a 200 mile stretch of a highway. The company is located b miles inside from one end of the stretch. Breakdowns of cars occur uniformly along the highway, and the towing trucks travel at a constant speed of 40mph. We assume that when the towing company is called, a truck starts immediately to travel from the company's location to the place of the breakdown.

   (a) (3 points) Compute (as a function of b) the probability that the waiting time for a towing truck exceeds 30 minutes. Which numerical value do you get for $b = 50$?

   (b) (2 points) Find the mean of the waiting time for a towing truck, again as a function of b. Which numerical value do you get for $b = 50$?

   (c) (5 points) The towing company would like to announce that their waiting time has a small standard deviation. What is then their best choice of location, and what is the numerical value of the resulting standard deviation?

Solution: Because the distribution of breakdowns is symmetric on $[0, 200]$, we can assume without loss of generality that b is measured from the left endpoint of the stretch and that this latter is at 0. Let X be the number of miles from the left endpoint that a breakdown occurs. Then $X \sim \mathcal{U}(0, 200)$, and the distance Y of the breakdown from the location of the towing company is $Y = |X - b|$. It will take the truck $Z = \frac{Y}{40} = \frac{|X-b|}{40}$ hours to reach the location of the breakdown.

(a)

$$
\begin{aligned}
\mathbb{P}\left[Z > \frac{1}{2}\right] &= \mathbb{P}\left[\frac{|X-b|}{40} > \frac{1}{2}\right] \\
&= \mathbb{P}\left[|X-b| > 20\right] \\
&= 1 - \mathbb{P}\left[|X-b| \leq 20\right] \\
&= 1 - \mathbb{P}\left[X \in [b-20, b+20]\right] \\
&= \begin{cases} 1 - \frac{40}{200}, & b \in [20, 180] \\ 1 - \frac{b+20}{200}, & b \in [0, 20] \\ 1 - \frac{200-(b-20)}{200}, & b \in [180, 200] \end{cases} \\
&= \begin{cases} \frac{4}{5}, & b \in [20, 180] \\ \frac{180-b}{200}, & b \in [0, 20] \\ \frac{b-20}{200}, & b \in [180, 200]. \end{cases}
\end{aligned}
$$

Setting $b = 50$ yields

$$
\mathbb{P}\left[Z > \frac{1}{2}\right] = 1 - \frac{40}{200} = \frac{4}{5} = 0.8.
$$

(b) We want the mean of Z. First,

$$\begin{aligned}
\mathbb{E}[Z] &= \mathbb{E}\left[\frac{|X-b|}{40}\right] = \frac{1}{40}\int_{-\infty}^{\infty}|x-b|f_X(x)dx = \frac{1}{40\cdot 200}\int_0^{200}|x-b|dx \\
&= \frac{1}{8000}\left(\int_0^b (b-x)dx + \int_b^{200}(x-b)dx\right) \\
&= \frac{1}{8000}\left((bx-\frac{x^2}{2})\Big|_{x=0}^{b} + (\frac{x^2}{2}-bx)\Big|_{x=b}^{200}\right) \\
&= \frac{1}{8000}\left(b^2-\frac{b^2}{2}+\frac{200^2}{2}-200b-\frac{b^2}{2}+b^2\right) \\
&= \frac{1}{8000}\left(b^2+\frac{200^2}{2}-200b\right) \\
&= \frac{1}{8000}\left(b^2-200b+20\,000\right) \text{ (hours)} \\
&= \frac{b^2}{8000}-\frac{b}{40}+\frac{5}{2} \text{ (hours)}.
\end{aligned}$$

This can also be written as

$$\mathbb{E}[Z] = \frac{1}{8000}(b-100)^2+\frac{5}{4} \text{ (hours)}.$$

Setting $b=50$ yields

$$\begin{aligned}
\mathbb{E}[Z] &= \frac{50^2}{40\cdot 200}-\frac{50}{4}+\frac{5}{2} \\
&= \frac{25}{80}-\frac{100}{80}+\frac{200}{80} \\
&= \frac{125}{80} = \frac{25}{16} = 1.5625 \text{ (hours)}.
\end{aligned}$$

(c) For the variance $\mathrm{Var}[Z] = \mathbb{E}[Z^2]-\mathbb{E}[Z]^2$, we need to compute the second moment first:

$$\mathbb{E}\left[Z^2\right] = \frac{1}{40^2}\mathbb{E}\left[(X-b)^2\right] = \frac{1}{40^2}\mathbb{E}\left[X^2 - 2Xb + b^2\right]$$

$$\begin{aligned}
&= \frac{1}{40^2}\frac{1}{200}\int_0^{200}(x^2 - 2bx + b^2)dx \\
&= \frac{1}{200\cdot 40^2}\left(\frac{x^3}{3} - \frac{2bx^2}{2} + b^2x\right)\Big|_{x=0}^{200} \\
&= \frac{1}{200\cdot 40^2}\left(\frac{200^3}{3} - 200^2 b + 200b^2\right) \\
&= \frac{1}{40^2}\left(\frac{200^2}{3} - 200b + b^2\right) \\
&= \frac{5^2}{3} - \frac{5}{40}b + \frac{1}{4^2 10^2}b^2 \\
&= \frac{25}{3} - \frac{1}{8}b + \frac{b^2}{1600} \text{ (hours}^2\text{)}
\end{aligned}$$

(this can also be written as $\mathbb{E}\left[Z^2\right] = \frac{1}{1600}(b-100)^2 + \frac{25}{12}$)
and therefore

$$\operatorname{Var}[Z] = \mathbb{E}\left[Z^2\right] - \mathbb{E}[Z]^2 = \frac{25}{3} - \frac{1}{8}b + \frac{b^2}{1600} - \left(\frac{b^2}{8000} - \frac{b}{40} + \frac{5}{2}\right)^2 \text{ (hours}^2\text{)}.$$

Since $\sqrt{\cdot}$ is a monotonous function on $[0,\infty)$, we can minimize $\operatorname{Var}[X]$ instead of $\sqrt{\operatorname{Var}[X]}$.
Setting the derivative

$$\begin{aligned}
\frac{d}{db}\operatorname{Var}[Z] &= \frac{d}{db}\left(\frac{25}{3} - \frac{1}{8}b + \frac{b^2}{1600} - \left(\frac{b^2}{8000} - \frac{b}{40} + \frac{5}{2}\right)^2\right) \\
&= -\frac{1}{8} + \frac{b}{800} - 2\left(\frac{b^2}{8000} - \frac{b}{40} + \frac{5}{2}\right)\left(\frac{b}{4000} - \frac{1}{40}\right) \\
&= -\frac{1}{8} + \frac{b}{800} - \left(\frac{b^2}{4000} - \frac{b}{20} + 5\right)\left(\frac{b}{4000} - \frac{1}{40}\right) \\
&= -\frac{1}{8} + \frac{b}{800} - \frac{b^3}{4000^2} + \frac{b^2}{20\cdot 4000} - \frac{b}{800} + \frac{b^2}{40\cdot 4000} - \frac{b}{800} + \frac{1}{8} \\
&= -\frac{b^3}{4000^2} + \frac{3b^2}{40\cdot 4000} - \frac{b}{800} \\
&= -\frac{b}{4000^2}\left(b^2 - 300b + 20\,000\right) \\
&= -\frac{b}{4000^2}(b-100)(b-200)
\end{aligned}$$

to zero leads either directly to the three solutions $b_1 = 0$, $b_2 = 100$, $b_3 = 200$ or (beside the solution $b_1 = 0$) to the solutions of the quadratic equation

$$b^2 - 300b + 20000 = 0.$$

We can use the formula for solving quadratic equations to get

$$\begin{aligned}
b_{2,3} &= 150 \pm \sqrt{150^2 - 20\,000}, \\
b_{2,3} &= 150 \pm \sqrt{22\,500 - 20\,000}, \\
b_{2,3} &= 150 \pm \sqrt{2500}, \\
b_{2,3} &= 150 \pm 50.
\end{aligned}$$

So the optimum $b^* \in \{0, 100, 200\}$ is one of those values.

By symmetry, $b = 0$ and $b = 200$ must lead to the same values. We compute for $b = 0$ that $\mathbb{E}[Z] = \frac{5}{2}$, $\mathbb{E}[Z^2] = \frac{25}{3}$ so that $\mathrm{Var}[Z] = 25\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{25}{12}$. On the other hand, $b = 100$ gives

$$\begin{aligned}
\mathbb{E}[Z] &= \frac{10\,000}{8000} - \frac{100}{40} + \frac{5}{2} = \frac{5}{4} - \frac{5}{2} + \frac{5}{2} = \frac{5}{4} \quad \text{and} \\
\mathbb{E}\left[Z^2\right] &= \frac{25}{3} - \frac{100}{8} + \frac{10\,000}{1600} = \frac{25}{3} - \frac{50}{8} = \frac{200 - 150}{24} = \frac{25}{12},
\end{aligned}$$

leading to $\mathrm{Var}[Z] = 25\left(\frac{1}{12} - \frac{1}{16}\right) = \frac{25}{48} < \frac{25}{12}$.

So the optimal position for the towing company is (not surprisingly) $b^* = 100$ (miles), i.e. in the middle of the stretch. The resulting standard deviation of the waiting time is then $\sqrt{\frac{25}{48}} = \frac{5}{4\sqrt{3}}$ (≈ 0.72 hours ≈ 43 minutes).

4. (10 points) To assess the reliability of a disease test procedure, we consider a group of people. We know that an individual chosen at random has the disease with a probability of $\frac{1}{3}$. The test procedure looks for certain markers in the blood, and it is found that the probability of a true positive (meaning that in an ill patient, markers are detected by the procedure) is $\frac{4}{5}$. It is also found that the probability of a false positive (meaning that markers are found in the blood of a healthy patient) is $\frac{3}{10}$.

   (a) (1 point) For two arbitrary events A, B, state the general definition of the conditional probability $\mathbb{P}[A|B]$.

   (b) (4 points) Compute the probability of the event C that a patient is ill and the test is positive.

   (c) (2 points) Compute the probability that the test outcome is positive.

   (d) (3 points) Given that markers were found in the blood (so that the test result is positive), what is the probability that the patient is ill?

Solution:

(a) For $\mathbb{P}[B] \neq 0$ the conditional probability is defined as

$$\mathbb{P}[A|B] := \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

For $\mathbb{P}[B] = 0$, $\mathbb{P}[A|B]$ is in general not defined.

(b) Let us first define the events

- D :={The patient has the disease}
- T :={The result of the test is positive}

By the problem formulation, we know that $\mathbb{P}[D] = \frac{1}{3}$, $\mathbb{P}[T|D] = \frac{4}{5}$ and $\mathbb{P}[T|D^c] = \frac{3}{10}$. So we obtain

$$\mathbb{P}[C] = \mathbb{P}[D \cap T] = \mathbb{P}[T|D]\,\mathbb{P}[D] = \frac{4}{5} \cdot \frac{1}{3} = \frac{4}{15}$$

(c) The law of total probability (or $\mathbb{P}[T] = \mathbb{P}[T \cap D] + \mathbb{P}[T \cap D^c]$) gives

$$\mathbb{P}[T] = \mathbb{P}[T|D]\,\mathbb{P}[D] + \mathbb{P}[T|D^c]\,\mathbb{P}[D^c] = \frac{4}{5} \cdot \frac{1}{3} + \frac{3}{10} \cdot \frac{2}{3} = \frac{7}{15}.$$

(d) We want to compute the probability $\mathbb{P}[D|T]$. For this, we use the definition of the conditional probability (or Bayes' theorem). We get

$$\begin{aligned}\mathbb{P}[D|T] &= \frac{\mathbb{P}[D \cap T]}{\mathbb{P}[T]} = \frac{\mathbb{P}[T|D]\,\mathbb{P}[D]}{\mathbb{P}[T|D]\,\mathbb{P}[D] + \mathbb{P}[T|D^c]\,\mathbb{P}[D^c]} \\ &= \frac{4}{15} \cdot \frac{15}{7} \\ &= \frac{4}{7}.\end{aligned}$$

5. (10 points) Each week, Frank and John spend the random amounts of time X (for Frank) and Y (for John) on their exercise sheets in probability and statistics. The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \frac{1}{y} I_{(0,1]}(y) I_{(0,y]}(x).$$

(a) (3 points) Find the marginal densities of X and Y.

(b) (1 point) Are X and Y independent? Argue your answer.

(c) (4 points) Compute the average amounts of time $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ spent by Frank and John. Who of the two is quicker on average?

(d) (2 points) Compute the covariance between X and Y.

Solution:

(a) Die Randdichte von X berechnet sich als Integral über die gemeinsame Dichte, also

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \\ &= \int_0^1 \frac{1}{y} 1_{(0,y]}(x)\,dy = I_{(0,1]}(x) \int_x^1 \frac{1}{y}\,dy \\ &= -I_{(0,1]}(x) \log x. \end{aligned}$$

Die Randdichte von Y berechnet sich als Integral über die gemeinsame Dichte, also

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \\ &= I_{(0,1]}(y) \int_0^y \frac{1}{y}\,dx = I_{(0,1]}(y). \end{aligned}$$

(b) Nein, sie sind abhängig, weil die gemeinsame Dichte ungleich dem Produkt der Randdichten ist.

Alternativ kann man auch die Abhängigkeit über (d) beweisen, da (d) zeigt, dass $\mathrm{Cov}(X,Y) \neq 0$.

(c) Nach (a) ist $Y \sim \mathcal{U}(0,1)$ und somit ist $\mathbb{E}[Y] = \frac{1}{2}$.

Alternative Lösung:

$$
\begin{aligned}
\mathbb{E}[Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\frac{1}{y} I_{(0,1]}(y) I_{(0,y]}(x)\,dx\,dy \\
&= \int_{-\infty}^{\infty} I_{(0,1]}(y)\int_{0}^{y} dx\,dy \\
&= \int_{-\infty}^{\infty} I_{(0,1]}(y) y\,dy \\
&= \int_{0}^{1} y\,dy \\
&= \frac{1^2}{2} - 0 = \frac{1}{2}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\frac{1}{y} I_{(0,1]}(y) I_{(0,y]}(x)\,dx\,dy \\
&= \int_{-\infty}^{\infty} \frac{1}{y} I_{(0,1]}(y)\int_{0}^{y} x\,dx\,dy \\
&= \int_{-\infty}^{\infty} \frac{1}{y} I_{(0,1]}(y)\frac{y^2}{2}\,dy \\
&= \int_{0}^{1} \frac{y}{2}\,dy \\
&= \frac{1^2}{2\cdot 2} - 0 = \frac{1}{4}.
\end{aligned}
$$

Alternative Lösung: Mit partieller Integration von $xf_X(x)$ folgt

$$
\begin{aligned}
\mathbb{E}[X] &= -\int_{0}^{1} x\log x\,dx \\
&= -\frac{x^2}{2}\log x\Big|_0^1 + \int_0^1 \frac{x^2}{2}\frac{1}{x}\,dx = 0 + \int_0^1 \frac{x^2}{2}\frac{1}{x}\,dx \\
&= \frac{x^2}{4}\Big|_0^1 = \frac{1}{4}.
\end{aligned}
$$

Bemerkung:

$$
\begin{aligned}
-\frac{x^2}{2}\log x\Big|_0^1 &= -\frac{1}{2}\underbrace{\log 1}_{=0} + \lim_{x\to 0}\frac{x^2}{2}\log x \\
&= \lim_{x\to 0}\frac{\log x}{2x^{-2}} \qquad \left(\text{unbestimmter Ausdruck des Typs } \frac{\infty}{\infty}\right) \\
&= \lim_{x\to 0}\frac{1/x}{-4x^{-3}} = \lim_{x\to 0} -x^2/4 = 0,
\end{aligned}
$$

wobei wir die l'Hôpitalsche Regel benutzt haben.

Somit ist Frank (X) schneller als John (Y), weil $\mathbb{E}[X] = \mathbb{E}[Y]/2$.

(d) Man hat

$$\begin{aligned}
\mathbb{E}[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy \\
&= \int_0^1 \int_0^y x\,dx\,dy \\
&= \int_0^1 \frac{y^2}{2}\,dy = \frac{1}{6},
\end{aligned}$$

$$\operatorname{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \frac{1}{6} - \frac{1}{2}\frac{1}{4} = \frac{1}{24}(\approx 0.0417).$$

6. (10 points) In a large manufactured lot, the fraction of defective items is 10%. We take a sample of n items and check them for defects, calling D_n the number of defective items in that sample.

   (a) (4 points) Give a probabilistic model to describe this situation. What is the distribution of D_n, and what are your assumptions?

   (b) (6 points) A customer tells us that the maximal proportion of defective items he will accept in his order of n items is 13%. How large must we choose n at least to make sure that we can satisfy this customer with an approximate probability of 99% under your assumptions in (a)?

   Hint: $\sqrt{5.41} \approx 2.3259$

Solution:

(a) Let $X_i = \begin{cases} 1 & \text{if the } i\text{th chosen item is defective,} \\ 0 & \text{otherwise.} \end{cases}$

Then $D_n = \sum_{i=1}^n X_i$, and if we assume that the X_i are i.i.d., $D_n \sim \text{Bin}(n,p)$ with $p = \mathbb{E}[X_i] = \mathbb{P}[X_i = 1] = 0.1$.

(b) Now $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i =$ sample proportion of defective items $= \frac{1}{n} D_n$.

The question is to find the minimal n such that

$$\mathbb{P}\left[\bar{X}_n < 0.13\right] \geq 0.99$$

approximately.

Here $\mathbb{E}[X_i] = \mu = 0.1$ and $\text{Var}[X_i] = (0.1)(1-0.1) = 0.09 = \sigma^2$, since the X_i are Bernoulli random variables.

By the central limit theorem,

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \overset{\text{approx}}{\sim} \mathcal{N}(0,1)$$

for large n.

Therefore, with $Z \sim \mathcal{N}(0,1)$,

$$\begin{aligned} \mathbb{P}\left[\bar{X}_n < 0.13\right] &= \mathbb{P}\left[\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} < \frac{(0.13-0.1)\sqrt{n}}{\sqrt{0.09}}\right] \\ &\approx \mathbb{P}\left[Z < \frac{0.03}{0.3}\sqrt{n}\right] \\ &= \mathbb{P}\left[Z < \frac{\sqrt{n}}{10}\right] = \Phi\left(\frac{\sqrt{n}}{10}\right), \end{aligned}$$

and so we require that $\Phi\left(\frac{\sqrt{n}}{10}\right) \geq 0.99 = \Phi(2.326)$. This is equivalent to $\frac{n}{100} \geq \underbrace{= 2.326^2}_{(\Phi^{\leftarrow}(0.99))^2}$, i.e., $n \geq \underbrace{= 2.326^2}_{(\Phi^{\leftarrow}(0.99))^2} \cdot 100 \approx 541.02$.

Therefore the smallest n we require is $\lceil \underbrace{= 2.326^2}_{(\Phi^{\leftarrow}(0.99))^2} \cdot 100 \rceil = 542$.

7. (10 points) Consider i.i.d. random variables $X_1, X_2, \dots$ that all have a uniform distribution on $[a, b]$ with $a \cancel{<} \cancel{0} < b$.

   (a) (4 points) Find the maximum likelihood estimator T_n for a on the basis of a sample $(X_1, \dots, X_n)$ of size n.

   Hint: The likelihood function is not differentiable.

   (b) (6 points) On the basis of the same sample $(X_1, \dots, X_n)$, we estimate b by the estimator $U_n := c_n \max_{i=1,\dots,n} X_i$ for some constant c_n. How must c_n be chosen to ensure that U_n is unbiased when $a = -b$?

Solution:

(a) Wir behandeln den allgemeinen Fall der uniformen Verteilung auf $[a, b]$ und bestimmen den Maximum-Likelihood-Schätzer für die Parameter a und b. Die gemeinsame Dichte von $X_1, \dots, X_n$ ist gleich

$$f^{a,b}_{X_1,\dots,X_n}(x_1, \dots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) = \frac{1}{(b-a)^n} \prod_{i=1}^{n} 1_{[a,b]}(x_i). \tag{1}$$

Nun muss $f^{a,b}_{X_1,\dots,X_n}(x_1, \dots, x_n)$ für feste $(x_1, \dots, x_n)$ bezüglich a und b maximiert werden. Seien $x_* := \min_{1 \le i \le n} x_i$ und $x^* := \max_{1 \le i \le n} x_i$. Falls x_* oder x^* ausserhalb $[a, b]$ liegen, dann verschwindet die rechte Seite von (1). Also muss die Maximumstelle $(\hat{a}_{\mathrm{ML}}, \hat{b}_{\mathrm{ML}})$ die Bedingungen $\hat{b}_{\mathrm{ML}} \ge x^* \ge x_* \ge \hat{a}_{\mathrm{ML}}$ erfüllen. Für $\hat{a}_{\mathrm{ML}} := x_* = \min_{1 \le i \le n} x_i$ und $\hat{b}_{\mathrm{ML}} := x^* = \max_{1 \le i \le n} x_i$ ist $f^{a,b}_{X_1,\dots,X_n} \le f^{\hat{a}_{\mathrm{ML}},\hat{b}_{\mathrm{ML}}}_{X_1,\dots,X_n}$ für alle a, b. Somit liefert das die Maximum-Likelihood-Schätzungen für a und b.

Der Maximum-Likelihood-Schätzer für a ist somit gegeben durch $T_n := X_* := \min(X_1, \dots, X_n)$.

(b) Die Zufallsvariable U_n wäre für $c_n \le 1$ nach oben beschränkt durch b; falls also die Varianz nicht verschwindet, müsste dann der Erwartungswert von U_n echt kleiner als b sein. Somit wäre der Schätzer nicht erwartungstreu für $c_n \le 1$.

Die Verteilungsfunktion von U_n ist wegen der i.i.d.-Voraussetzung

$$\begin{aligned} F_{U_n}(t) &= \mathbb{P}\left[c_n X^* \le t\right] = \mathbb{P}\left[X_i \le \frac{t}{c_n}, \quad i = 1, \dots, n\right] \\ &= \left(\frac{\frac{t}{c_n} - a}{b-a}\right)^n = \left(\frac{t - c_n a}{c_n(b-a)}\right)^n \quad \text{für } \frac{t}{c_n} \in [a, b]. \end{aligned}$$

Durch Ableiten erhält man die Dichtefunktion $n \frac{1}{(c_n(b-a))^n}(t - c_n a)^{n-1} 1_{[c_n a, c_n b]}(t)$.

Damit U_n erwartungstreu wird, muss gelten

$$\begin{aligned}
b = \mathbb{E}[U_n] &= \int_{c_n a}^{c_n b} tn \frac{1}{(c_n(b-a))^n}(t - c_n a)^{n-1}\,dt \\
&= n\frac{1}{(c_n(b-a))^n}\int_0^{c_n(b-a)} (s + c_n a)s^{n-1}\,ds \\
&= n\frac{1}{(c_n(b-a))^n}\int_0^{c_n(b-a)} (s^n + c_n a s^{n-1})\,ds \\
&= n\frac{1}{(c_n(b-a))^n}\left(\frac{1}{n+1}s^{n+1} + \frac{c_n a}{n}s^n\right)\Big|_{s=0}^{c_n(b-a)} \\
&= n\frac{1}{(c_n(b-a))^n}\left(s^n\left(\frac{1}{n+1}s + \frac{c_n a}{n}\right)\right)\Big|_{s=0}^{c_n(b-a)} \\
&= n\frac{(c_n(b-a))^n}{(c_n(b-a))^n}\left(\frac{c_n(b-a)}{n+1} + \frac{c_n a}{n}\right) \\
&= \frac{n}{n+1}c_n(b-a) + c_n a \\
&= c_n\left(\frac{n}{n+1}b + \frac{1}{n+1}a\right) = \frac{c_n}{n+1}(nb + a)
\end{aligned}$$

Für $a = -b$ liefert das also die Bedingung $b = c_n \frac{n-1}{n+1} b$ und damit $c_n = \frac{n+1}{n-1}$.

8. (10 points) In a sleep laboratory, two different types of sleeping pills were tested. 11 persons participated in the study; each of them tried to sleep one night with sleeping pill I and another night with sleeping pill II. The resulting hours of sleep per night were

person	1	2	3	4	5	6	7	8	9	10	11
sleeping pill I	3.8	5.5	5.6	6.3	6.9	7.1	7.4	7.8	7.9	8.0	8.1
sleeping pill II	3.0	4.2	6.4	4.5	4.6	4.6	7.0	6.7	7.3	6.7	6.9

Before this study, it was not clear which sleeping pill has the stronger effect.

(a) (1 point) Are we dealing with a paired or an unpaired sample? Argue your answer.

(b) (3 points) Which test would be appropriate if you do not want to assume a specific distribution? What are its assumptions?

(c) (6 points) Formulate the null hypothesis H_0 for this situation. Can you reject H_0 at the significance level $\alpha = 0.05$ by performing the test from (b)?

Solution:

(a) This is a paired sample—the connection between the data are the persons, which does give natural pairs.

(b) Let us call X_i and Y_i the hours of sleep for person i with pills I and II, respectively. The sign test does not assume a specific distribution for X_i and Y_i. The assumption for this test is that all differences $X_1-Y_1, \ldots, X_n-Y_n$ are i.i.d. with an arbitrary (unknown) continuous distribution F with $F(0) = \frac{1}{2}$.

(c) The null hypothesis is that both sleeping pills have the same effect, which translates into $F(0) = \frac{1}{2}$. Under H_0, the test statistic

$$T := \sum_{i=1}^{n} I_{\{X_i - Y_i > 0\}}$$

follows a $\mathrm{Bin}(n, p = \frac{1}{2})$-distribution. For the given data, we obtain $T(\omega) = 10$ by counting the number of persons for whom sleeping pill I was more effective than sleeping pill II. The realized P-value

$$\begin{aligned}
\pi(\mathbf{X}, \mathbf{Y})(\omega) &= \mathbb{P}_{H_0}\left[\left|T - \frac{11}{2}\right| \geq 10 - \frac{11}{2}\right] \\
&= 2 \sum_{k=10}^{n} \binom{n}{k} 2^{-n} \\
&= 2 \sum_{k=10}^{11} \binom{11}{k} 2^{-11} \\
&= 2 \left(\binom{11}{11} + \binom{11}{10}\right) 2^{-11} \\
&= (1 + 11)\, 2^{-10} = \frac{12}{1024} \approx 0.012
\end{aligned}$$

is clearly smaller than 0.05. Hence we can reject H_0.

9. (10 points) We consider a sewage treatment plant located near a big river. At a point downriver from the plant, we take 16 water samples and measure their ammonium concentration (measured in $\mu gNH_4^+/\ell$). We choose the measurement times randomly and are allowed to assume that the measurement results follow a normal distribution. The average concentration in the sample is given by $\overline{x}_{16} = 204.2$, and we also know that the empirical variance in the sample is $s_{16}^2 = (9.8)^2 = 96.04$.

   The legally critical value for the concentration is 200 $\mu gNH_4^+/\ell$. We want to know whether our data indicate an exceedance of that pollution level at a significance level of 5%.

   (a) (2 points) Suppose that the standard deviation of measurements is known from earlier studies and is given by 10 $\mu gNH_4^+/\ell$. Find a suitable statistical test to test the above hypothesis. What are the model assumptions that you make for your test?

   (b) (4 points) Now perform the test you have chosen in (a). Specify explicitly your null hypothesis and alternative, the test statistic T_n, the realised value of T_n, the rejection region and the test result.

   (c) (3 points) Suppose that the actual ammonium concentration in the water is 205 $\mu gNH_4^+/\ell$. What is the probability that the test chosen above is able to detect this exceedance of the legally critical value?

   (d) (1 point) Suppose that the actual ammonium concentration in the water is 200 $\mu gNH_4^+/\ell$. What is the probability that the test chosen above will indicate an exceedance of the legally critical value?

Solution:

(a) Da die Standardabweichung der Messungen als bekannt vorausgesetzt wird, entscheiden wir uns für einen z-Test. Wir nehmen an, dass die Ammoniumkonzentrationen X_i unabhängig und identisch normalverteilt sind für $i = 1, \ldots, 16$, mit $\sigma^2 = 100$ und μ unbekannt.

(b)

Nullhypothese H_0:	X_i i.i.d. $\sim \mathcal{N}(\mu_0, \sigma^2)$ mit $\mu_0 = 200, \sigma = 10$
Alternative H_A:	X_i i.i.d. $\sim \mathcal{N}(\mu, \sigma^2)$ mit $\mu > 200, \sigma = 10$ (einseitig)
Teststatistik:	$Z = \frac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$ unter H_0
Verwerfungsbereich:	Aus der Normalverteilungstabelle: $\mathcal{K} = \{z \geq z_{1-\alpha}\} = [1.64, \infty)$. (Dies entspricht dem Verwerfungsbereich $[204.1, \infty)$ auf der Skala von $\overline{X}_n$.)
Wert der Teststatistik:	$z = \frac{\overline{x}_n - \mu_0}{\sigma/\sqrt{n}} = \frac{204.2-200}{\sigma/\sqrt{16}} = \frac{4.2}{2.5} = 1.68$
Testentscheid:	$1.68 \in \mathcal{K}$, also wird die Nullhypothese verworfen. Die Überschreitung des Grenzwertes ist auf dem 5%-Niveau signifikant.

(c) Aus (a) folgt: Die Nullhypothese kann verworfen werden, falls der Mittelwert aller Messungen grösser als 204.1 ist, also falls

$$\overline{X}_n > 204.1.$$

Um die Wahrscheinlichkeit zu berechnen, dass eine Grenzwertüberschreitung nachgewiesen werden kann (H_0 verworfen werden kann), geht man wieder zu einer standardisierten normalverteilten Zufallsvariable über. Mit $\mu_A = 205$ und $\sigma = 10$ erhält man

$$\begin{aligned}
\mathbb{P}_{\mu_A}[\overline{X}_n > 204.1] &= \mathbb{P}_{\mu_A}\left[\frac{\overline{X}_n - \mu_A}{\sigma/\sqrt{n}} > \frac{204.1 - \mu_A}{\sigma/\sqrt{n}}\right] \\
&= \mathbb{P}_{\mu_A}\left[\frac{\overline{X}_n - \mu_A}{\sigma/\sqrt{n}} > -0.36\right] \\
&= \mathbb{P}\left[Z > -0.36\right].
\end{aligned}$$

Dies entspricht also der Wahrscheinlichkeit, dass eine normalverteilte Zufallsvariable Z mit Varianz 1,

$$Z \sim \mathcal{N}(0, 1),$$

einen Wert grösser als -0.36 annimmt. Diese Wahrscheinlichkeit ist wegen der Symmetrie der Normalverteilung gleich

$$\mathbb{P}\left[Z \leq 0.36\right] = 0.6406,$$

wie aus der Tabelle entnommen werden kann. Die gesuchte Wahrscheinlichkeit (die Macht des Tests an der Stelle $\mu = \mu_A = 205$) ist also ungefähr 64%.

(d) Dies ist genau das Niveau des Testes und war als 5% vorgegeben.

10. (10 points) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Poisson-distributed random variables. Each X_n has a Poisson parameter λ_n, and we assume that $0 < \lambda_n \leq C$ for some constant $C \in (0, \infty)$.

   (a) (3 points) Prove that

$$\mathbb{P}[X_n \geq n \text{ for infinitely many } n] = 0.$$

   (b) (3 points) Discuss in detail if or when the sum $S_n := \sum_{i=1}^n X_i$ has a Poisson distribution. You have to prove your assertions in full detail, without using results on Poisson distributions from the lecture.

   (c) (4 points) Show that for all n,

$$\mathbb{P}[X_n \geq b] \leq e^{b - b\log\frac{b}{C} - C}$$

   for $b > C$.

Solution:

(a)

$$\sum_{n=1}^{\infty} \mathbb{P}[X_n \geq n] = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \mathbb{P}[X_n = k] = \sum_{k=1}^{\infty}\sum_{n=1}^{k} \mathbb{P}[X_n = k]$$
$$= \sum_{k=1}^{\infty}\sum_{n=1}^{k} e^{-\lambda_n}\frac{\lambda_n^k}{k!} \leq \sum_{k=1}^{\infty}\frac{C^k}{k!} = e^C - 1 < \infty.$$

Daher folgt nach dem ersten Teil des Lemmas von Borel-Cantelli die Behauptung.

(b) (i) S_n is not always Poisson-distributed for a general sequence $(X_n)_{n\in\mathbb{N}}$ of Poisson-distributed random variables.

Proof. We show a counter-example. Let $n = 2$ and $X_1 = X_2$. Then $S_2 = X_1 + X_2 = 2X_1$ can never attain uneven values—in particular $\mathbb{P}[S_2 = 1] = 0$, since Poisson-distributed random variables attain almost surely only integer values. If S_2 were Poisson-distributed, $\mathbb{P}[S_2 = 1] = e^{-\lambda}\frac{\lambda^1}{1!} \neq 0$ could not be zero. (Note that $\lambda > 0$ has to hold for every Poisson distribution.) So S_2 is not Poisson-distributed in this example. □

(ii) If $(X_n)_{n\in\mathbb{N}}$ is a sequence of *independent* Poisson-distributed random variables, S_n is Poisson-distributed as well.

Proof. For $n = 2$, set $X = X_1 + X_2$. The law of total probability gives us because of independence

$$\begin{aligned}
\mathbb{P}[X=k] &= \sum_{j=0}^{k} \mathbb{P}[X_1=j, X_2=k-j] \\
&= \sum_{j=0}^{k} \mathbb{P}[X_1=j]\,\mathbb{P}[X_2=k-j] \\
&= \sum_{j=0}^{k} e^{-\lambda_1}\frac{\lambda_1^j}{j!} e^{-\lambda_2}\frac{\lambda_2^{k-j}}{(k-j)!} \\
&= e^{-(\lambda_1+\lambda_2)}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda_1^j\lambda_2^{k-j} \\
&= e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^k}{k!},
\end{aligned}$$

which corresponds again to a Poisson distribution (with parameter $\lambda = \lambda_1 + \lambda_2$).

(Alternatively this could be shown with the help of characteristic functions.)

We can use induction to prove the statement for arbitrary $n \in \mathbb{N}$. □

(c) By the Markov inequality,

$$\mathbb{P}[X_n \geq b] \leq \frac{\mathbb{E}\left[e^{\alpha X_n}\right]}{e^{\alpha b}}$$

holds for $\alpha > 0$. Then we compute

$$\mathbb{E}\left[e^{\alpha X_n}\right] = \sum_{k=0}^{\infty} e^{-\lambda_n}\frac{\lambda_n^k}{k!}e^{\alpha k} = e^{-\lambda_n}e^{\lambda_n e^{\alpha}} = e^{\lambda_n(e^{\alpha}-1)} \leq e^{C(e^{\alpha}-1)}.$$

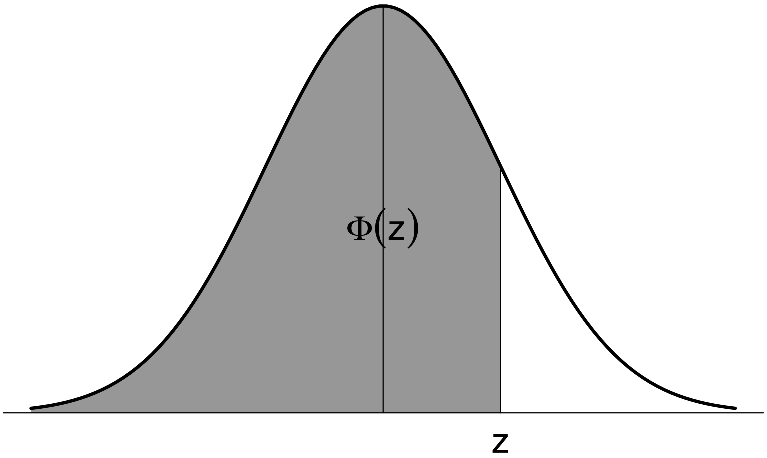
So we have $\mathbb{P}[X_n \geq b] \leq e^{Ce^{\alpha}-\alpha b-C}$. Optimising over $\alpha > 0$ gives $0 = Ce^{\alpha} - b$, thus $Ce^{\alpha} = b$ and $\alpha = \log\frac{b}{C} > 0$ for $b > C$. Then we get

$$\mathbb{P}[X_n \geq b] \leq e^{b-b\log\frac{b}{C}-C}.$$

⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆

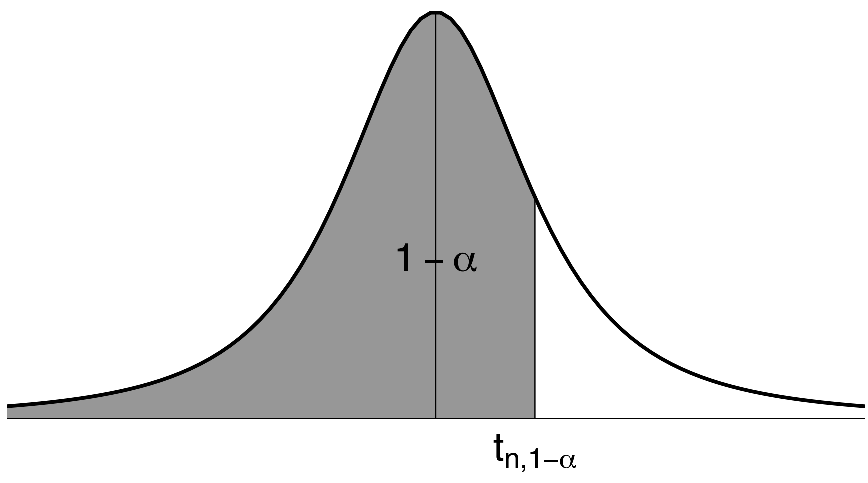
⋆ ⋆ ⋆ Good luck! ⋆ ⋆ ⋆

Tabellen



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

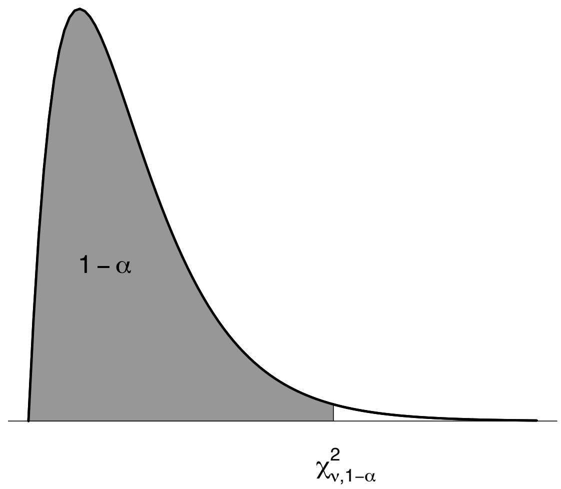
Tabelle der Standard-Normalverteilungsfunktion $\Phi(z) = P[Z \leq z]$ mit $Z \sim \mathcal{N}(0,1)$



1 − α
$t_{n,1-\alpha}$

df	$t_{0.60}$	$t_{0.70}$	$t_{0.80}$	$t_{0.90}$	$t_{0.95}$	$t_{0.975}$	$t_{0.99}$	$t_{0.995}$
1	0.325	0.727	1.376	3.078	6.314	12.706	31.821	63.657
2	0.289	0.617	1.061	1.886	2.920	4.303	6.965	9.925
3	0.277	0.584	0.978	1.638	2.353	3.182	4.541	5.841
4	0.271	0.569	0.941	1.533	2.132	2.776	3.747	4.604
5	0.267	0.559	0.920	1.476	2.015	2.571	3.365	4.032
6	0.265	0.553	0.906	1.440	1.943	2.447	3.143	3.707
7	0.263	0.549	0.896	1.415	1.895	2.365	2.998	3.499
8	0.262	0.546	0.889	1.397	1.860	2.306	2.896	3.355
9	0.261	0.543	0.883	1.383	1.833	2.262	2.821	3.250
10	0.260	0.542	0.879	1.372	1.812	2.228	2.764	3.169
11	0.260	0.540	0.876	1.363	1.796	2.201	2.718	3.106
12	0.259	0.539	0.873	1.356	1.782	2.179	2.681	3.055
13	0.259	0.538	0.870	1.350	1.771	2.160	2.650	3.012
14	0.258	0.537	0.868	1.345	1.761	2.145	2.624	2.977
15	0.258	0.536	0.866	1.341	1.753	2.131	2.602	2.947
16	0.258	0.535	0.865	1.337	1.746	2.120	2.583	2.921
17	0.257	0.534	0.863	1.333	1.740	2.110	2.567	2.898
18	0.257	0.534	0.862	1.330	1.734	2.101	2.552	2.878
19	0.257	0.533	0.861	1.328	1.729	2.093	2.539	2.861
20	0.257	0.533	0.860	1.325	1.725	2.086	2.528	2.845
21	0.257	0.532	0.859	1.323	1.721	2.080	2.518	2.831
22	0.256	0.532	0.858	1.321	1.717	2.074	2.508	2.819
23	0.256	0.532	0.858	1.319	1.714	2.069	2.500	2.807
24	0.256	0.531	0.857	1.318	1.711	2.064	2.492	2.797
25	0.256	0.531	0.856	1.316	1.708	2.060	2.485	2.787
26	0.256	0.531	0.856	1.315	1.706	2.056	2.479	2.779
27	0.256	0.531	0.855	1.314	1.703	2.052	2.473	2.771
28	0.256	0.530	0.855	1.313	1.701	2.048	2.467	2.763
29	0.256	0.530	0.854	1.311	1.699	2.045	2.462	2.756
30	0.256	0.530	0.854	1.310	1.697	2.042	2.457	2.750
31	0.255	0.530	0.853	1.309	1.696	2.040	2.452	2.744
32	0.255	0.530	0.853	1.309	1.694	2.037	2.449	2.738
33	0.255	0.530	0.853	1.308	1.693	2.035	2.445	2.733
34	0.255	0.529	0.852	1.307	1.691	2.032	2.441	2.728
35	0.255	0.529	0.852	1.306	1.690	2.030	2.438	2.724
40	0.255	0.529	0.851	1.303	1.684	2.021	2.423	2.704
60	0.254	0.527	0.848	1.296	1.671	2.000	2.390	2.660
120	0.254	0.526	0.845	1.289	1.658	1.980	2.358	2.617
∞	0.253	0.524	0.842	1.282	1.645	1.960	2.326	2.576

Ausgewählte Quantile $t_{n,1-\alpha}$ der t-Verteilung; in der Tabelle ist $n = df$.
Für $df = \infty$ erhält man die Quantile $z_{1-\alpha}$ der Standard-Normalverteilung.



	$p = 0.90$	$p = 0.95$	$p = 0.975$	$p = 0.999$	$p = 0.9995$
$\nu = 1$	2.7055	3.8415	5.0239	10.8276	12.1157
$\nu = 2$	4.6052	5.9915	7.3778	13.8155	15.2018
$\nu = 3$	6.2514	7.8147	9.3484	16.2662	17.7300
$\nu = 4$	7.7794	9.4877	11.1433	18.4668	19.9974
$\nu = 5$	9.2364	11.0705	12.8325	20.5150	22.1053
$\nu = 6$	10.6446	12.5916	14.4494	22.4577	24.1028
$\nu = 7$	12.0170	14.0671	16.0128	24.3219	26.0178
$\nu = 8$	13.3616	15.5073	17.5345	26.1245	27.8680
$\nu = 9$	14.6837	16.9190	19.0228	27.8772	29.6658
$\nu = 10$	15.9872	18.3070	20.4832	29.5883	31.4198
$\nu = 11$	17.2750	19.6751	21.9200	31.2641	33.1366
$\nu = 12$	18.5493	21.0261	23.3367	32.9095	34.8213

Ausgewählte Quantile $\chi^2_{\nu,1-\alpha}$ der Chiquadrat-Verteilung; in der Tabelle ist $p = 1 - \alpha$.

Grade	From	To
1,00	0,0	1,5
1,25	2,0	5,5
1,50	6,0	9,0
1,75	9,5	13,0
2,00	13,5	16,5
2,25	17,0	20,5
2,50	21,0	24,0
2,75	24,5	28,0
3,00	28,5	31,5
3,25	32,0	35,5
3,50	36,0	39,0
3,75	39,5	43,0
4,00	43,5	47,5
4,25	48,0	52,5
4,50	53,0	58,0
4,75	58,5	63,0
5,00	63,5	68,5
5,25	69,0	73,5
5,50	74,0	79,0
5,75	79,5	84,0
6,00	84,5	100,0