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A.12 Axiom of Determinacy

1. Introduction

Rem: (Notation). In today's lecture we will work once again in the Baire space $\mathcal{N} = \omega^\omega = \{\langle a_n\rangle_{n\in\mathbb{N}} \mid a_n \in \mathbb{N}\ \forall n\}$ of all infinite sequences of natural numbers, where $\omega = \mathbb{N}$ is taken to contain 0, $0 \in \omega$. Write $Seq_n := \{\langle k_1, k_2, \dots, k_n\rangle \mid k_i \in \mathbb{N}\ \forall 1 \leq i \leq n\}$ for the set of all finite sequences in $\mathbb{N}$ of length n, with $Seq_0 := \emptyset$. Also let $Seq_{even} := \bigcup_{n\in\mathbb{N}} Seq_{2n}$ resp. $Seq_{odd} := \bigcup_{n\in\mathbb{N}} Seq_{2n+1}$ (whose elements are resp. finite seq. of even and odd length). Thus $Seq_n = \omega^n$ & $Seq := \bigcup_{n\in\mathbb{N}} Seq_n$ (the set of all finite seq.). If we wish for $k_i \in S$, S any set, write $Seq_n(S) := \{\langle k_1, \dots, k_n\rangle \mid k_i \in S\ \forall 1 \leq i \leq n\}$ (in practice we'll use only $S := \{0,1\}$).

Goal of today's lecture is to play games! Given $A \subseteq \mathcal{N}$ any subset, we define a game G_A played by two players I & II as follows: I starts by choosing some $a_0 \in \omega$, then II picks some $b_0 \in \omega$, then I some $a_1 \in \omega$, II some $b_1 \in \omega$, ... so that after ω steps we get a $\langle a_0, b_0, a_1, b_1, \dots\rangle \in \mathcal{N}$. I wins if actually $\langle a_0, b_0, a_1, b_1, \dots\rangle \in A$, otherwise (if $\in \mathcal{N}\setminus A$) II wins. G_A is said determined if one of the players has a "strategy" which enables him to always win (no matter what the other does!).

↳ "The Axiom of Determinacy (AD) states that $\forall A \subseteq \mathcal{N}$ the associated game G_A is determined".

... wait, what?! This surely looks like a good way to win the big money (if only you could live countably infinitely many years), thus we want to understand what is the cost of AD and which are the consequences (at least some of them).

2. Determinacy and Choice

Def 2.1: Let $A \subseteq \mathcal{N}$, with associated game G_A.

- A play is any $\langle a_0, b_0, a_1, b_1, \dots\rangle \in \mathcal{N}$ constructed according to the rule of G_A stated above. So a_n is the n-th move of I, b_n the n-th move of II.
- A strategy for I is a function $\sigma: Seq_{even} \to \omega,\ s \mapsto \sigma(s)$ (interpret as "given a partial play $s = \langle a_0, b_0, a_1, b_1, \dots, a_n, b_n\rangle$, I's next move is $a_{n+1} = \omega(s)$").
  A strategy for II is a function $\tau: Seq_{odd} \to \omega,\ s \mapsto \tau(s)$ (interpret as "given a partial play $s = \langle a_0, b_0, a_1, b_2, \dots, a_n\rangle$, II's next move is $b_n = \tau(s)$").

- For $b = \langle b_0, b_1, \dots \rangle \in \mathcal{N}$, Ⓘ plays $\langle a_0, b_0, a_1, b_1, \dots \rangle$ by the strategy σ if $a_0 = \sigma(\emptyset)$ & $a_1 = \sigma(\langle a_0, b_0 \rangle)$ & $a_2 = \sigma(\langle a_0, b_0, a_1, b_1 \rangle)$ & …, so that the play $\langle a_0, b_0, a_1, b_1, \dots \rangle$ is actually determined by only σ & b. Thus we write $\langle a_0, b_0, a_1, b_1, \dots \rangle =: \sigma * b$ ($\equiv \langle a_0, a_1, \dots \rangle$ occasionally).

  For $a = \langle a_0, a_1, \dots \rangle \in \mathcal{N}$, Ⓘ plays $\langle a_0, b_0, a_1, b_1, \dots \rangle =: a * \tau$ by the strategy τ if $b_0 = \tau(\langle a_0 \rangle)$ & $b_1 = \tau(\langle a_0, b_0, a_1 \rangle)$ & $b_2 = \tau(\langle a_0, b_0, a_1, b_1, a_2 \rangle)$ & …

- σ is a winning strategy for Ⓘ if $\{\sigma * b \mid b \in \mathcal{N}\} \subseteq A$ (interpret as "no matter what Ⓘⓘ plays, $\sigma * b \in A \ \forall b \in \mathcal{N}$", precisely the winning condition for Ⓘ).

  τ is a winning strategy for Ⓘⓘ if $\{a * \tau \mid a \in \mathcal{N}\} \subseteq \mathcal{N} \setminus A$ (i.e. Ⓘⓘ always wins).

Lem 2.2: Assuming the Axiom of Choice (AC), there exists some $A \subseteq \mathcal{N}$ s.t. G_A is not determined. Therefore the Axiom of Choice is incompatible with the Axiom of Determinacy (asserting that G_A is always determined).

Proof: Assuming AC, the number of strategies is $2^{\aleph_0}$.[1] Enumerate (thus order) them as $\Sigma := \{\sigma_\alpha\}_{\alpha < 2^{\aleph_0}}$ for Ⓘ resp. $T := \{\tau_\alpha\}_{\alpha < 2^{\aleph_0}}$ for Ⓘⓘ. Now, construct sets of plays $X := \{x_\alpha \mid \alpha < 2^{\aleph_0}\} \subseteq \mathcal{N}$ of Ⓘ resp. $Y := \{y_\alpha \mid \alpha < 2^{\aleph_0}\} \subseteq \mathcal{N}$ of Ⓘⓘ as follows: given the partial $X_\alpha := \{x_\beta \mid \beta < \alpha\}$, $Y_\alpha := \{y_\beta \mid \beta < \alpha\} \subseteq \mathcal{N}$, pick $y_\alpha \in \mathcal{N}$ s.t. $y_\alpha = \sigma_\alpha * b$ for a suited $b \in \mathcal{N}$ & $y_\alpha \notin X_\alpha$ (possible as $|\{\sigma_\alpha * b \mid b \in \mathcal{N}\}| = |\mathcal{N}| = 2^{\aleph_0} > \alpha$). Set $Y_{\alpha+1} = Y_\alpha \cup \{y_\alpha\}$ and pick $x_\alpha \in \mathcal{N}$ s.t. $x_\alpha = a * \tau_\alpha$ for a suited $a \in \mathcal{N}$ & $x_\alpha \notin Y_{\alpha+1}$. Set $X_{\alpha+1} := X_\alpha \cup \{x_\alpha\}$ and proceed iteratively for $y_{\alpha+1}, x_{\alpha+1}, \dots$

So $X \cap Y = \emptyset$, $\forall y_\alpha \in Y \ \exists b \in \mathcal{N}$ s.t. $\sigma_\alpha * b = y_\alpha \notin X_\alpha$ (i.e. $\forall$ strategies $\sigma_\alpha \ \exists b \in \mathcal{N}$ s.t. $\sigma_\alpha * b \notin X \Rightarrow$ Ⓘ has no winning strategy) and $\forall x_\alpha \in X \ \exists a \in \mathcal{N}$ s.t. $a * \tau_\alpha = x_\alpha \in X_{\alpha+1}$ (i.e. $\forall$ strategies $\tau_\alpha \ \exists a \in \mathcal{N}$ s.t. $a * \tau_\alpha \in X \Rightarrow$ Ⓘⓘ has no winning strategy). Therefore, playing G_A for $A := X$, neither Ⓘ nor Ⓘⓘ has a winning strategy, i.e. G_A is undetermined. □

Lem 2.3: The Axiom of Determinacy implies a weak form of AC, namely that every countable family of non-empty sets of real numbers has a choice function.

Proof: Set-theoretically $\mathbb{R} \cong \mathcal{N}$ (both have cardinality $\aleph_1$), thus what we really want to show is: for any $\mathcal{X} = \{X_n \subseteq \mathcal{N} \mid X_n \neq \emptyset, n \in \omega\} \ \exists f : \mathcal{X} \to \mathcal{X}$ s.t. $f(X_n) \in X_n \ \forall n \in \omega$. Consider the game: if Ⓘ plays $a := \langle a_0, a_1, a_2, \dots \rangle \in \mathcal{N}$ & $b := \langle b_0, b_1, b_2, \dots \rangle \in \mathcal{N}$ so Ⓘⓘ wins iff $b \in X_{a_0} \rightsquigarrow$ Ⓘ has no winning strategy: once played a_0, Ⓘⓘ chooses any $b \in X_{a_0} \overset{\text{ass.}}{\neq} \emptyset$ and plays it, winning (as required by AD ✓), which determines a winning strategy τ. Replacing $a_0 \leftarrow n$, set $\langle a_0, b_0, a_1, b_1, \dots \rangle :=$

[1] A proof can be found in the reference [1] of Seminar A, namely Cor 3.26 (b)-(c).

$\langle n,0,0,0,\dots\rangle * \tau$ and finally define $f(X_n) := \langle b_0, b_1, \dots\rangle = b \in X_n$, $\forall n \in \omega$ ✓. $\square$

3. Consequences of AD: about Lebesgue measurability

Note: in the rest of the lecture we will work in ZF+DC.

Def 3.1: (Lebesgue outer measure). Let $I_n := (a_n, b_n) \subset \mathbb{R}$ denote an interval of length $\ell(I_n) = b_n - a_n \in \mathbb{R}^+$. The Lebesgue outer measure of $A \subset \mathbb{R}$ is given by $\mu^*(A) := \inf\{\sum_{n=1}^{\infty} \ell(I_n) \mid A \subseteq \bigcup_{n=1}^{\infty} I_n\}$. We say that $X \subset \mathbb{R}$ is Lebesgue measurable if $\mu^*(A) = \mu^*(A \cap X) + \mu^*(A \setminus X)$ $\forall A \subset \mathbb{R}$. It has zero measure if $\mu^*(X) = 0$.

Def 3.2: (The covering game). Let $\varepsilon > 0$, $S \subset [0,1]$ set. We define a game as follows. If $\langle a_0, a_1, a_2, \dots\rangle \in \mathrm{Seq}(\{0,1\})$, set $a := \sum_{n=0}^{\infty} \frac{a_n}{2^{n+1}} \in [0,1]$. For each $n \in \omega$, let $K_n = \{G \mid G = \bigcup_{i=1}^{m} (a_i, b_i) \subset [0,1]$ s.t. $a_i, b_i \in \mathbb{Q}$ $\forall 1 \leq i \leq m \in \mathbb{N}$, $\mu(G) \leq \frac{\varepsilon}{2^{2(n+1)}}\}$ and enumerate it as $K_n = \{G^n_k\}_{k \in \omega}$. These are the rules:

- (I) plays some $\langle a_0, a_1, a_2, \dots\rangle \in \mathrm{Seq}(\{0,1\})$ s.t. the corresponding $a \in S$;
- (II) plays some $\langle b_0, b_1, b_2, \dots\rangle \in \mathcal{N}$, yielding a selection $H_n = G^n_{b_n} \in K_n$ $\forall n \in \omega$

The play $\langle a_0, b_0, a_1, b_1, \dots\rangle$ is won by:

- (I) if $a \notin \bigcup_{n=0}^{\infty} G^n_{b_n}$
- (II) if $a \in \bigcup_{n=0}^{\infty} G^n_{b_n}$

Lem 3.3: Assuming AD, let $S \subset \mathbb{R}$ be set s.t. ($Z \subset S$ Lebesgue measurable $\Rightarrow \mu^*(Z) = 0$) holds. Then $\mu^*(S) = 0$.

Proof: Through suited bijection, take wlog $S \subset [0,1]$. We want to show that $\forall \varepsilon > 0$ $\mu^*(S) < \varepsilon$. Fix some $\varepsilon > 0$. Then we can play the covering game from 3.2.

Claim: (I) does not have a winning strategy.

Proof: Assume he does, namely $\sigma : \mathrm{Seq}_{even} \to \{0,1\}$, then the function $F : \mathcal{N} \to \mathcal{N} \cong \mathbb{R}$, $b = \langle b_0, b_1, b_2, \dots\rangle \mapsto \sigma * b = \langle a_0, a_1, a_2, \dots\rangle \equiv a$ (winning play) is continuous ($\Leftarrow$ small variations of a correspond to small variations of b) $\Rightarrow Z := F(\mathcal{N}) \subset \overbrace{[0,1]}^{\text{Polish space}}$ is by def. analytic $\Rightarrow Z \overset{a \in S}{\subseteq} S$ measurable $\overset{\text{ass.}}{\Rightarrow} \mu^*(Z) = 0$, thus can surely be covered by some $\bigcup_{n=0}^{\infty} H_n$ s.t. $H_n \in K_n$ $\forall n \in \omega$. Set $G^n_{b_n} := H_n$, then (II) wins by playing $\langle b_0, b_1, b_2, \dots\rangle \in \mathcal{N}$ since $a = \sigma * b \in Z = \bigcup_{n=0}^{\infty} G^n_{b_n}$. Hence σ is no winning strategy ↯

$\Rightarrow$ (I) has no winning strategy. $\square$

We play the ace: assuming AD, then (II) must have a winning strategy, say τ. For each $s_n = \langle a_0, a_1, \dots, a_n\rangle \in \mathrm{Seq}_{n+1}(\{0,1\}) = \{0,1\}^{n+1}$ let $G_{s_n} := G^n_{b_n} \in K_n$, where $b_n = \tau(\langle a_0, b_0, a_1, b_1, \dots, a_n\rangle)$. τ winning $\Rightarrow \forall a \in S$ is $a \in \bigcup_{n \in \omega} \{G_{s_n} \mid s_n \subseteq a$ as subseq.$\}$ $\xrightarrow{\text{vary } a}$ $S \subseteq \bigcup\{G_s \mid s \in$

$\mathrm{Seq}(\{0,1\})\} = \bigcup_{n=0}^{\infty} \bigcup_{s_n \in \{0,1\}^{n+1}} G_{s_n}$. But $\mu\left(\bigcup_{n=0}^{\infty} \bigcup_{s_n \in \{0,1\}^{n+1}} G_{s_n}\right) \overset{\text{add.}}{\leq} \sum_{n=0}^{\infty} \sum_{s_n \in \{0,1\}^{n+1}} \mu(G_{s_n}) \overset{\text{def. game}}{\leq} \sum_{n=0}^{\infty} \frac{\varepsilon}{2^{2(n+1)}} \cdot \Big(\overbrace{\sum_{s_n \in \{0,1\}^{n+1}} 1}^{=2^{n+1}}\Big) =$

$= \sum_{n=0}^{\infty} \frac{\varepsilon}{2^{n+1}} = \varepsilon \Rightarrow \mu^*(S) \leq \varepsilon \xrightarrow{\varepsilon>0 \text{ any}} \mu^*(S) = 0$, as desired. $\square$

Thm. 3.4: Assume AD, then every set $X \subseteq \mathbb{R}$ is Lebesgue measurable.

Proof: By adding suited sets of zero measure, $\exists A \supseteq X$ measurable in $\mathbb{R}$ s.t. ($Z \subseteq A \setminus X$ measurable $\Rightarrow \mu^*(Z) = 0$) holds[1]. Taking $S := A \setminus X$, Lem 3.3 gives $\mu^*(A \setminus X) = 0$. Now, $\mu^*(B) = \mu^*(B \cap X \cup B \setminus X) \overset{\text{subadd.}}{\leq} \mu^*(B \cap X) + \mu^*(B \setminus X)$ readily $\forall B \subseteq \mathbb{R}$ subset. For "$\geq$", one proves it first for subsets $C \subseteq A$ & $D \subseteq \mathbb{R}$ s.t. $A \cap D = \emptyset$ (see Appendix A). $\square$

Rem 3.5: This is not an obvious statement. Indeed, assuming the Axiom of Choice (by 2.2 incompatible with AD!) one can prove the existence of non-measurable sets of reals (e.g. the Vitali sets). So 3.4 shows one nice benefit in assuming AD instead!

4. Consequences of AD about the Baire property

Def 4.1: Let X be a topological space.

- A set $S \subseteq X$ is nowhere dense if $(\bar{S})^\circ = \emptyset$ (i.e. its closure has no interior).
- A set $M \subseteq X$ is meager if $\exists S_i \subseteq X$ nowhere dense s.t. $M = \bigcup_{i=1}^{\infty} S_i$ (countable union).
- A set $B \subseteq X$ has the Baire property if $\exists G \subseteq X$ open s.t. $B \triangle G = B \setminus G \cup G \setminus B$ is meager.

Def 4.2: (The Banach-Mazur game). Let $X \subseteq \mathcal{N}$. The rules are: I plays some finite $s_0 \in \mathrm{Seq}_n$, II extends it via some finite $t_0 \in \mathrm{Seq}_m$ $(m > n)$ s.t. $t_0 \supseteq s_0$, I extends by $s_1 \supseteq t_0, \ldots$ forming an increasing chain of finite seq. $s_0 \subseteq t_0 \subseteq s_1 \subseteq t_1 \subseteq \ldots$ tending to some $x \in \mathcal{N}$. The play is won by:

- I if $x \in X$
- II if $x \in \mathcal{N} \setminus X$

Rem 4.3: Since we want to apply AD, we check that the game in 4.2 can be rephrased as one like in 2.1. Enumerate $\mathrm{Seq} = \{u_k\}_{k \in \omega}$, define $A := \{\langle a_0, b_0, a_1, b_1, \ldots \rangle \in \mathcal{N} \mid$ either $\exists n \in \omega$ s.t. $u_{a_0} \subseteq u_{b_0} \subseteq \ldots \subseteq u_{a_n} \not\subseteq u_{b_n}$ or $u_{a_0} \subseteq u_{b_0} \subseteq \ldots \longrightarrow x \in X\}$ (so that in both cases I wins, the former because II fails to extend the seq. by a suited u_{b_n}). Therefore: I wins the Banach-Mazur game $\Longleftrightarrow$ I wins at G_A.

Lem 4.4: Let $X \subseteq \mathcal{N}$. Then II has a winning strategy in the Banach-Mazur game $\Longleftrightarrow$ X is meager.

Proof: First we make $\mathcal{N}$ into a top. space by declaring $O(s) := \{x \in \mathcal{N} \mid s \subseteq x\} \subseteq \mathcal{N}$ for $s \in \mathrm{Seq}_n$ to be open & a basis el. for the topology[2]. The concepts in 4.1 now make sense.

"$\Longleftarrow$" Let $Y := \mathcal{N} \setminus X$, $X = \bigcup_{i=0}^{\infty} S_i$ be meager $\Rightarrow Y = \bigcap_{i=0}^{\infty} \mathcal{N} \setminus S_i$, where each $\mathcal{N} \setminus S_i$

[1] I challenge you to prove this! My thanks to Chris for providing a nice proof of this non-trivial fact.

[2] If $s = \langle s_0, t_0, \ldots, s_i, t_i \rangle \in \mathrm{Seq}_{i+1}$, so we can also define $U_{t_i} := \{x \in \mathcal{N} \mid s \subseteq x\} \subseteq \mathcal{N}$.

fulfills $\overline{\mathcal{N}\setminus S_i} = \mathcal{N}$ (otherwise $O := \mathcal{N}\setminus(\overline{\mathcal{N}\setminus S_i}) \neq \emptyset$ open $\Rightarrow \overline{S_i}^\circ \supseteq S_i^\circ = (\mathcal{N}\setminus(\overline{\mathcal{N}\setminus S_i}))^\circ \supseteq O \neq \emptyset$, ↯ to S_i nowhere dense). So $G_i := (\mathcal{N}\setminus S_i)^\circ$ is open, dense & $Y \supseteq \bigcap_{i=0}^{\infty} G_i$. Now, if (I) plays s_0, set $\tau(\langle s_0\rangle) := t_0 \supseteq s_0$ s.t. $U_{t_0} \subseteq G_0$ (t_0 $\exists$ by density of G_0!); if (I) answers with $s_1 \supseteq t_0$, set $\tau(\langle s_0,t_0,s_1\rangle) := t_1 \supseteq s_1$ s.t. $U_{t_1} \subseteq G_1$ (t_1 $\exists$ by density of G_1!); ... obtaining $s_0 \subseteq t_0 \subseteq s_1 \subseteq \ldots \to x \in \bigcap_{i=0}^{\infty} G_i \subseteq Y = \mathcal{N}\setminus X$, i.e. (II) always wins by playing by such τ ✓.

"$\Longrightarrow$" Assume (II) has a winning strategy τ. Write again $Y := \mathcal{N}\setminus X$.

Claim: Let $x \in \mathcal{N}$ and assume that $\forall$ correct positions $p = \langle s_0, t_0, \ldots, s_n, t_n\rangle$ ("$= \langle s_0,\ldots,s_n\rangle * \tau$", i.e. partial play by τ) s.t. $t_n \subseteq x$ $\exists\, s \supseteq t_n$ s.t. $\tau(\underbrace{p^\frown s}_{:=\langle s_0,t_0,\ldots,t_n,s\rangle}) \subseteq x$. Then $x \in Y$.

Proof: By ass. for $p = \emptyset$ (correct pos.) $\exists\, s_0$ s.t. $\tau(\emptyset^\frown s_0) = \tau(\langle s_0\rangle) \subseteq x$. Set $t_0 := \tau(\langle s_0\rangle)$ & $p \leftarrow \langle s_0, t_0\rangle$. Then for $p = \langle s_0, t_0\rangle$ (correct) $\exists\, s_1 \supseteq t_0$ s.t. $t_1 := \tau(p^\frown s_1) = \tau(\langle s_0,t_0,s_1\rangle) \subseteq x$. Iterating we get $s_0 \subseteq t_0 \subseteq s_1 \subseteq t_1 \subseteq \ldots \to x$, where (II) played by τ by constr. $\xRightarrow{\tau \text{ winning}}$ (II) wins, i.e. $x \in Y$ forcedly. □

For a correct pos. $p = \langle s_0, t_0, \ldots, s_n, t_n\rangle$ let $F_p := \{x \in \mathcal{N} \mid t_n \subseteq x,\ \tau(p^\frown s) \not\subseteq x\ \forall s \supseteq t_n\}$. Contraposing the claim, $\forall x \in \mathcal{N}$ $\exists$ correct p s.t. $t_n \subseteq x$ & $\forall s \supseteq t_n$ is $\tau(p^\frown s) \not\subseteq x$, i.e. s.t. $x \in F_p \Longrightarrow X \subseteq \bigcup\{F_p \mid p \text{ correct pos.}\}$, actually a countable union.[1] But $O(t_n)\setminus F_p = \{x \in \mathcal{N} \mid t_n \subseteq x, \exists\, s \supseteq t_n \text{ s.t. } \tau(p^\frown s) \subseteq x\} \subseteq \mathcal{N}$ is clearly dense ("almost all" $x \in \mathcal{N}$ fulfill the condition!) $\Longrightarrow F_p$ is nowhere dense, thus X meager (since contained in a countable union of nowhere dense sets). □

Cor 4.5: Let $X \subseteq \mathcal{N}$. Then (I) has a winning strategy in the Banach-Mazur game $\Longleftrightarrow$ $\exists\, s \in Seq_n$ s.t. $O(s)\setminus X$ is meager.

Proof: Inverting the roles, given $s \in Seq_n$, let (I) play $t_0 \supseteq s$, (II) $s_0 \supseteq t_0$, (I) $t_1 \supseteq s_0$, ... with (I) winning if $t_0 \subseteq s_0 \subseteq t_1 \subseteq \ldots \to x \in O(s)\setminus X$. By Lem 4.4 (II) always wins here iff $O(s)\setminus X$ is meager. But (II) is now playing the "original hand" of (I) in the standard Banach-Mazur game (indices of $\langle s_0, s_1, \ldots\rangle$ are just shifted by 1!) □

Thm 4.6: Assume AD, then every set $X \subseteq \mathbb{R}$ has the Baire property.

Proof: The Banach-Mazur game, translatable to some G_A according to Rem 4.3, must be determined, i.e. either (I) or (II) has a winning strategy $\overset{4.3+4.4}{\Longleftrightarrow}$ either $X \subseteq \mathcal{N} \subseteq \mathbb{R}$ or $O(s)\setminus X$ for some $s \in Seq_n$ is meager.

- if X is meager, the open $\emptyset \subseteq \mathcal{N}$ makes $X \triangle \emptyset = X\setminus\emptyset \cup \emptyset\setminus X = X$ meager $\Rightarrow X$ has the Baire property ✓.
- otherwise, set $G := \bigcup\{O(s) \mid O(s)\setminus X \text{ meager}\} \overset{\text{ass.}}{\neq} \emptyset$ and open (as union of open sets).

[1] Any correct p just depends on a finite $\langle s_0, \ldots, s_n\rangle$, thus there are at most $|Fin(\omega)|$-many of them ($Fin(\omega)$ being the set of all finite subsets of ω). Again Cor 3.26 from [1] tells us that $|Fin(\omega)| = |\omega| = \aleph_0$.

Note that $G \setminus X = \bigcup\{O(s) \setminus X \mid O(s) \setminus X \text{ meager}\}$ is meager (since union of meager sets) and $X \setminus G$ too (otherwise by above $\exists$ seq. s s.t. $O(s) \setminus (X \setminus G) = O(s) \setminus X \cup O(s) \cap G$ is meager, impossible since both $O(s)$ & $O(s) \setminus X$ are non-meager in $\mathcal{N}$![1]). Thus we have $X \Delta G = X \setminus G \cup G \setminus X$ is meager $\Longrightarrow$ X has the Baire property ✓. □

Rem 4.7: Again, assuming AC last theorem is false!

5. Consequences of AD: about perfect subsets

Def 5.1: Let X be a topological space.

- A point $x \in X$ is isolated if $\{x\} \subseteq X$ is open.
- A subset $\emptyset \neq S \subseteq X$ is perfect if it's closed & has no isolated points.

Def 5.2: $\mathcal{C} = \{0,1\}^\omega$ is the Cantor space, arising from the well-known construction (and using base-2 decimals). It has the following properties (see any course in Topology or Measure Theory): is closed in $\mathbb{R}$, has no isolated points, is in bijection with $\mathbb{R}$ (Cantor-Bernstein Thm).

Def 5.3: (The perfect set game). Let $X \subseteq \mathcal{C}$. The rules are: I plays $s_0 \in \mathrm{Seq}(\{0,1\}) \cup \{\emptyset\}$, II plays $n_0 \in \{0,1\}$, I plays $s_1 \in \mathrm{Seq}(\{0,1\})$, ... forming $x = s_0{}^\frown n_0{}^\frown s_1{}^\frown n_1{}^\frown \ldots \in \mathcal{C}$.

The play is won by:

- I if $x \in X$
- II if $x \notin X$

(Arguing as in 4.3, this reformulates to a game G_A for $A \subseteq \mathcal{N}$).

Lem 5.4: Let $X \subseteq \mathcal{C}$, let II have a winning strategy in the associated perfect set game. Then X is countable.

Proof: Take the winning strategy τ for II and proceed as in the proof of 4.4. Then $X = \bigcup\{F_p \mid p = \langle s_0, n_0, s_1, n_1, \ldots, s_k, n_k\rangle \text{ is correct}\} = \bigcup\{x \in \mathcal{C} \mid s_0{}^\frown n_0{}^\frown \ldots{}^\frown n_k \subseteq x,\ s_0{}^\frown \ldots{}^\frown n_k{}^\frown s{}^\frown \tau(p^\frown s) \not\subseteq x\ \forall s \supseteq s_0{}^\frown \ldots{}^\frown n_k\}$, a countable union.

Claim: $|F_p| = 1$ $\forall$ correct pos. p

Proof: Clearly $\exists\, \ell \in \omega$ s.t. $\langle x_0, \ldots, x_{\ell-1}\rangle = s_0{}^\frown \ldots{}^\frown n_k$ (namely $\ell = \mathrm{length}(s_0{}^\frown \ldots{}^\frown n_k)$) $\Longrightarrow x_\ell = 1 - \tau(p^\frown \underbrace{\emptyset}_{= s_{k+1}})$, since $n_{k+1} = \tau(p^\frown \emptyset)$ is set (by def. of F_p) by the condition that $\forall s$ (also $\emptyset$) $p^\frown s^\frown \tau(p^\frown s) = p^\frown s^\frown n_{k+1} \not\subseteq x$, so that we deduce: $n_{k+1} = \begin{cases} 0 \Rightarrow x_\ell = 1 = 1-0 \\ 1 \Rightarrow x_\ell = 0 = 1-1 \end{cases}$ ✓.

Hence $x_{\ell+1} = 1 - \tau(p^\frown \langle x_\ell\rangle)$, and iteratively each entry of x is uniquely set. □

Therefore X is contained in a countable union of singleton, i.e. is countable. □

Thm 5.5: Assume AD, then every uncountable set $X \subseteq \mathbb{R}$ contains a perfect subset.

Proof: By 5.2, $\mathbb{R} \cong \mathcal{C}$ set-theoretically, thus we can restrict to $X \subseteq \mathcal{C}$. Contraposing 5.4,

[1] More precisely: either $O(s) \subseteq G$, so $O(s) \setminus (X \setminus G) = O(s) = \mathcal{N}$ is not meager (any $\neq \emptyset$, open subset of a Baire space is not meager!), or $O(s) \not\subseteq G$, so $O(s) \setminus X$ is not meager (otherwise $\subseteq G \setminus X$ ↯) and then $O(s) \setminus (X \setminus G)$ can't possibly be meager by definition.

X uncountable $\Rightarrow$ II doesn't have a winning strategy $\xrightarrow{AD}$ I has a winning strategy σ. For a play $x = \langle n_0, n_1, \dots \rangle \in \mathcal{C}$ by II set $F(x) := s_0{}^\frown n_0{}^\frown s_1{}^\frown \dots = \sigma(\emptyset)^\frown n_0{}^\frown \sigma(\langle s_0{}^\frown n_0 \rangle)^\frown \dots \overset{\sigma \text{ winning}}{\in} X$. Such $F : \mathcal{C} \to X$ is continuous (by constr.) & bijective (as $x \leftrightarrow \sigma * x$ bijectively). Therefore $F(\mathcal{C}) \neq \emptyset$ is closed (if not $F^{-1}(F(\mathcal{C})) = \mathcal{C}$ would not be closed either ↯) and has no isolated points (otherwise F bij. & cont. $\Rightarrow$ ($x \in F(\mathcal{C})$ isolated $\Rightarrow \overbrace{F^{-1}(x)}^{\text{a point}} \in \mathcal{C}$ isolated) ↯ to 5.2). So $F(\mathcal{C}) \overset{\sigma \text{ winning}}{\subseteq} X$ is a perfect subset. $\square$

<u>Cor 5.6:</u> Assume AD, then $\aleph_1$ is inaccessible in $L[a]$ $\forall a \subseteq \omega$.

6. Further consequences of AD: about large cardinals

Here we just mention a couple more interesting results involving $\aleph_1$ & $\aleph_2$, without proof. The interested reader can find more about this (and the Determinacy topic in general!) in [4] § 33 of the Seminar A references.

<u>Thm 6.1:</u> (Solovay). AD implies that:

(i) $\aleph_1$ is a measurable cardinal (and its closed unbounded filter is an ultrafilter).

(ii) $\aleph_2$ is a measurable cardinal as well.

Appendix A

About the <u>proof of $\mu^*(B) \overset{(\dagger)}{\geq} \mu^*(B \cap X) + \mu^*(B \setminus X)$</u> for any $B \subseteq \mathbb{R}$ subset (Thm 3.4): we choose subsets $C \subseteq A$ & $D \subseteq \mathbb{R}$ s.t. $A \cap D = \emptyset$, and claim that these fulfill ($\dagger$). Indeed $\mu^*(\overbrace{C \setminus X}^{\subseteq A \setminus X}) + \mu^*(\overbrace{C \cap X}^{\subseteq C \cap A}) \overset{\text{monot.}}{\leq} \mu^*(A \setminus X) + \mu^*(C \cap A) \overset{\text{def. } A}{=} \mu^*(C)$ ✓ and $\mu^*(\overbrace{D \setminus X}^{= D}) + \mu^*(\overbrace{D \cap X}^{\subseteq D \cap A}) \leq \mu^*(D) + \mu^*(\emptyset) = \mu^*(D)$. As "$\geq$" was proven in 3.4, these are in fact equalities. For $C := A \cap B \subseteq A$ & $D := B \setminus A$ (s.t. $D \cap A = \emptyset$ ✓) we thus get $\mu^*(A \cap B) = \mu^*((A \cap B) \setminus X) + \mu^*((A \cap B) \cap X)$ resp. $\mu^*(B \setminus A) = \mu^*((B \setminus A) \setminus X) + \mu^*((B \setminus A) \cap X)$, and finally $\mu^*(B \cap X) + \mu^*(B \setminus X) = \mu^*((B \cap X) \cap A \cup (B \cap X) \setminus A) + \mu^*((B \setminus X) \cap A \cup (B \setminus X) \setminus A) \leq \mu^*((B \cap X) \cap A) + \mu^*((B \cap X) \setminus A) + \mu^*(\overbrace{(B \setminus X) \cap A}^{= (A \cap B) \setminus X}) + \mu^*(\overbrace{(B \setminus X) \setminus A}^{= (B \setminus A) \setminus X}) \overset{\text{reorder}}{=} \big(\mu^*((A \cap B) \setminus X) + \mu^*((A \cap B) \cap X)\big) + \big(\mu^*((B \setminus A) \setminus X) + \mu^*((B \setminus A) \cap X)\big) = \mu^*(A \cap B) + \mu^*(B \setminus A) = \mu^*(B)$, last equality because A is measurable by construction (again cfr. proof of 3.4).

So ($\dagger$) is fulfilled $\forall B \subseteq \mathbb{R}$ subset, meaning (together with "$\leq$") that X is measurable.

That's it for today's topic. Thank you for reading along... and enjoy your summer!