0. Prologue

We start introducing the following proposition, which together with the Axiom of Choice guarantees us the existence of a bijection between any set and a certain ordinal.

Prop. 0

Let A be an arbitrary set and let R be an well-ordering on A. Then, there exists a unique ordinal number $\gamma \in \Omega$ for which there is a bijection $w: \gamma \longrightarrow A$ s.t. for all $\alpha, \beta \in \gamma$

$$\alpha \in \beta \iff w(\alpha) \, R \, w(\beta)$$

γ is called the order type of the well-ordering R.

1. Cardinality. First definitions

From now on we assume the Axiom of Choice.

Def. Cardinality of a set

The cardinality of a set A is given by:

$$|A| := \bigcap \{ \alpha \in \Omega : \exists f : \alpha \longrightarrow A \text{ bijective} \}$$

Rmk:

Since the Axiom of Choice is equivalent to the existence of an well-ordering for any A, we have at least the existence of a bijection between an ordinal and A by the Prop. 0, what guarantees that such a definition is well-defined.

Now we define the cardinal numbers.

<u>Def</u>

An ordinal number $K \in \Omega$ such that $|K| = K$ is said to be a cardinal number or just a cardinal.

<u>Rmk:</u>

- We have $||A|| = |A|$, so any $|A|$ is obviously cardinal.
- Not all ordinals are cardinals. As an example, we take $\omega+1$.

<u>Claim:</u>

$$\underset{(1)}{|\omega+1| =} \omega \underset{(2)}{\neq} \omega+1$$

(1) Consider the bijection $f : \omega+1 \longrightarrow \omega$ given by

$$f(\alpha) = \begin{cases} 0, & \alpha = \omega \\ \alpha+1, & \alpha \in \omega \end{cases}$$

Since ω is a cardinal itself $|\omega+1| = \omega$

(2) If $\omega = \omega+1 = \omega \cup \{\omega\}$, then $\omega \in \omega$ which is in contradiction with Fact 3.2

- The collection of the cardinals is well-ordered by $\in$.

Indeed, since they are a collection of ordinals, by Thm 3.3 they are also well-ordered by $\in$.

Hence we can write $K < \lambda$ if $\lambda \ni K$.

2. The class of infinite cardinals

We define in this section as it has been done for the ordinals, the existence of successor and limit cardinals and provide the construction of the class of infinite cardinals, that is, cardinals $\alpha \notin \omega$.

Def

- A cardinal μ is said to be a successor cardinal if there exists a cardinal K s.t.:

$$\mu = K^{+} := \bigcap \{\alpha \in \Omega : K < |\alpha|\}$$

- Otherwise, it is called a limit cardinal.

Construction of W

Let W be the following class and ω_α defined as follows:

$$W = \{\omega_\alpha \in \Omega : \alpha \in \Omega\}$$

$$\omega_0 := \omega$$

$$\omega_{\alpha+1} := \omega_\alpha^{+} \text{ if } \alpha \text{ is a successor ordinal}$$

$$\omega_\alpha := \bigcup_{\delta\in\alpha} \omega_\delta \text{ if } \alpha \text{ is a limit ordinal}$$

Rmks:

- $\bigcup_{\delta\in\alpha} \omega_\delta$ is a cardinal if α is a limit ordinal:

Suppose $\exists\, \gamma$ cardinal s.t. $|\omega_\alpha| = |\bigcup_{\delta\in\alpha} \omega_\delta| = \gamma$. Then $\gamma \in \omega_\alpha$, so $\exists\, \delta \in \alpha$ s.t. $\gamma \in \omega_\delta$. Since α is a limit ordinal, $\omega_{\delta+1} \subseteq \omega_\alpha$ and

$$\gamma = |\gamma| \preceq \omega_\delta < \omega_{\delta+1} \leq |\bigcup_{\delta\in\alpha} \omega_\delta| = \gamma \quad \lightning$$

Prop.

(1) The collection W is the class of all infinite cardinals.
(2) W is a proper class.

Proof:

We will only prove (2) for brevity.
By definition of W, there is a bijection $f: \Omega \longrightarrow W$, given by $f(\alpha) = \omega_\alpha$. If we assume that W is a set, then by Prop. 0 there exists an ordinal β and a bijection $g: W \longrightarrow \beta$.
Hence $g \circ f: \Omega \longrightarrow \beta$ is a bijection. However, since $2^\alpha \in \Omega$ and Ω is transitive, $2^\alpha \subset \Omega$ and $|2^\alpha| = |g(f(2^\alpha))| \leq |\alpha|$, which is a contradiction since $|\alpha| < |2^\alpha|$ by Cantor's Thm.

3. Cardinal Operations

We can define the following operations in the class of the cardinals:

- Cardinal addition: $\alpha + \beta := |\alpha \times \{0\} \dot{\cup} \beta \times \{1\}|$
- C. Multiplication: $\alpha \cdot \beta := |\alpha \times \beta|$
- C. Exponentiation: $\alpha^\beta := |{}^\beta\alpha|$

The following fact sums up some of the immediate properties of these operations.

Fact 1

- C. Addition and multiplication are commutative and ~~additive~~ associative.
- C. Multiplication is distributive over addition.
- For cardinal exponentiation we have:

(a) $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$ (b) $\kappa^{\mu \cdot \lambda} = (\kappa^\lambda)^\mu$ (c) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$

P1:

(1) Commutativity is obvious for both operations.
The associative property is also immediate from the definition

(2) $\alpha \cdot (\beta + \gamma) = \alpha \cdot |\beta \times \{0\} \dot\cup \gamma \times \{1\}| = |\alpha \times |\beta \times \{0\} \dot\cup \gamma \times \{1\}|| =$
$= |\alpha \times \beta \times \{0\} \dot\cup \alpha \times \gamma \times \{1\}| = \alpha \cdot \beta + \alpha \cdot \gamma$

(3) (a) $K^{\lambda+\mu} = K^{\lambda} \cdot K^{\mu}$

Given $f \in {}^{\lambda+\mu}K$, $f(\alpha) = \begin{cases} f_\lambda(\alpha), & \alpha \in \lambda \times \{0\} \\ f_\mu(\alpha), & \alpha \in \mu \times \{1\} \end{cases}$

~~$\varphi : K^{\lambda+\mu} \longrightarrow K$~~ $\quad \varphi : {}^{\lambda+\mu}K \longrightarrow {}^{\lambda}K \times {}^{\mu}K$ is a bijection
$f \longmapsto \langle f_\lambda, f_\mu \rangle$

- Injectivity: if $\varphi(f) = \langle 0, 0 \rangle$, then $f_\lambda(\alpha), f_\mu(\alpha) = 0 \ \forall \alpha$ so $f \equiv 0$.
- Surjectivity: ~~$\forall f \in {}^{\lambda}K$~~ $\forall g \in {}^{\lambda}K$, $h \in {}^{\mu}K$, we can define $\tilde{g} \in {}^{\lambda \times \{0\}}K$ and $\tilde{h} \in {}^{\mu \times \{1\}}K$ by $\tilde{g}(\alpha \times \{0\}) = g(\alpha)$, $\tilde{h}(\alpha \times \{1\}) = h(\alpha)$. So, we have

$f(x) = \begin{cases} \tilde{g}(x), & x \in \lambda \times \{0\} \\ \tilde{h}(x), & x \in \mu \times \{1\} \end{cases}$, $f : \lambda + \mu \longrightarrow K$

s.t. $\varphi(f) = \langle g, h \rangle$.

(b) For any function $f \in {}^{\mu}({}^{\lambda}K)$, we define $\hat{f} : \mu \times \lambda \longrightarrow K$ s.t. $\hat{f}(\langle \alpha, \beta \rangle) = f(\alpha)(\beta)$ and $\bar{\varphi} : {}^{\mu}({}^{\lambda}K) \longrightarrow {}^{\mu \times \lambda}K$ given by $\bar{\varphi}(f) = \hat{f}$

Claim: $\bar{\varphi}$ is bijective

- Injectivity: if $\tilde{f}(\langle\alpha,\beta\rangle)=0$ for $\alpha\in\mu$, $\beta\in\lambda$, then $f(\alpha)\equiv 0$ $\forall\alpha\in\mu$, so $f\equiv 0$.
- Surjectivity: for any $\tilde{f}\in{}^{\mu\times\lambda}K$, we define $f:\mu\longrightarrow{}^{\lambda}K$ as $f(\alpha)=\tilde{f}(\langle\alpha,-\rangle)$

(c) If $f\in{}^{\mu}K\times\lambda$, then $\exists\, f_K:\mu\longrightarrow K$, $f_\lambda:\mu\longrightarrow\lambda$. Then we can define the bijection $\psi:{}^{\mu}(K\times\lambda)\longrightarrow{}^{\mu}K\times{}^{\mu}\lambda$ by $\psi(f)=\langle f_K,f_\lambda\rangle$.

- Injectivity: if $\langle f_K,f_\lambda\rangle=0\Rightarrow$ ~~$\psi(f)$~~ $f\equiv 0$
- Surjectivity $\langle f_K,f_\lambda\rangle:\mu\longrightarrow K\times\lambda$.

Now we introduce the following ~~thm~~ theorem that shows how easy it is to add and multiply cardinals.

Thm. 1

For any ordinal numbers $\alpha,\beta\in\Omega$, we have

$$\omega_\alpha+\omega_\beta=\omega_\alpha\cdot\omega_\beta=\cancel{\omega_\alpha\cup\omega_\beta}\ \omega_{\alpha\cup\beta}=\max\{\omega_\alpha,\omega_\beta\}$$

Pf:

It is enough to prove that $\omega_\alpha\cdot\omega_\alpha=\omega_\alpha$

Indeed, if $\beta\in\alpha$

$$\omega_\alpha\cdot\omega_\beta=|\omega_\alpha\times\omega_\beta|\le|\omega_\alpha\times\omega_\alpha|=\omega_\alpha\cdot\omega_\alpha=\omega_\alpha$$
$$\ge|\omega_\alpha\times\{0\}|=\omega_\alpha$$

$$\omega_\alpha + \omega_\beta = |\omega_\alpha \times \{0\} \dot\cup \omega_\beta \times \{1\}| \leq |\omega_\alpha \times \{0\} \dot\cup \omega_\alpha \times \{1\}| \leq |\omega_\alpha \times \omega_\alpha| = \omega_\alpha$$
$$\geq |\omega_\alpha \times \{0\}| = \omega_\alpha .$$

and $\omega_{\alpha \cup \beta} = \max\{\omega_\alpha, \omega_\beta\}$ since the collection of cardinals is well-ordered with $\in$.

Now, let's prove the claim:

~~α_0~~ We have that $\omega_0 \cdot \omega_0 = \omega \cdot \omega = \omega$. Indeed, if we take the bijection

$$f : \omega \times \omega \longrightarrow \omega, \quad f(\alpha) = \begin{cases} \beta + \beta, & \alpha = \omega + \beta \\ \alpha + \alpha + 1, & \alpha \in \omega \end{cases}$$

it is clear that $\omega \cdot \omega = \omega$.

Now suppose there exists α_0 minimal s.t. $\omega_{\alpha_0} \cdot \omega_{\alpha_0} > \omega_{\alpha_0}$:

$$\alpha_0 = \bigcap \{\alpha \in \Omega : \omega_\alpha \cdot \omega_\alpha > \omega_\alpha\}$$

We define on $\omega_{\alpha_0} \times \omega_{\alpha_0}$ an ordering $\leq$ "$\prec$" in the following manner:

$$\langle \gamma_1, \delta_1 \rangle \prec \langle \gamma_2, \delta_2 \rangle \iff \begin{cases} \gamma_1 \cup \delta_1 \in \gamma_2 \cup \delta_2 & \text{or} \\ \gamma_1 \cup \delta_1 = \gamma_2 \cup \delta_2 \wedge \gamma_1 \in \gamma_2 & \text{or} \\ \gamma_1 \cup \delta_1 = \gamma_2 \cup \delta_2 \wedge \gamma_1 \in \gamma_2 \wedge \delta_1 \in \delta_2 \end{cases}$$

This ordering is a linear one since $\in$ is a linear ordering on Ω.

Besides, it is also a well-ordering. We show that for any subset $S \subset \omega_{\alpha_0} \times \omega_{\alpha_0}$, there exists a minimal element with the above ordering. Let us define

$$T := \{\alpha \cup \beta \in \Omega ; \langle \alpha, \beta \rangle \in S\} \subset \Omega$$

T has an $\in$-minimal element by Thm 3.3 which will be denoted by γ.

$$U := \{\langle \alpha, \beta \rangle \in S ; \alpha \cup \beta = \gamma\} \subset \Omega$$

Taking the projection on the first factor of the cartesian product which we denote by $\pi^{2}_{\#1}(\mathcal{U})$ and is contained in $\mathcal{U}\,\Omega$ has also an $\in$-minimal element, which we will call δ_1.

Then, we define $\mathcal{Z} := \{\langle \alpha, \beta \rangle \in \mathcal{S} : \alpha \cup \beta = \gamma \wedge \alpha = \delta_1\}$, and take the projection π_2 on the second factor of the cartesian product. $\pi_2(\mathcal{Z})$ has by the same argument an $\in$-minimal element, denoted by δ_2. Thus, $\langle \delta_1, \delta_2 \rangle$ is a minimal element for the ordering $\prec$ on $\mathcal{S}$.

Now, by Prop. 3.20 $\exists \eta \in \Omega$ and an unique $f : \eta \to A$ bijective s.t. $\alpha \in \beta$ iff $\alpha \prec \beta$. Then we have

- $|\eta| = |A| = |\omega_{\alpha_0} \times \omega_{\alpha_0}| > \omega_{\alpha_0}$, so $\omega_{\alpha_0} \in \eta$.
- We take $f(\omega_{\alpha_0}) = \langle \gamma, \delta \rangle$ and denote by $\nu = \max\{\gamma, \delta\} \in \omega_{\alpha_0}$

Since $\nu \in \omega_{\alpha_0}$, $|\nu| < \omega_{\alpha_0}$ but due to the bijectivity of f $\omega_{\alpha_0} \leq |\nu \times \nu|$. Taking $\omega_\beta = |\nu|$ we arrive to a contradiction.

Thanks to the previous theorem we are able to formulate the following corollary:

<u>Cor. 2</u>

If κ is an infinite cardinal,

(a) $\forall n \in \omega$, $\kappa^{n+1} = \kappa$

(b) $|\mathrm{seq}(\kappa)| = \kappa$

(c) $\kappa^{\kappa} = 2^{\kappa}$

Proof:

We proof (a) by induction on ω:

- If $n=0$, $K^{0+1}=K$
- If it is true for n, then $K^{(n+1)+1} = K^{1+1} = K$ by the theorem.

(b) $|\mathrm{seq}\, K| = |\bigcup_{n\in\omega} K^n| = |\{\emptyset\}\times\{0\} \cup K\times\{1\} \cup \dots \cup K^n\times\{n\} \cup \dots| =$

$= 1 + K + K^2 + \dots + K^n + \dots = 1 + \omega K = 1 + K = K.$

(c) $\geq]$ Since $2 < K$, $\cancel{|K^K| \geq |2^K|}$ $|{}^K K| \geq |{}^K 2|$.

$\leq]$ If we identify the graph of $f\in {}^K K$ with its graph, which is a subset of $K\times K$, we have $|{}^K K| \leq |\mathcal{P}(K\times K)| = |\mathcal{P}(K)| = 2^K$

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Def.: Let λ be an infinite limit ordinal. $C \subseteq \lambda$ is called <u>cofinal</u> in λ if $\bigcup C = \lambda$. The <u>cofinality</u> of λ, denoted $cf(\lambda)$, is the cardinality of a smallest cofinal set $C \subseteq \lambda$:

$$cf(\lambda) := \min \{|C| : C \text{ cofinal in } \lambda\}$$

where the minimum is to be understood as the (unique) $\in$-minimal element of the collection $\{|C| : C \text{ cofinal in } \lambda\}$. Thm. 3.3. (d) and the fact that $\lambda = \bigcup \lambda$ guarantee well-definedness.

Remark: By definition, $cf(\lambda)$ is a cardinal number.

Construction of cofinal sequences: λ an infinite limit ordinal, $C = \{\beta_\xi : \xi \in cf(\lambda)\} \subseteq \lambda$ cofinal in λ.

$\forall \nu \in cf(\lambda)$ we define $\alpha_\nu := \bigcup \{\beta_\xi : \xi \in \nu\}$ (note that $\forall \nu \in cf(\lambda)\ (\alpha_\nu \in \lambda)$)

$\Rightarrow \langle \alpha_\nu : \nu \in cf(\lambda) \rangle$ is an inc. sequence (meaning that $\forall \mu, \nu \in cf(\lambda)\ (\mu \in \nu \rightarrow \alpha_\mu \in \alpha_\nu)$) of length $cf(\lambda)$ with $\bigcup \{\alpha_\nu : \nu \in cf(\lambda)\} = \lambda$

$\Rightarrow \langle \alpha_\nu : \nu \in cf(\lambda) \rangle$ is a cofinal sequence of λ

Remark: Every infinite cardinal is an infinite limit ordinal.

Def.: An infinite cardinal κ is called <u>regular</u> if $cf(\kappa) = \kappa$; otherwise, κ is called <u>singular</u>.

Example: (1) ω is regular.

(2) $\omega_\omega := \bigcup_{n \in \omega} \omega_n$ is singular because $\{\omega_n : n \in \omega\} \subseteq \omega_\omega$ is cofinal.

(3) $\forall$ limit ordinals $\lambda \neq \emptyset$: $cf(\omega_\lambda) = cf(\lambda)$.

(4) $cf(\omega_\omega) = cf(\omega_{\omega+\omega}) = cf(\omega_{\omega_\omega}) = \omega$.

Fact 3.27.: $\forall$ infinite limite ordinals λ, the cardinal $cf(\lambda)$ is regular, i.e. $cf(cf(\lambda)) = cf(\lambda)$.

Proposition 3.28.: If κ is an infinite cardinal, then $\kappa^+ := \bigcap \{\alpha \in \Omega : \kappa < |\alpha|\}$ is regular.

Proof: Assume towards a contradiction that $\exists$ cofinal subset $C \subseteq \kappa^+$ with $|C| < \kappa$, i.e., $|C| \leq \kappa$ ($|C| \in \kappa \vee |C| = \kappa$). $C \subseteq \kappa^+ \Rightarrow \forall \alpha \in C\ (|\alpha| \leq \kappa)$. $AC \Rightarrow \forall \alpha \in C\ \exists f_\alpha : \alpha \hookrightarrow \kappa$. In particular, $\forall \alpha \in C\ \forall \nu \in \alpha\ (f_\alpha(\nu) \in \kappa)$. Furthermore, $\exists g : C \hookrightarrow \kappa$.

$\Rightarrow \{\langle g(\alpha), f_\alpha(\nu) \rangle : \alpha \in C \wedge \nu \in \alpha\} \subseteq \kappa \times \kappa \Rightarrow |\bigcup C| = |\kappa \times \kappa| = \kappa$

$\Rightarrow C$ is not cofinal in κ^+ ↯

⊣

Def.: Let $I \neq \emptyset$ be a set and $\{\kappa_\iota : \iota \in I\}$ a family of cardinals.

(1) $\sum_{\iota \in I} \kappa_\iota := |\bigcup_{\iota \in I} A_\iota|$ where $\{A_\iota : \iota \in I\}$ is a family of pairwise disjoint sets s.t. $|A_\iota| = \kappa_\iota$ $\forall \iota \in I$, e.g., $A_\iota := \kappa_\iota \times \{\iota\}$ works.

(2) $\prod_{\iota \in I} \kappa_\iota := |\bigtimes_{\iota \in I} A_\iota|$ where $\{A_\iota : \iota \in I\}$ is a family of sets s.t. $|A_\iota| = \kappa_\iota$ $\forall \iota \in I$, e.g., $A_\iota = \kappa_\iota$ works.

Theorem 3.29. (Inequality of König-Jourdain-Zermelo): Let I be a non-empty set and $\{\kappa_\iota : \iota \in I\}$ and $\{\lambda_\iota : \iota \in I\}$ families of cardinal numbers s.t. $\kappa_\iota < \lambda_\iota$ $\forall \iota \in I$. Then

$$\sum_{\iota \in I} \kappa_\iota < \prod_{\iota \in I} \lambda_\iota .$$

Proof: $\{A_\iota : \iota \in I\}$ a family of pairwise disjoint sets s.t. $|A_\iota| = \kappa_\iota$ $\forall \iota \in I$.
$\forall \iota \in I$ $\exists$ injection $f_\iota : A_\iota \hookrightarrow \lambda_\iota$ and $\exists y_\iota \in \lambda_\iota \setminus f_\iota[A_\iota]$. Existence of y_ι is guaranteed because $|A_\iota| < \lambda_\iota$ and hence $\lambda_\iota \setminus f_\iota[A_\iota] \neq \emptyset$.

(1) $\sum_{\iota \in I} \kappa_\iota \leq \prod_{\iota \in I} \lambda_\iota$: $\forall \iota \in I$ define $\bar{f}_\iota : \bigcup_{\iota \in I} A_\iota \to \lambda_\iota$, $x \mapsto \bar{f}_\iota(x) := \begin{cases} f_\iota(x) & \text{if } x \in A_\iota \\ y_\iota & \text{otherwise.} \end{cases}$

And define $\bar{f} : \bigcup_{\iota \in I} A_\iota \to \bigtimes_{\iota \in I} \lambda_\iota$, $x \mapsto \langle \bar{f}_\iota(x) : \iota \in I \rangle$. By definition, $\bar{f}$ is injective.

$\Rightarrow \sum_{\iota \in I} \kappa_\iota < \prod_{\iota \in I} \lambda_\iota$

(2) $\sum_{\iota \in I} \kappa_\iota < \prod_{\iota \in I} \lambda_\iota$: It is sufficient to show that $\nexists$ bijection between $\bigcup_{\iota \in I} A_\iota$ and $\bigtimes_{\iota \in I} \lambda_\iota$.

Take any function $g : \bigcup_{\iota \in I} A_\iota \to \bigtimes_{\iota \in I} \lambda_\iota$, and $\forall \iota \in I$, let $P_\iota(g[A_\iota])$ be the projection of $g[A_\iota]$ on λ_ι. $|A_\iota| < \lambda_\iota$ $\forall \iota \in I$ $\Rightarrow \exists z_\iota \in \lambda_\iota \setminus P(g[A_\iota]) \Rightarrow \langle z_\iota : \iota \in I \rangle \in \bigtimes_{\iota \in I} \lambda_\iota \setminus g[\bigcup_{\iota \in I} A_\iota]$
$\Rightarrow g$ is not surjective $\Rightarrow g$ is not bijective $\dashv$

Corollary 3.30.: $\forall$ infinite cardinal κ : (1) $\kappa < \kappa^{cf(\kappa)}$

(2) $cf(2^\kappa) > \kappa$.

In particular: $cf(2^\omega) > \omega$.

Proof: (1): $\langle \alpha_\nu : \nu \in cf(\kappa) \rangle$ a cofinal sequence of κ

$$\Rightarrow \quad \kappa = \left| \bigcup_{\nu \in cf(\kappa)} \alpha_\nu \right| \le \sum_{\nu \in cf(\kappa)} |\alpha_\nu| \le cf(\kappa) \cdot \kappa = \kappa$$

$$\Rightarrow \quad \kappa = \sum_{\nu \in cf(\kappa)} |\alpha_\nu|$$

$$\forall \nu \in cf(\kappa) : |\alpha_\nu| < \kappa \overset{\text{Thm. 3.29.}}{\Rightarrow} \sum_{\nu \in cf(\kappa)} |\alpha_\nu| < \prod_{\nu \in cf(\kappa)} \kappa = \kappa^{cf(\kappa)}$$

$$\Rightarrow \quad \kappa < \kappa^{cf(\kappa)}$$

(2): Assume towards a contradiction that $cf(2^\kappa) \le \kappa$.

$$cf(2^\kappa) \le \kappa \Rightarrow (2^\kappa)^{cf(2^\kappa)} \le (2^\kappa)^\kappa = 2^{\kappa \cdot \kappa} = 2^\kappa$$

$\Rightarrow (2^\kappa)^{cf(2^\kappa)} \le 2^\kappa$ but by (1) $2^\kappa < (2^\kappa)^{cf(2^\kappa)}$ ↯ ⊣