A.9 - P_n and the Prime Ideal Theorem

Axiom of Choice Seminar A

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1 Recall

Last week's presentation showed the equivalence of the Prime Ideal Theorem to several other theorems and principles, in particular the Ultrafilter theorem. We recall the Prime Ideal Theorem (PIT) and the Ultrafilter Theorem (UT).

Theorem 1 (Prime Ideal Theorem)**.** *If I is an ideal in a Boolean algebra, then I can be extended to a prime ideal.*

Theorem 2 (Ultrafilter Theorem)**.** *If F is a filter over a set S, then F can be extended to an ultrafilter.*

This week, we want to give a graph-theoretical statement which we will show is equivalent to the PIT.

2 Introduction

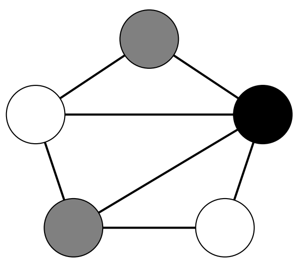
We introduce the graph-theoretical notion of P_n. For this we define what a subgraph is, and what n-colourable means.

Definition 3 (Subgraph)**.** $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$, and for all $x, y \in V'$,

$$\{x, y\} \in E' \iff \{x, y\} \in E.$$

Definition 4 (n-colourable)**.** A graph $G = (V, E)$ is n-colourable it there exists a colouring function $\gamma : V \to \{1, \ldots, n\}$ such that for all $x, y \in E$ with $x \neq y$, $\gamma(x) \neq \gamma(y)$.

For example, the following graph is 3-colourable with the colours white, grey and black.



Now we can define P_n for positive integers n.

P_n: If every finite subgraph of G is n-colourable, then G is n-colourable.

3 Theorem 6.10

Today, our goal is to show the following theorem from [1].

Theorem 5 (Theorem 6.10)**.** *P_3 is equivalent to the Prime Ideal Theorem.*

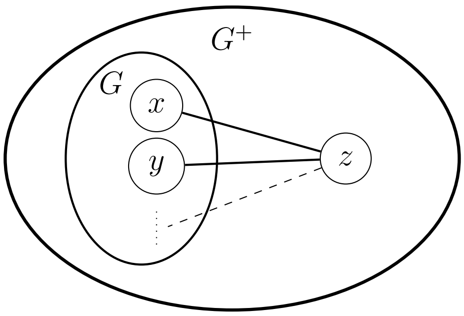
In order to prove Theorem 6.10, we first state and prove some auxiliary results.

Fact 6 (Fact 6.8)**.**

$$P_{n+1} \implies P_n$$

Proof of Fact 6. Let $G_n = (V, E)$ be a graph such that every finite subgraph is n-colourable. We are going to extend this graph in a fashion such that the extended graph is $(n+1)$-colourable, and from that we can extract an n-colouring for G.

We choose a new vertex $z \notin V$, which will have edges to all the vertices in V. The new graph $G^+ := (V \cup \{z\}, E \cup \{(x, z) : x \in V\})$ looks like this:



Now, we want to show that G^+ is $(n+1)$-colourable. Therefore, we look at finite subgraphs of G^+. A finite subgraph of G^+ either contains only vertices of V, or includes the vertex z. In the first case, the subgraph is n-colourable, and thus, also $(n+1)$-colourable. In the second case, we know that z has to have a different colour than the other vertices because z is connected to all the other vertices. As the subgraph of this subgraph without z is n-colourable, the subgraph with the vertex z is $(n+1)$-colourable with the n-colouring of before and z has the new colour. Therefore, we can apply P_{n+1} to G^+, and obtain that G^+ is $(n+1)$-colourable.

As mentioned before, z has to have a different colour than the rest of the vertices. By leaving out that colour and ignoring the vertex z, we get an n-colouring of G. This concludes the proof of Fact 6.8. □

Lemma 7 (Lemma 6.9)**.** *The Ultrafilter Theorem implies P_n for all integers n.*

Proof of Lemma 7. Let $G = (V, E)$ be a graph such that all finite subgraphs are n-colourable. We define the set S to be the set of all functions $f : V \to \{1, \ldots, n\}$. The idea of the proof is to define a filter over this set S, which we can extend to an ultrafilter using the Ultrafilter Theorem. Then, we have to show that from this ultrafilter we can extract an n-colouring for the graph G.

We start by defining the sets of n-colourings for finite subgraphs. For $A \in \text{fin}(V)$, i.e. for a finite subset of V, we set

$$\chi_A \coloneqq \{f \in S : f|_A \text{ is an } n\text{-colouring of } G|_A\}$$

For the empty set we have $\chi_\emptyset = S$. Now let

$$\mathcal{F} \coloneqq \{X \subseteq S : \exists A \in \text{fin}(V) \text{ s.t. } \chi_A \subseteq X\}$$

We want to show that $\mathcal{F}$ is a filter. We are showing the properties of a filter step by step:

1. $\emptyset \notin \mathcal{F}$: since G is a graph where all finite subgraphs are n-colourable, χ_A is not empty for all $A \in \text{fin}(V)$. Thus, $\emptyset \notin \mathcal{F}$.

2. $S \in \mathcal{F}$: for $X = S$, the condition is satisfied with $A = \emptyset$ because $\chi_\emptyset = S$.

3. $(X \in \mathcal{F} \vee Y \in \mathcal{F}) \to X \cup Y \in \mathcal{F}$: w.l.o.g. we assume $X \in \mathcal{F}$. Then there exists an $A \in \text{fin}(V)$ such that $\chi_A \subseteq X$. It also holds that $\chi_A \subseteq X \cup Y$. Therefore, this condition is satisfied.

4. $(X \in \mathcal{F} \wedge Y \in \mathcal{F}) \to X \cap Y \in \mathcal{F}$: if $X, Y \in \mathcal{F}$, then there exist $A, B \in \text{fin}(V)$ such that $\chi_A \subseteq X$ and $\chi_B \subseteq Y$. The property follows from the fact that $\chi_{A \cup B} \subseteq X \cap Y$ because $\chi_{A \cup B} \subseteq \chi_A \cap \chi_B$. This is the case because an n-colouring of the graph with vertices $A \cup B$ is also an n-colouring for all its subgraphs. Thus, also for the subgraphs with vertices A and B.

Using the UT, we can extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$. From this ultrafilter we are going to extract an n-colouring for G. First, we take a look at the functions from S which map a certain vertex to a certain colour. In particular, for $x \in V$ and for $i \in \{1, \ldots, n\}$, $u_{x,i} \coloneqq \{f \in S : f(x) = i\}$. These sets are a partition of S, i.e.

$$\bigcup_{i=1}^{n} u_{x,i} = S,$$

and for $i \neq j$

$$u_{x,i} \cap u_{x,j} = \emptyset.$$

Since the $u_{x,i}$'s are disjoint for $x \in V$, and $\emptyset \notin \mathcal{F}$, also not in $\mathcal{U}$, we can conclude from the filter properties that either no $u_{x,i}$ is in $\mathcal{U}$, or there exists one $i \in \{1, \ldots, n\}$ such that $u_{x,i} \in \mathcal{U}$.

To show that we can find a colour for each $x \in V$, we take a look at the complement of $u_{x,i}$, namely

$$\bar{u}_{x,i} \coloneqq \{f \in S : f(x) \neq i\}$$

Because $\mathcal{U}$ is an ultrafilter, we have that eiter $u_{x,i} \in \mathcal{U}$, or $\bar{u}_{x,i} \in \mathcal{U}$. If there is an $i_x \in \{1, \ldots, n\}$ such that $u_{x,i} \in \mathcal{U}$, we have what we want. Now suppose that $\bar{u}_{x,i} \in \mathcal{U}$ for all $i \in \{1, \ldots, n\}$. Then the intersection of all these sets is the empty set, i.e. $\bigcap_{i=1}^{n} \bar{u}_{x,i} = \emptyset$, which is not in the ultrafilter $\mathcal{U}$. However, because all the $\bar{u}_{x,i}$'s are in $\mathcal{U}$, the intersection has to be in $\mathcal{U}$ as well, due to the filter properties. Thus, we obtain a contradiction, and for all $x \in V$ there exists an $i_x \in \{1, \ldots, n\}$ such that $u_{x,i_x} \in \mathcal{U}$. This i_x is unique because above we showed that there exists at most one such colour. Note that the uniqueness also directly follows from Fact 6.5 (a).

At last, we can define the following n-colouring:

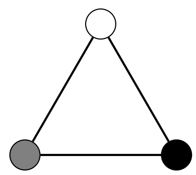
$$\begin{aligned} \gamma : \quad & V \to n \\ & x \mapsto i_x \end{aligned}$$

This is indeed an n-colouring because for all finite sets $A \in \operatorname{fin}(V)$, $\gamma|_A$ is an n-colouring by construction of our ultrafilter. Thus, for $x, y \in E$ with $x \neq y$, $\gamma(x) \neq \gamma(y)$. This concludes the proof of Lemma 6.9. □

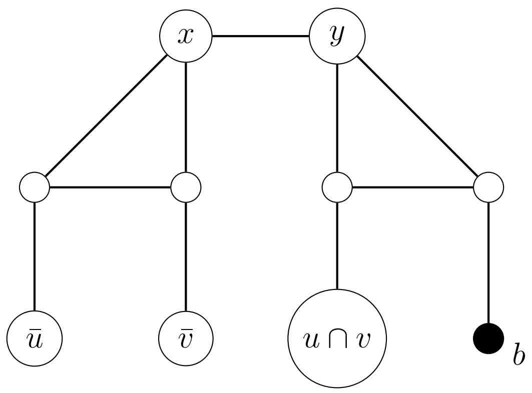
Knowing the Ultrafilter Theorem to be equivalent to the PIT, it suffices to show that P_3 implies UT. The converse implication follows by Lemma 7.
Hence, assuming P_3, we show how a filter $\mathcal{F}$ over a set S can be extended to an ultrafilter $\mathcal{U}$. First, starting from the vertex set $\mathscr{P}(S)$, we construct a graph $G_\mathcal{F} = (V_\mathcal{F}, E_\mathcal{F})$ in such a way that the vertices contained in $\mathcal{F}$ are colored in the same color. We then show that every finite subgraph of $G_\mathcal{F}$ is 3-colorable. We can then define the ultrafilter $\mathcal{U}$ using the 3-coloring of $G_\mathcal{F}$ we obtain from P_3.

Proof of Theorem 6.10. Let $S \neq \varnothing$ and let $\mathcal{F} \subseteq \mathscr{P}(S)$ be a filter over S. We start with the graph $G = (V, E)$ on the vertex set $V := \mathscr{P}(S)$ and edge set E obtained by connecting every vertex $u \subseteq S$ to its complement $\bar{u} := S \setminus u$, i.e. $E := \{\{u, \bar{u}\} : u \in \mathscr{P}(S)\}$.

To G we add the *fundamental triangle* Δ: this consists of 3 connected vertices, each colored differently, say one black (b), one white (w) and one gray (g).



We now connect every vertex in V with the gray vertex of Δ, and for each pair of vertices $u, v \subseteq S, u \neq v$, we add the following graph:

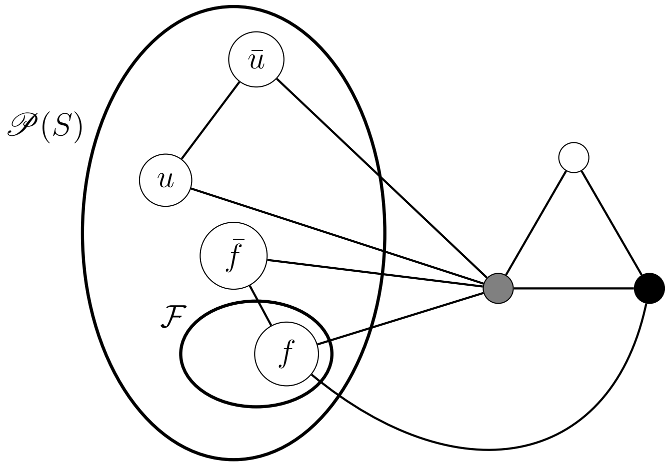


(†)

Nota bene: in each of these graphs, the vertex b is the black-colored vertex of Δ.

Lastly, we connect every vertex $f \in \mathcal{F}$ with the vertices (b) and (g) of Δ. The resulting graph is $G_{\mathcal{F}} = (V_{\mathcal{F}}, E_{\mathcal{F}})$.

Up until now we did not pay much attention to coloring our graph, so now we turn our focus to this, and show how to get every vertex in $\mathcal{F}$ to be colored white.
Notice first that for any vertex $u \subseteq S$ and any vertex $f \in \mathcal{F}$ we have the following situation:



We can see that by connecting every vertex $u \subseteq S$ to the gray vertex of Δ, the two vertices u and $\bar{u}$ are colored either black or white. In particular, no vertex $u \subseteq S$ gets to be colored gray.
Similarly, for every pair $u, v \subseteq S$, the construction (†) guarantees that at least one of $\bar{u}, \bar{v}$ and $u \cap v$ is assigned the color white. Indeed, since $\bar{u}, \bar{v}, u \cap v \subseteq S$, none of them is colored gray, and assuming all of them to be colored black would imply that the vertices denoted in (†) by x and y would also be colored black, what would constitute a contradiction.
Lastly, by connecting every vertex $f \in \mathcal{F}$ to both black and gray vertices of Δ, makes sure that all the vertices in the filter $\mathcal{F}$ are colored white. Notice that by the definition of a filter, this is in fact possible and does not lead to any contradictions. Indeed, if $f \in \mathcal{F}$ then $\bar{f} \notin \mathcal{F}$, hence every triangle $(f, \bar{f}, (g))$ is well-colored, and because $f, h \in \mathcal{F}$ implies $f \cap h \in \mathcal{F}$, the bottom row of vertices in (†) cannot all be colored black. Hence the graph (†) is properly 3-colored, as argued above.

We now state the following

Claim. *Every finite subgraph of $G_{\mathcal{F}}$ is 3-colorable.*

Assuming this claim, the entire graph $G_{\mathcal{F}}$ is 3-colorable by P_3.
Let $\gamma : V_{\mathcal{F}} \to \{g, b, w\}$ be such a 3-coloring of $G_{\mathcal{F}}$, and define

$$\mathcal{U} := \{u \subseteq S : \ \gamma(u) = w\}.$$

By the construction of $G_{\mathcal{F}}$, it holds that $\mathcal{F} \subseteq \mathcal{U}$, and we can also notice that since $\forall u \subseteq S, \ \{u, \bar{u}\} \in E_{\mathcal{F}}$, we have for each vertex $u \subseteq S$ of $G_{\mathcal{F}}$ that either $u \in \mathcal{U}$ or $\bar{u} \in \mathcal{U}$. Moreover, it holds that for any $u, v \in \mathcal{U}$, also $u \cap v \in \mathcal{U}$ (else the bottom row of (†) would all be colored black, a contradiction as previously discussed).
Finally, from the definition of filter, $S \in \mathcal{F}$. Hence the vertex $S \in V_{\mathcal{F}}$ is colored white and, in particular, $\varnothing \notin \mathcal{U}$. This implies that if $u \in \mathcal{U}$ and $u \subseteq v$, then also $v \in \mathcal{U}$ (indeed we would otherwise have $\bar{v} = S \setminus v \in \mathcal{U}$ and hence $u \cap \bar{v} = \varnothing \in \mathcal{U}$, a contradiction). This shows that $\mathcal{U}$ is an ultrafilter extending $\mathcal{F}$, proving the UT.

In order to fully complete the proof of the theorem, it remains to prove the claim.

From our construction, all vertices in $\mathcal{F}$ are necessarily colored white, while all their complements have to be colored black. All other vertices $u \subseteq S$ can be colored either way, black or white. Therefore we can divide a finite set of vertices A into those we already have colored, i.e. the vertices in $\mathcal{F}$ and their complements, which we denote by $\bar{\mathcal{F}} := \{\bar{u} : \ u \in \mathcal{F}\}$, and those vertices whose color is yet to be determined. Because we have a finite number of such vertices, we can verify step by step what color to assign to each of them, obtaining a coloring which can be extended to a 3-coloring of the subgraph.

Proof of Claim. Let $A \subseteq V_{\mathcal{F}}$ be an arbitrary *finite* set of vertices of $G_{\mathcal{F}}$. Without loss of generality we can assume that with every vertex $u \subseteq S$ contained in A, also $\bar{u} \in A$. We define

$$U := A \cap \mathcal{F} \qquad V := (A \cap \mathscr{P}(S)) \setminus (\mathcal{F} \cup \bar{\mathcal{F}}).$$

These two sets are the vertices in A which are colored white and the set of vertices for which we still have to choose their color, respectively.
Notice that since U is a finite subset of the filter $\mathcal{F}$, it holds that $\varnothing \neq \bigcap U \in \mathcal{F}$, and since V is finite, $V = \{v_0, ..., v_k\}$ for some $k \in \omega$.

Define $w_0 := v_0 \cap \bigcap U$.
If $w_0 \neq \varnothing$, we color v_0 white, else $\bar{v}_0$ is assigned the color white. Let us denote by $\hat{v_0}$ the vertex we color white between v_0 and $\bar{v}_0$.

Proceeding to v_1, we define $w_1 := v_1 \cap \hat{v_0} \cap \bigcap U$.
Analogously as above, if $w_1 \neq \varnothing$, we assign the color white to v_1, else to $\bar{v_1}$.

We can continue this way until all of the vertices of V are assigned a color, and can therefore extend this coloring to a 3-coloring of the subgraph $G_{\mathcal{F}}|_A$. Since A was arbitrary, this shows that every finite subgraph of $G_{\mathcal{F}}$ is 3-colorable.

■

This completes the proof of the theorem. □

A direct application of the theorem, together with Lemma 7 and Fact 6 shows the following

Corollary. *For all* $n, m \geq 3$ *we have* $P_n \Leftrightarrow P_m$.

It should be noted that it has been shown that $P_2 \Rightarrow P_3$ is not provable in ZF.

This is reminiscent of computational complexity theory. In this context, one can prove that the problem n-col is a so called *NP-complete* problem for every $n \geq 3$. Intuitively speaking, this means that all these problems are equally hard to solve. The 2-col problem, however, stands out for being the only problem (of this type) for which we know an algorithm solving it, and intuitively is seen to be easier than 3-col. It seems not possible, however, to reduce the 3-col problem to the 2-col one, what would answer the open question known as "$P \stackrel{?}{=} NP$".

References

[1] Lorenz J. Halbeisen. *Combinatorial set theory, with a gentle introduction to forcing*. Springer, 2nd edition, 2017.