B5: König's Lemma and other choice principles (part 1, Daiki Brender)

1. Introduction

Why do we need the axiom of choice?

Let us look at König's lemma (KL): Every infinite, finitely branching tree contains an infinite branch.

In order to prove this, we might say let v_0 be the root. Since the tree is infinite, but finitely branching, $\exists$ neighbour of v_0 from which we can reach infinitely many vertices without going back to v_0. Let this neighbour be denoted by v_1. Then we do the same procedure from v_1 to pick a new neighbour v_2 and so on. At the end, we get the infinite branch $(v_0, v_1, v_2, \dots)$.

But here is the catch (in fact, there are two subtilities):

1) In order to prove that the set of neighbours of v_0 from which we reach infinitely many vertices without going back to v_0 is not empty, we need an infinite version of the Pigeon-Hole Principle:
   If infinitely many objects are coloured with finitely many colours, then infinitely many objects have the same colour.
2) Now we know that the set as mentioned in 1) is non-empty. But the next question is which element should we choose from this set?
   For this, we need some kind of choice function which selects infinitely often one vertex from a given non-empty set of vertices.
   crucial

The goal of this section is to show the following implication graph (the abbreviations will be defined in the next page):

(AC = axiom of choice)

$$\begin{tikzcd}
 & AC \arrow[d, Rightarrow] \arrow[drdr, Rightarrow, to path={}] \arrow[ddrr, Rightarrow] & & \\
 & \text{countable choice} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & & \\
C(\aleph_0,\aleph_0) \arrow[d, Rightarrow] & RPP \arrow[r, Rightarrow] \arrow[d, Rightarrow] & CaCP & \text{finite choice} \arrow[d, Rightarrow] \\
C(\aleph_0,<\aleph_0) \arrow[r, Leftrightarrow] \arrow[dr, Rightarrow] & KL \arrow[d, Rightarrow] & & C(\omega,n) \arrow[dll, Rightarrow] \\
 & C(\aleph_0,n) \arrow[r, Rightarrow] & C(\aleph_0,m) \text{ if } m \text{ divides } n &
\end{tikzcd}$$

König's lemma and other choice principles

Now let us define some choice principles:

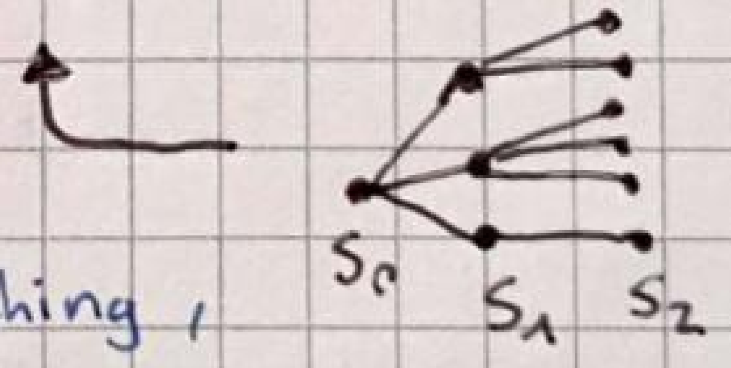
- $C(\aleph_0, \infty)$: Every countable family of non-empty sets has a choice fct. (usually called countable axiom of choice)
- $C(\aleph_0, \aleph_0)$: Every countable family of non-empty countable sets has a choice fct.
- $C(\aleph_0, < \aleph_0)$: Every countable family of non-empty finite sets has a choice fct.
- $C(\aleph_0, n)$: Every countable family of n-element sets, where $n \in \omega$, has a choice fct.
- $C(\infty, \aleph_0)$: Every family of non-empty finite sets has a choice fct. (usually called axiom of choice for finite sets)
- $C(\infty, n)$: Every family of n-element sets, where $n \in \omega$, has a choice fct. (usually denoted by C_n)
- Ramseyan Partition Principle (RPP): If X is an infinite set and $[X]^2$ is 2-coloured, then $\exists$ infinite subset Y of X s.t. $[Y]^2$ is monochromatic.
  - $[X]^2$: set of all 2-elements subsets
  - 2-coloured: for each element in the set, we assign one of the two colours
- Chain anti-chain principle (CaCP): Every infinite partially ordered set contains an infinite chain or an infinite antichain.

Prop.: $C(\aleph_0, < \aleph_0) \iff KL$

↳ Proof: "⇒" Let $T = (V, E)$ be an infinite, finitely branching tree with vertex set V, edge set E and let v_0 be the root. We note that the edge set E can be considered as a subset of $V \times V$, i.e. as a set of ordered pairs of vertices indicating the direction from the root to the top of the tree.

Let $S_0 := \{v_0\}$, and for $n \in \omega$, $S_{n+1} := \{v \in V : \exists u \in S_n (\langle u, v \rangle \in E)\}$ (order matters)

and let $S := \bigcup_{n \in \omega} S_n$



Since T is infinite and finitely branching, S is infinite and for every $n \in \omega$, S_n is a non-empty finite set.

(otherwise the tree would stop and is not infinite anymore)

For every $v \in S$, let $S(v) = \{u \in S : \exists$ non-empty finite seq. $s \in seq(S)$ of length $k+1$ (for some $k \in \omega$) with $s(0) = v$ and $s(k) = u$, and for all $i < k$, $\langle s(i), s(i+1) \rangle \in E\}$

(i.e. $S(v)$ is the set of all vertices which can be reached from v.)

(Note that $(S(v), E|_{S(v)})$ is a subtree of T)

Since S is infinite and for all $n \in \omega$, $\bigcup_{i \in n} S_i$ is finite (each S_i is finite)

$\Rightarrow \forall n \in \omega \ \exists v \in S_n$ s.t. $S(v)$ is infinite

- $\forall n \in \omega$: for each step in the tree
- $\exists v \in S_n$: where we have from this edge onward an infinite tree
- we need this part to make sure that we don't already have infiniteness (so it has to be infinite after)

for any two elements of S_n we have an order
and any non-empty subset of S_n has a minimal element w.r.t. w

$\forall n \in \omega$, let $W_n := \{ w \subseteq \underbrace{S_n \times S_n}_{\text{finite}} : w \text{ is a well-ordering of } S_n \}$

Then, $\mathcal{W} := \{ W_n : \underbrace{n \in \omega}_{\text{countable}} \}$ is a countable family of finite sets.

By our assumption that $C(\aleph_0, < \aleph_0)$ holds, $\exists$ choice fct. for $\mathcal{W}$ which selects from each set W_n a well-ordering $w_n \in W_n$.

Finally, for each $n \in \omega$, construct a branch $v_0, v_1, \dots, v_n, \dots$ through T, where

$\forall n \in \omega$, v_{n+1} is the w_n-minimal element of the non-empty set $\{ v \in S_{n+1} : \langle v_n, v \rangle \in E \wedge \text{"} S(v) \text{ is infinite"} \}$

(This ensures us that we have constructed an infinite branch)

"$\Leftarrow$" let $\mathcal{F} = \{ \underbrace{F_n}_{\text{some finite set}} : n \in \omega \}$ be a countable family of non-empty finite sets.

let $V = \bigcup_{k \in \omega} \left(\bigtimes_{n \in k} F_n \right)$ and $E \subseteq V \times V$ be the set of all ordered pairs $\langle s, t \rangle$ of the forms $s = \langle x_0, \dots, x_n \rangle$ and $t = \langle x_0, \dots, x_n, x_{n+1} \rangle$, where for each $i \in n+2$, $x_i \in F_i$.

(i.e. seq. t is obtained by adding an element of F_{n+1} to s) (Think of $\bigtimes_{n \in k} F_n$ as a path which has the form $\langle x_0, \dots, x_n \rangle \in F_1 \times \dots \times F_n$ and $\bigcup_{k \in \omega} \left(\bigtimes_{n \in k} F_n \right)$ is the union of all paths. So F_1 = "root", F_2 = "next step" and so on)

Then $T := (V, E)$ is an infinite, finitely branching tree.

$\uparrow$ b/c. countable union of non-empty finite set

$\uparrow$ b/c. F_i is finite for all i

König's lemma
$\parallel$

By KL, $\exists$ infinite branch, say $\langle a_n : n \in \omega \rangle$. Since, for all $n \in \omega$, $a_n(n) \in F_n$,

the fct. : $f : \mathcal{F} \to \bigcup \mathcal{F}$ is a choice fct. for $\mathcal{F}$.

$F_n \mapsto a_n(n)$

Since the countable family of finite sets $\mathcal{F}$ was arbitrary, we get $C(\aleph_0, < \aleph_0)$ $\square$

Rmk. :
- $C(\aleph_0, < \aleph_0) \Rightarrow C(\aleph_0, n)$ for all positive integers $n \in \omega$
- For each $n \geq 2$, $C(\aleph_0, n)$ is not provable with ZF (case $n = 2$, see Prop. 8.7 in book)

Lemma 6.13: $C(\aleph_0, \omega) \Rightarrow \forall$ infinite set X $\exists$ inj. $f : \omega \hookrightarrow X$.

$\hookrightarrow$ Proof: let X be an infinite set and $\forall n \in \omega$, F_n = set of all injections from n into X.

(For $n = 0$, $F_0 = \{ \emptyset \}$). Consider the family $\mathcal{F} := \{ F_n : n \in \omega \}$. Since X is infinite,

$\mathcal{F}$ is a countable (b/c. of ω) family of non-empty (since X is infinite) sets. By $C(\aleph_0, \omega)$, $\exists$ choice fct. g for $\mathcal{F}$.

$\forall n \in \omega$, let $f_n := g(F_n)$ (Now we have countably many injections f_n)

Thus we can construct an injection $f : \omega \hookrightarrow X$ $\square$

$n \in \omega \mapsto f_n$

($\uparrow$ If $f_{n_1} = f_{n_2} \Rightarrow g(F_{n_1}) = g(F_{n_2}) \overset{g \text{ is choice fct.}}{\Rightarrow} n_1 = n_2$)

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Recall:

set of all 2-element subsets (so all comb. when picking 2 elements of it)

Ramseyan Partition Principle (RPP): If X is an infinite set and $[X]^2$ is 2-coloured, then $\exists$ infinite subset Y of X s.t. $[Y]^2$ is monochromatic.

for each element in the set, we assign one of the two colours

Thrm.: $C(\aleph_0,\infty) \Rightarrow RPP \Rightarrow KL \Rightarrow C(\aleph_0,n)$

↳ Proof: T.S: $C(\aleph_0,\infty) \Rightarrow RPP$: let X be an arbitrary infinite set. By prev. Lemma, $\exists$ injection $f: \omega \hookrightarrow X$. For $n \in \omega$, let $a_n := f(n)$ and $S := \{a_n : n \in \omega\} \subseteq X$.

Notice that every 2-colouring of $[X]^2$ induces a 2-colouring of $[S]^2$.

By Ramsey's Thrm. (see Thrm. 4.1 in the book), $\exists$ infinite subset Y of S s.t. $[Y]^2$ is monochromatic. (Rmk.: No non-trivial form of choice is needed to establish Ramsey's Thrm. for countable sets.)

T.S: $RPP \Rightarrow KL$: let $T=(V,E)$ be an infinite, finitely branching tree and $S_0 := \{v_0\}$, where v_0 is root, and $S_{n+1} := \{v \in V : \exists u \in S_n (\langle u,v \rangle \in E)\}$ for $n \in \omega$.

Define the colouring $\pi : [V]^2 \to \{0,1\}$ as follows: $\pi(\{u,v\}) = 0 \Leftrightarrow \{u,v\} \subseteq S_n$ for some $n \in \omega$.

If the pair is on the same level, colour it by 0

By RPP, $\exists$ infinite subset $Y \subseteq V$ s.t. $[Y]^2$ is monochromatic.

Since T is finitely branching, $[Y]^2$ is of colour 1 (suppose $[Y]^2$ is of colour 0, then to be infinite, we must have $[Y]^2 = \bigcup_{n \in \omega} S_n = V$, but by assumption V has 2-colouring. ↯)

(This means that no two distinct elements of Y are in the same set S_n, i.e. not on the same level.)

Since Y is infinite, we can construct a branch $y_0, \dots, y_n, \dots$ through T, where for all $n \in \omega$, $y_n \in Y$ is the unique element in S_n. (For each level, we only have one element $y_n \in Y$ in S_n.)

T.S: $KL \to C(\aleph_0,n)$: $KL \Leftrightarrow C(\aleph_0,<\aleph_0)$ and $C(\aleph_0,<\aleph_0) \Rightarrow C(\aleph_0,n)$

□

Das Auswahlaxiom B.5

König's Lemma und weitere Choice-Prinzipien, 2. Teil

Ruben Scherrer

The goal of the second part of the presentation is to show further conditionals concerning certain forms of choice.

1 Chain Antichain Principle

Firstly, we consider the so called Chain Antichain Principle. For this, recall the following simple definitions:

Partially Ordered Set A set with a binary relation $(P, \leq)$ that is reflexive, anti-symmetric and transitive.

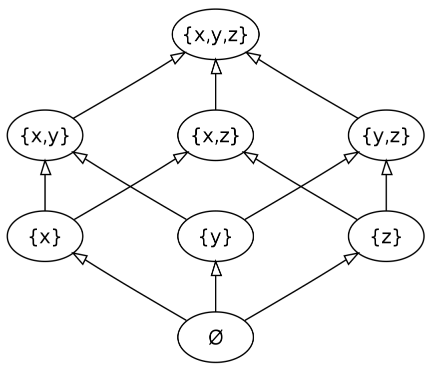
Total Ordered Set A partially ordered set $(P, \leq)$ such that for all elements $p, q \in P$ it is $p \leq q$ or $q \leq p$.

Comparable Two distinct elements $p, q \in P$ of a partial order are comparable if either $p \leq q$ or $q \leq p$

Chain Nonempty subset $C \subseteq P$ that is linearly ordered (i.e. C is a total order).

Antichain Nonempty subset $A \subseteq P$ such that all elements in A are pairwise incomparable (i.e. not comparable).

So for an easy example consider the powerset of $\{x, y, z\}$ with $\subseteq$ as partial ordering.



Here a chain is just the set of subsets on a path following arrows in the specified direction, for example $\emptyset \to \{x\} \to \{x, y\} \to \{x, y, z\}$. Antichains are elements that are not connected via a path with arrows going in the specified direction, for example $\{\{x\}, \{y\}, \{z\}\}$ or also $\{\{x\}, \{y, z\}$. In this case, all the nontrivial antichains are the nonemtpy subsets of $\mathcal{P}(\{x, y, z\})$ that do not share any elements. With this background, we can consider the Chain Antichain Principle:

Principle 1. *(CaCP): Every infinite partially ordered set contains an infinite chain or an infinite antichain.*

It is easy to see that the RPP implies the CaCP (see the following theorem), which implies that the CaCP is also a consequence of countable choice.

Theorem 1. *RPP $\Rightarrow$ CaCP*

Proof. Let $(P, \leq)$ be an infinite partially ordered set and consider $[P]^2$, i.e. all the two-element subsets of P.
We will define a partition of $[P]^2$ into two parts and use RPP to show that there is an infinite chain or an infinite antichain.
$C := \{\{x, y\} \in [P]^2 : x \leq y$ or $y \leq x\}$ which is the set of 2-sets of comparable elements.
$A := [P]^2 \setminus C$ which is the set of 2-sets of incomparable elements.
By definition, these two sets form a partition. Now colour C red and colour A blue, then we have a 2-colouring of $[P]^2$ and can apply RPP to get an infinite subset $Y \subseteq P$ such that $[Y]^2$ is monochromatic (which means that the two-element subsets of Y belong either all to C or all to A.) Then Y is either an infinite chain or an infinite antichain. To see this more clearly, consider that if all the 2-element sets of a set are contained fully in C, then all the elements are pairwise comparable which implies a linear ordering and therefore a chain of maximal length. $\square$

2 Choice on finite sets

Secondly we consider choice functions for finite sets (or more precisely for n-element sets). Recall the following choice principle for $n \in \omega$:

Principle 2. *($C(\infty, n) = C_n$) Every family of n-element sets has a choice function.*

(Note that this definition makes sense only for $n > 0$.) There are two important things to say about this principle. First, without any further assumptions from ZF we cannot prove C_n for any $n \in \omega$, in particular we cannot prove C_2 from ZF. Second, if we have C_n for all $n \in \omega$ this does not imply Finite Choice! (These results follow from so called permutation models, about which you will hear more in future sesseions of this Seminar.) However, we can investigate the relations between C_n for different $n \in \omega$. We will prove the following relations:

- $m, n \in \omega, m|n \Rightarrow (C_n \Rightarrow C_m)$ ("Downwards")
- $C_2 \Rightarrow C_4$ ("Upwards")
- Generalized: An ordered pair (m, n) satsifies some condition (S) (defined later), then C_k for all $k \leq m$ implies C_n. ("Upwards")

Let's face them in this order. Consider the first fact, which implies that if we have choice functions for a family of n-element sets, we also have choice functions for families of p-element sets for any divisor $p|n$.

Proposition 1. $m, n \in \omega, m|n \Rightarrow (C_n \Rightarrow C_m)$

Proof. Let $\mathcal{F}_m := \{A_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of m-element sets (note that we use Λ as index set since the family must not be finite or even countable). Our goal is to find a choice function on $\mathcal{F}_m$.
Further, define $k := \frac{n}{m}$ which is a positive integer by assumption and for any m-element set $A_\lambda \in \mathcal{F}_m$ define the set

$$A_\lambda^k := \{\langle x, i \rangle : x \in A_\lambda, i \in k\}$$

Since any A_λ has cardinality m, the sets A_λ^k all have cardinality $m \cdot k = n$. Then the set of all these A_λ^k, i.e. $\mathcal{F}_n := \{A_\lambda^k : \lambda \in \Lambda\}$ is a family of n-element sets.
Therefore if we have C_n we have a choice function $f : \mathcal{F}_n \to \bigcup \mathcal{F}_n$ such that $f(A_\lambda^k) \in A_\lambda^k$.
So finally, we can define the choice function $g : \mathcal{F}_n \to \bigcup \mathcal{F}_m$ as follows:

$$g(A_\lambda) = x \Leftrightarrow \exists i \in k \text{ st. } f(A_\lambda^k) = \langle x, i \rangle$$

□

So what we're doing for the downwards-direction is basically just expanding our original sets by any factor k and then using the choice function on the bigger set that implies a unique element in the original set.
The upwards direction is more involved: We will first consider the set of 2-element subsets of our original 4-sets and then use combinatorial properties to define a suitable choice function.

Proposition 2. $C_2 \Rightarrow C_4$

Proof. Let again $\mathcal{F}_4 = \{A_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of 4-element sets, again our goal is to find a choice function on $\mathcal{F}_4$. Now define the following set consisting of all 2-element subsets of A_λ for any $\lambda \in \Lambda$:

$$E_2 = \bigcup \{[A_\lambda]^2 : A_\lambda \in \mathcal{F}_4\}$$

This is a family of 2-element sets and therefore has a choice function by C_2. Denote this function f and note that if f is a choice function on E_2 it is also a choice function on any subset of E_2. In particular for any $A \in \mathcal{F}_4$, f is a choice function on $[A]^2 \subseteq E_2$, i.e. for any $\{x, y\} \in [A]^2$ we have $f(\{x, y\}) \in \{x, y\}$. Since A has 4 elements, the set $[A]^2$ will have $\binom{4}{2} = 6$ elements. Denote for example $A = \{x_0, x_1, x_2, x_3\}$, then we have

$$[A]^2 = \{\{x_0, x_1\}, \{x_0, x_2\}, \{x_0, x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\}$$

Now consider $f|_{[A]^2}$, which will meet 6 choices (one choice for any set in $[A]^2$). Now there is only a limited number of possibilities for the distribution of chosen elements:

- One element gets chosen three times.
- Three elements get chosen twice.
- Two elements get chosen twice and two elements get chosen once.

These cases are mutually exclusive and allow the definition of a suitable choice function g on $\mathcal{F}_4$ as follows for any $A \in \mathcal{F}_4$:

1. One element x gets chosen three times, then set $g(A) = x$.
2. Three elements gets chosen twice, then one element y gets never chosen, then set $g(A) = y$.
3. Two elements get chosen twice and two elements z_1, z_2 get chosen once, then set $g(A) = f(\{x, y\})$

□

Finally, we can generalize the proof of the above proposition and meet statements about more general upwards-direction implications. For this we need another definition:

Condition (S) An ordered pair $\langle m, n\rangle \in \mathbb{N} \times \mathbb{N}$ satisfies condition (S) iff there is no decomposition of n into a sum of primes $p_1, ..., p_s$ such that $\forall 1 \leq i \leq s$ it is $p_i > m$ (i.e. $n = \sum_{i=1}^{s} p_i$).

Thus we set up the theorem:

Theorem 2. *If an ordered pair $\langle m, n\rangle$ satisfies condition (S) and C_k holds for all $k \leq m$, then C_n holds as well.*

Proof. We prove by induction over n. C_1 holds trivially (choose the only element available). Then we set up the:
Induction Hypothesis: Let $m < n$ (if $m \geq n$, the statement holds automatically) such that $\langle m, n\rangle$ satisfies condition (S) and C_l holds for all $l < n$. We will show that this implies that also C_n holds in multiple steps:

1. *Step: Investigate n:* Since $\langle m, n\rangle$ satisfies (S), n itself cannot be prime (as otherwise $n = n > m$). Further n has a prime factor smaller or equal to m, as otherwise $n = p + ... + p$ with $p > m$ which contradicts condition (S). Since p is a factor of n we can write $n = kp^{a+1}$ with $k, a \in \mathbb{N}$ and $coprime(k, p)$.

2. *Step: Setup Goal:* Consider $\mathcal{F}_n = \{A_\lambda : \lambda \in \Lambda\}$ arbitrary family of n-element sets. Again our goal is to find a choice function f on $\mathcal{F}_n$ that, given $A \in \mathcal{F}_n$, finds $f(A) = x \in A$.

3. *Step: Define Stuff:* Let $A \in \mathcal{F}_n$ be an abitrary n-element set for which we want to define $f(A)$. Consider the set $[A]^p$ of p-element subsets of A. This set has cardinality $\binom{n}{p}$. Since $p \leq m$ by induction hypothesis we have C_p and therefore a choice function $g : [A]^p \to \bigcup[A]^p$ with $g(X) \in X$ for $X \in [A]^p$, in particular $g(X) \in A$. Also note that an element $a \in A$ can be chosen multiple times by the choice function g, since it is contained in many different p-element subsets of A.
Further, define for any element $a \in A$ the number $q(a)$ of times that it gets chosen by the choice function g. Define also q_0 as the least nonzero $q(a)$ for any $a \in A$ (which must be at least one and at most $\binom{n}{p}$). And finally define B as the set of all elements of A such that $q(a) = q_0$ holds, which is a nonempty subset of A.
Summary:
$$q(a) := |\{X \in [A]^p : g(X) = a\}|$$
$$q_0 := \min\{q(a) : a \in A, q(a) \neq 0\}$$
$$B := \{a \in A : q(a) = q_0\}$$
For visualizing these definitions, recall the situation in the last proof: We had $A = \{x_0, x_1, x_2, x_3\}$ being a 4-element subset, which fits the situation perfectly for

$\langle m, n \rangle = \langle 2, 4 \rangle$ and $p = 2$. Then we have for example the case $q(x_0) = 3, q(x_1) = 1, q(x_2) = 2, q(x_3) = 0$ which would fall into the first case of the three cases at the end of the proof (i.e. one element gets chosen three times). Then q_0 would be 1 and $B = \{x_1\}$.

4. Step: Show $A \setminus B$ is nonempty: For proving this claim recall the properties from step 1. In particular we have

$$\binom{n}{p} = \frac{n!}{p!(n-p)!} = \frac{kp^{a+1}}{p} \cdot \frac{(n-1) \cdot ... \cdot (n-p+1)}{(p-1)!} = kp^a \binom{n-1}{p-1}$$

But then, since $p \not| \binom{n-1}{p-1}$ we get that $p^a | \binom{n}{p}$ but $p^{a+1} \not| \binom{n}{p}$ which further implies that $n = kp^{a+1}$ cannot divide $\binom{n}{p}$.

Now assume for a contradiction that $A \setminus B = \emptyset$, then all the elements $a \in A$ get chosen the same amount of times by g and therefore $|[A]^p| = \binom{n}{p} = nq_0$ and therefore n must be a divisor of $\binom{n}{p}$ which contradicts the last paragraph. Therefore $A \setminus B$ must be nonempty. Now define $|B| = l_1$ and $|A \setminus B| = l_2$ and note that $l_1, l_2 > 0$ and $l_1 + l_2 = n$.

Step 5: Show that $\langle m, l_1 \rangle$ or $\langle m, l_2 \rangle$ satisfy (S): Assume for a contradiction that neither $\langle m, l_1 \rangle$ nor $\langle m, l_2 \rangle$ satisfy (S). Then there are decompositions $l_1 = p_1 + ... + p_s$ and $l_2 = p'_1 + ... + p'_t$ with $p_i, p'_i > m$. But then we get that $n = \sum_{i=1}^{s} p_i + \sum_{i=1}^{t} p'_i$ which contradicts our assumptions.

Step 6: Define Choice Function Let $i \in \{1, 2\}$ such that $\langle m, l_i \rangle$ satisfies (S) according to the last step. Then by assumption C_{l_i} holds and there is a choice function f' that chooses an element of B if $i = 1$ or an element of $A \setminus B$ if $i = 2$. Therefore we can define our choice function f to send $A \mapsto f(A) = f'(A) \in A$.

$\square$

Now, what does this theorem mean? Firstly, it does NOT mean that if we have C_k for k the prime decomposition of a number n, that it follows C_n (see the simple counterexample $12 = 3 \cdot 2^2$ but $12 = 5 + 7$). However, it does mean that if n itself is not prime, then C_k for all $k < n$ implies that C_n holds (using the ordered pair $\langle n-1, n \rangle$ which must satisfy (S) by n not being prime). But even more can be derived: If n is not prime and C_k holds for all $k \leq \lfloor n/2 \rfloor$, then C_n holds (using the ordered pair $\langle \lfloor n/2 \rfloor, n \rangle$ that satisfies condition (S) as the composition of n into a sum of primes larger than $\lfloor n/2 \rfloor$ cannot consist any prime twice.) On the other hand, it can be shown that if $\langle m, n \rangle$ don't satisfy condition (S), it is possible that C_k holds for all $k \leq m$ but C_n fails. This means, in particular, that if n is a prime, there is no conjunction of C_k with $k < n$ that would imply C_n.