Seminar

# Axiom of Choice

## $2^{2^{\mathfrak{m}}} + 2^{2^{\mathfrak{m}}} = 2^{2^{\mathfrak{m}}}$, whenever $\mathfrak{m}$ is infinite

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# Preliminaries

**Theorem 0.0.1. (Cantor-Bernstein Theorem)** *Let $A$ and $B$ be any sets. If $|A| \leq |B|$ and $|A| \geq |B|$, then*

$$|A| = |B|.$$

**Corollary 0.0.2. (Finite Ramsey Theorem)** *For all $m, n, r \in \omega$, where $r \geq 1$ and $n \leq m$, there exists an $N \in \omega$, where $N \geq m$, such that for every colouring of $[N]^n$ with $r$ colours, there exists a set $H \in [N]^m$, all of whose $n$-element subsets have the same colour.*

**Definition 0.0.3.** For any $n \in \omega$ and any set $S$, let $[S]^n$ denote the set of all $n$-element subsets of S. Slightly more formally,

$$[S]^n := \{x \in \mathcal{P}(S) : \text{there exists a bijection between } x \text{ and } n\}.$$

**Definition 0.0.4.** The set of all finite subsets of a set $S$ denoted by $\text{fin}(S)$ is given by

$$\text{fin}(S) := \bigcup_{n \in \omega} [S]^n.$$

**Fact 0.0.5.** *Let $\mathfrak{m}$ be a cardinal and let $A$ be a set of cardinality $\mathfrak{m}$. Then*

- $2^{\mathfrak{m}} := |\mathcal{P}(\text{A})|$
- $\text{fin}(\mathfrak{m}) := |\text{fin}(\text{A})|$
- $\text{fin}(\text{A}) \subset \mathcal{P}(\text{A})$

**Fact 0.0.6.** *If $\mathfrak{m}$ is an infinite cardinal, then*

$$2^{\aleph_0} \leq 2^{\text{fin}(\mathfrak{m})}.$$

# Statement & Proof

**Lemma 0.0.7. (Läuchli's Lemma)** *If* $\mathfrak{m}$ *is an infinite cardinal, then*

$$\left(2^{\text{fin}(\mathfrak{m})}\right)^{\aleph_0} = 2^{\text{fin}(\mathfrak{m})}.$$

*Proof.* In order to prove the Lemma, we have to show that $2^{\text{fin}(\mathfrak{m})} \leq \left(2^{\text{fin}(\mathfrak{m})}\right)^{\aleph_0}$ and $2^{\text{fin}(\mathfrak{m})} \geq \left(2^{\text{fin}(\mathfrak{m})}\right)^{\aleph_0}$ and conclude with the Cantor-Bernstein Theorem (0.0.1).

Claim: $2^{\text{fin}(\mathfrak{m})} \leq (2^{\text{fin}(\mathfrak{m})})^{\aleph_0}$
Proof: this inequality is obviously true. □

Claim: $2^{\text{fin}(\mathfrak{m})} \geq (2^{\text{fin}(\mathfrak{m})})^{\aleph_0}$
Proof: Let $A$ be an arbitrary, but fixed set of cardinality $\mathfrak{m}$. To prove this inequality, we will define several functions and use their properties to construct an injection from $\mathcal{P}(\text{fin}(A))^{\omega}$ into $\mathcal{P}(\text{fin}(A))$.

We start by defining the help functions. For $n, k \in \omega$ such that $k \geq n$, we define for any $X \subseteq [A]^n$:

$$g_{n,k}(X) := \{y \in [A]^n : \forall z \in [A]^k (y \subseteq z \rightarrow \exists x \in X(x \subseteq z))\}$$
$$d_{n,k}(X) := g_{n,k}(X) \setminus X$$

Essentially, any $y \in g_{n,k}(X)$ is an $n$-element subset of $A$ such that when you add any $k - n$ new elements to $y$, it covers an $x \in X$. This also means that $X \subseteq g_{n,k}(X)$, which will be seen in a little more detail soon.

**Example 0.0.8.** $n = 2, k = 3$, $\{a_1, a_2\} \in [A]^2$ and $X = \{\{a_1, x\} : x \in A \setminus \{a_1, a_2\}\}$.
In this case, $y = \{a_1, a_2\}$ will cover an element of $X$ no matter what we add and thus, $g_{2,3}(X)$ contains all 2-element sets that contain $a_1$. Any other element of $[A]^2$ can be complemented with an element that is not $a_1$ and thus does not cover an element of $X$ and is not an element of $g_{2,3}(X)$. Hence, we have

$$g_{2,3}(X) = \{y \in [A]^2 : a_1 \in y\} \text{ and } d_{2,3}(X) = \{\{a_1, a_2\}\}.$$

**Example 0.0.9.** $n = 2, k = 4$, $\{a_1, a_2\} \in [A]^2$ and $X = \{x \in [A]^2 : x \cap \{a_1, a_2\} = \emptyset\}$.
Note that any 4-element subset of $A$ thus contains an element of $X$ as a subset. So

$$g_{2,4}(X) = [A]^2 \text{ and } Y := d_{2,4}(X) = \{y \in [A]^2 : y \cap \{a_1, a_2\} \neq \emptyset\}.$$

Arguing similarly to the previous example, we get $g_{2,4}(Y) = Y$ which means $d_{2,4}(Y) = d_{2,4}(d_{2,4}(X)) = \emptyset$.

We will prove and use the following properties of $g_{n,k}$ and $d_{n,k}$:

1. For all $X \subseteq [A]^n$ we have $X \subseteq g_{n,k}(X)$
2. For all $X \subseteq [A]^n$ we have $g_{n,k}(g_{n,k}(X)) = g_{n,k}(X)$, i.e. $g_{n,k} \circ g_{n,k} = g_{n,k}$
3. For all $X \subseteq [A]^n$ we have $g_{n,k}(X) \subseteq g_{n,k'}(X)$ whenever $k' \geq k$

4. For all $X \subseteq [A]^n$ we have $d^j_{n,k}(X) = \left(g_{n,k} \circ d^j_{n,k}\right)(X) \setminus d^{j+1}_{n,k}(X)$

5. For all $X \subseteq [A]^n$ we have $d^n_{n,k}(X) = \left(g_{n,k} \circ d^n_{n,k}\right)(X)$

6. $I_{n,k'} \subseteq I_{n,k}$ whenever $k' \geq k$

where

$$I_{n,k} := \{X \subseteq [A]^n : g_{n,k} = X\}.$$

Proof: In order to prove $X \subseteq g_{n,k}(X)$ for all $X \subseteq [A]^n$, observe that for any $x \in X,\ z \in [A]^k$ such that $x \subseteq z$, we have $x \subseteq z$ so $x \in g_{n,k}(X)$. Thus, $X \subseteq g_{n,k}(X)$ giving us 1.

Next, we are going to prove that for all $X \subseteq [A]^n$ we have $g_{n,k}(g_{n,k}(X)) = g_{n,k}(X)$. Therefore observe that for any $y \in g_{n,k}(g_{n,k}(X)),\ z \in [A]^k$ such that $y \subseteq z$, we have a $y' \in g_{n,k}(X)$ such that $y' \subseteq z$. By definition of $g_{n,k}(X)$ we have an $x \in X$ such that $x \subseteq z$ and we get $y \in g_{n,k}(X)$. Thus, $g_{n,k} \circ g_{n,k}(X) \subseteq g_{n,k}(X)$ together with 1. this proves 2.

In the following, we want to prove that $g_{n,k}(X) \subseteq g_{n,k'}(X)$ whenever $k' \geq k$ for all $X \subseteq [A]^n$. Note that for any $y \in g_{n,k}(X)$, any $k'$-element superset $z$ of $y$ contains a $k$-element superset $z'$ of $y$, i.e. $y \subseteq z' \subseteq z$. Thus, by definition of $g_{n,k}(X)$ we have an $x \in X$ such that $x \subseteq z' \subseteq z$ and we conclude $y \in g_{n,k'}$. Hence, for $X \subseteq [A]^n$, $g_{n,k}(X) \subseteq g_{n,k'}(X)$ and we have 3.

Next, we want to prove that for all $X \subseteq [A]^n$ we have $d^j_{n,k}(X) = (g_{n,k} \circ d^j_{n,k})(X) \setminus d^{j+1}_{n,k}(X)$. If we define via induction $d^{j+1}_{n,k} := d_{n,k} \circ d^j_{n,k}$ with $d^0_{n,k}$ the identity map, we get $d^{j+1}_{n,k} := (g_{n,k} \circ d^j_{n,k}) \setminus d^j_{n,k}$ as for any $X \subseteq [A]^n$ we have $d^{j+1}_{n,k}(X) = d_{n,k}(d^j_{n,k}(X)) = g_{n,k}(d^j_{n,k}(X)) \setminus d^j_{n,k}(X)$. By 1. we have that $d^j_{n,k}(X) \subseteq g_{n,k} \circ d^j_{n,k}(X)$ giving us $d^j_{n,k}(X) = (g_{n,k} \circ d^j_{n,k})(X) \setminus d^{j+1}_{n,k}(X)$ which is 4.

In order to prove $d^n_{n,k}(X) = \left(g_{n,k} \circ d^n_{n,k}\right)(X)$ for all $X \subseteq [A]^n$, we show a combinatorial result by applying the Finite Ramsey Theorem (0.0.2) in a first step. The claim follows in a second step by applying the first result. For fixed $n, k \in \omega$ with $k \geq n$, for $U \subseteq A$ with $|U| \leq n$, and for any $X \in [A]^n$, we define the following statements $\psi(U, X, W)$ and $\varphi(U, X)$ as

$$\psi(U, X, W) \equiv W \subseteq A \setminus U \wedge \forall V \in [W]^{n-|U|}(U \cup V \in X)$$
$$\varphi(U, X) \equiv \forall m \in \omega\ \exists W \subseteq A(|W| \geq m \wedge \psi(U, X, W))$$

In essence, $\psi(U, X, W)$ means that $W$ and $U$ are disjoint subsets of $A$ and that constructing an $n$-element subset of $A$ by adding elements from $W$ to $U$ always results in an element of $X$. Additionally, $\varphi(X, U)$ means that for any finite cardinality $m$ you can find such $W$ with cadinality greater than $m$. Notably, for any $U \in X,\ W \subseteq A \setminus U$ we thus have $\psi(U, X, W)$ as nothing needs to be added.

**Example 0.0.10.** Let $n = 2, k = 4$ and $X$ as in 0.0.9, so $X = \{\{a_1, x\} : x \in A \setminus \{a_1, a_2\}\}$, additionally let $b \in A \setminus \{a_1, a_2\}$ and $U = \{a_1, b\}$, as $U \cup \{a_1, a_2\} \neq \emptyset$ we have $U \in d_{2,4}$ and thus $\varphi(U, d_{2,4}(X))$.
Further, let $U' = \{b\}$, as adding any element to U' that is not $a_1, a_2$ or $b$ makes it an element of $X$ we get for any $m \in \omega, m \geq 1, W \in [A \setminus \{a_1, a_2, b\}]^m$ that $\psi(U', X, W)$ holds and we get $\varphi(U', X)$.

Claim 1: If we have $\varphi(U, d_{n,k}(X))$, then there is a set $U'$ with $|U'| < |U|$ such that we have $\varphi(U', X)$. In particular, we see that $\varphi(\emptyset, d_{n,k}(X))$ fails.
Proof 1: Assume that $\varphi(U, d_{n,k}(X))$ holds for $U \subseteq A$ with $|U| \leq n$ and some set $X \subseteq [A]^n$. It is enough to show that for any integer $m \geq k$ there is a proper subset $U'$ of $U$ and $W \in [A]^n$ such that $\psi(U', X, W)$ holds. Indeed, we find that $\varphi(U', X)$ holds.

By the Finite Ramsey Theorem (0.0.2), for all $p, i, j \in \omega$, where $j \geq 1$ and $i \leq p$, there exists a smallest integer $N_{p,i,j} \geq p$ such that for each $j$-colouring of $[N_{p,i,j}]^i$ there is an $p$-element subset of $N_{p,i,j}$, all of whose $i$-element subsets have the same colour. Now let $p \geq k$, $p' := \max\{N_{p,i,2} : 0 \leq i \leq n\}$ and $p'' := N_{p',k-r,2^r}$, where $r = |U|$. Note that

$$p'' \geq p' \geq p \geq k \geq n \geq r.$$

By

$$\varphi(U, d_{n,k}(X)) \equiv \forall m \in \omega\, \exists W \subseteq A\, (|W| \geq m \wedge \psi(U, d_{n,k}(X), W))$$

there exists a set $S \subseteq A$ with $|S| = p''$ such that the statement

$$\psi(U, d_{n,k}(X), S) \equiv S \subseteq A \backslash U \wedge \forall V \in [S]^{n-|U|}(U \cup V \in d_{n,k}(X))$$

holds. In particular, $S \subseteq A \backslash U$. For every subset $U'$ of $U$ the set denoted by $X(U')$ is defined by

$$X(U') = \left\{Y \in [S]^{k-r} : \exists V' \subseteq Y(U' \cup V' \in X)\right\}.$$

Claim: $\bigcup_{U' \subset U} X(U') = [S]^{k-r}$
Proof: First, let $P \in \bigcup_{U' \subset U} X(U')$. Then by definition of $X(U')$, $P$ clearly is an element of $[S]^{k-r}$. Next, let $P \in [S]^{k-r}$. From above, we know that the following statement holds

$$\psi(U, d_{n,k}(X), S) \equiv S \subseteq A \backslash U \wedge \forall V \in [S]^{n-|U|}\, (U \cup V \in d_{n,k}(X)). \tag{1}$$

Since $P \subset S$ and because of the first half of the statement, we can conclude that $P \cap U = \emptyset$. Therefore,

$$|U \cup P| = |U| + |P| = r + k - r = k$$

as $|U| = r$. Note that $P$ is a subset of $A \backslash U$ as well. Additionally, since the second half of the statement (1) holds for any $V \in [S]^{n-|U|}$, $V$ can also be an element of $[P]^{n-|U|}$ and the statement is still true, i.e.

$$\psi(U, d_{n,k}(X), P) \equiv P \subseteq A \backslash U \wedge \forall\, V \in [P]^{n-|U|}\, (U \cup V \in d_{n,k}(X)). \tag{2}$$

By assumption, $k \geq n$ which yields that $k - r \geq n - r$. Hence, there is a set $Q \in [P]^{n-r}$. In particular,

$$U \cup Q \in d_{n,k}(X) = g_{n,k}(X) \backslash X = \left\{y \in [A]^n : \forall z \in [A]^k (y \subseteq z \to \exists x \in X(x \subseteq z))\right\} \backslash X.$$

Since $U \cup Q \subseteq U \cup P$, by definition of $g_{n,k}(X)$ there exists $x \in X$ such that $x \subseteq U \cup P$. If we let $U' = U \cap x$ and $V' = P \cap x$, then we have found a $V' \subseteq P$ such that $U' \cup V' \in X$. Hence, by definition of $X(U')$, $P \in X(U') \subseteq \bigcup_{U' \subset U} X(U')$. $\square$

Since $|S| = p'' = N_{p',k-r,2^r} \geq p'$, there is a set $T \in [S]^{p'}$ and a set $U' \subseteq U$ such that

$$[T]^{k-r} \subseteq X(U').$$

Note that $[T]^{k-r} \neq \emptyset$ since $p' \geq k \geq r$ implies that $p' \geq k - r$. Moreover, by definition of $X(U')$ we have $[T]^{k-r} \subseteq [S]^{k-r}$. Next, let $s = |U'|$, $Z := \{V'' \in [T]^{n-s} : U' \cup V'' \in X\}$ and $Z' := [T]^{n-s} \backslash Z$. Further, because $|T| = p' \geq N_{p,n-s,2} \geq p$, there is a set $W \in [T]^p$ such that either

$$[W]^{n-s} \subseteq Z \quad \text{or} \quad [W]^{n-s} \subseteq Z'.$$

Additionally, we note that each element $w$ of $[W]^{k-r}$ is an element of $[T]^{k-r}$ and therefore also an element of $X(U')$. Thus, by definition of $X(U')$ there is a $V' \subseteq w$ such that

$$U' \cup V' \in X.$$

Since each element of $X$ has cardinality $n$ and $|U'| = s$, there exists a subset $V''$ of $V'$ with $|V''| = n - s$ and $U' \cap V'' = \emptyset$ such that

$$U' \cup V'' \in X.$$

Note that $V'' \subseteq w \in [W]^{k-r}$ implies that $V'' \in [W]^{n-s}$. Moreover, since $[W]^{n-s} \subseteq [T]^{n-s}$ we also have that $V'' \in [T]^{n-s}$ and therefore, $V'' \in Z$. In particular, $Z \cap [W]^{n-s} \neq \emptyset$ and thus, $[W]^{n-s} \subseteq Z$. Finally, the statement

$$\psi(U', X, W) \equiv W \subseteq A \backslash U' \wedge \forall V \in [W]^{n-|U'|} (U' \cup V \in X). \tag{3}$$

holds, where $|W| = p$.

Claim: $U' \neq U$

Proof by contradiction: First, we note that the following statement holds

$$\psi(U, d_{n,k}(X), S) \equiv S \subseteq A \backslash U \wedge \forall V \in [S]^{n-|U|} (U \cup V \in d_{n,k}(X)). \tag{4}$$

Since $W \subseteq S$, $W$ is also a subset of $A \backslash U$. Additionally, since the second half of the statement (4) holds for any $V \in [S]^{n-|U|}$, $V$ can also be an element of $[W]^{n-|U|}$ and the statement is still true. Thus,

$$\psi(U, d_{n,k}(X), W) \equiv W \subseteq A \backslash U \wedge \forall V \in [W]^{n-|U|} (U \cup V \in d_{n,k}(X)). \tag{5}$$

holds as well. Now we assume that $U' = U$. Then by (3) we also have

$$\psi(U, X, W) \equiv W \subseteq A \backslash U \wedge \forall V \in [W]^{n-|U|} (U \cup V \in X). \tag{6}$$

By comparing (5) and (6), we note that we can summarize these two statements as follows

$$\psi(U, d_{n,k}(X) \cap X, W) \equiv W \subseteq A \backslash U \wedge \forall V \in [W]^{n-|U|} (U \cup V \in d_{n,k}(X) \cap X).$$

But by definition, $d_{n,k}$ is given by $g_{n,k}(X) \backslash X$ which yields that $d_{n,k}(X) \cap X = \emptyset$. Hence, both $U$ and $V$ are equal to the empty set. Therefore, the set $[W]^{n-r}$ is empty which is only the case when $|W| < n - r$. But this is a contradiction because $|W| = p \geq k \geq n \geq r > 0$ which means in particular, that $|W| \geq n - r$. Thus, $U' \neq U$. □

Claim 2: If $d^l_{n,k}(X) \neq \emptyset$ for some set $X \subset [A]^n$, then $l \leq n$.

Proof 2: Let $U \in d^l_{n,k}(X) \subset [A]^n$. This implies that we have $\psi(U, d^l_{n,k}(X), W)$ for every $W \in A \backslash U$ and consequently $\varphi(U, d^l_{n.k}(X))$, as we have already seen right before Example (0.0.10). Note that $d^l_{n,k}(X) := (d_{n.k} \circ d^{l-1}_{n,k})(X)$. Hence, we can apply Claim 1 to $\varphi(U, (d_{n.k}(d^{l-1}_{n,k}(X)))$ and obtain that there is a set $U'$ with $|U'| < |U|$ such that we have $\varphi(U, d^{l-1}_{n,k}(X))$. By iterating this process $l - 1$ times, we get a sequence $U = U_l, U' = U_{l-1}, \cdots, U_0$ with $|U_j| < |U_{j+1}|$. Thus, $|U_j| \geq j$ for all $j \in \{0, \cdots, l\}$. In particular, $|U| = |U_l| \geq l$. Since $|U| = n$, we obtain that $l \leq n$. □

As a consequence of 4. and Claim 2 we get $d^n_{n,k}(X) = (g_{n,k} \circ d^n_{n,k})(X) \backslash d^{n+1}_{n,k}(X) = (g_{n,k} \circ d^n_{n,k})(X)$, which is 5.

Finally, we are going to show that $I_{n,k'} \subseteq I_{n,k}$ whenever $k' \geq k$. Therefore let $k' \geq k, X \in I_{n,k'}$ by 1. we get $X \subseteq g_{n,k}(X)$ and by 3. we get $g_{n,k}(X) \subseteq g_{n,k'}(X) = X$ and thus $X \subseteq I_{n,k}$ and we get $I_{n,k'} \subseteq I_{n,k}$ whenever $k' \geq k$. □

Now, we can define a further function $f_{n,k} : \mathcal{P}([A]^n) \to \mathcal{P}\left([A]^k\right)$ by:

$$f_{n,k}(X) = \left\{z \in [A]^k : \exists x \in X (x \subseteq z)\right\}.$$

Consider now $\bar{f}_{n,k} := f_{n,k}|_{I_{n,k}}$.

Claim: $\bar{f}_{n,k}$ is injective.
Proof: Let $X, X' \in I_{n,k}$, so we have $X = g_{n,k}(X), X' = g_{n,k}(X')$, such that $\bar{f}_{n,k}(X) = \bar{f}_{n,k}(X')$. Now let $x \in X$, for any $z \in [A]^k$ such that $x \subseteq z$ we have $z \in \bar{f}_{n,k}(X) = \bar{f}_{n,k}(X')$ which means $\exists x' \in X$ such that $x' \subseteq z$ and thus $x \in g_{n,k}(X') = X'$ and we have $X \subseteq X'$. Analogously, we obtain $X' \subseteq X$ and can conclude that $X = X'$. □

So for the sets in $I_{n,k}$ we can define the inverse of $\bar{f}_{n,k}$ via:

$$\bar{f}_{n,k}^{-1}(\bar{f}_{n,k}(X)) = X.$$

Now we have all the tools needed to construct an injective function $F$ from $\mathcal{P}(\text{fin}(A))^\omega$ into $\mathcal{P}(\text{fin}(A))$. Let $X \in \mathcal{P}(\text{fin}(A))^\omega$, i.e. $X = \{X_s : s \in \omega\}$ where for any $s \in \omega : X_s \in \mathcal{P}(\text{fin}(A))$ (note $X$ is uniquely determined by the $X_s$). We then define $F$ as follows:

$$F(X) := \bigcup_{s\in\omega} \bigcup_{n\in\omega} \left( \bigcup_{0\le j\le n} f_{n,k(s,n,j)} \circ g_{n,k(s,n,n)} \circ d^j_{n,k(s,n,n)} \left(X_s \cap [A]^n\right) \right),$$

where we have $k(s,n,j) = 2^s \cdot 3^n \cdot 5^j$. By definition, $F$ is a function from $\mathcal{P}(\text{fin}(A))^\omega$ to $\mathcal{P}(\text{fin}(A))$ so we will show that $F$ is injective by showing that a given $F(X)$ has a unique element in it's preimage, working backwards from $F(X)$. To help with legibiliy we will introduce some new notation:

$$\begin{aligned} X_{s,n} &= X_s \cap [A]^n \\ X_{s,n,j} &= g_{n,k(s,n,n)} \circ d^j_{n,k(s,n,n)}(X_{s,n}) \\ Y_{s,n,j} &= f_{n,k(s,n,j)}(X_{s,n,j}) \end{aligned}$$

This gives us the simplified expression:

$$F(X) = \bigcup_{s\in\omega} \bigcup_{n\in\omega} \left( \bigcup_{0\le j\le n} Y_{s,n,j} \right).$$

As $Y_{s,n,j}$ is in the image of $f_{n,k(s,n,j)}$, we have $Y_{s,n,j} \subseteq F(X) \cap [A]^{k(s,n,j)}$. Combined with the fact that $(s,n,j) \mapsto k(s,n,j)$ is an injective map, we get

$$Y_{s,n,j} = F(X) \cap [A]^{k(s,n,j)}$$

and is thus uniquely determined by $F(X)$. We observe now that by 2. we have $g_{n,k(s,n,n)}(X_{s,n,j}) = X_{s,n,j}$. Thus, $X_{s,n,j} \in I_{n,k(s,n,n)}$. Additionally, as $j \le n$ we have $k(s,n,j) \le k(s,n,n)$. So 6. then implies that $X_{s,n,j} \in I_{n,k(s,n,n)} \subseteq I_{n,k(s,n,j)}$ and is then in the domain of $\bar{f}^{-1}$. It follows that

$$X_{s,n,j} = \bar{f}^{-1}_{n,k(s,n,j)}(Y_{s,n,j}).$$

Now as $d^0_{n,k(s,n,n)}$ is the identity and using 4. we get

$$\begin{aligned} X_{s,n} &= d^0_{n,k(s,n,n)}(X_{s,n}) \\ &= (g_{n,k(s,n,n)} \circ d^0_{n,k(s,n,n)}(X_{s,n})) \setminus d^1_{n,k(s,n,n}(X_{s,n}) \\ &= X_{s,n,0} \setminus d^1_{n,k(s,n,n)}(X_{s,n}) \\ &= X_{s,n,0} \setminus ((g_{n,k(s,n,n)} \circ d^1_{n,k(s,n,n)}(X_{s,n})) \setminus d^2_{n,k(s,n,n}(X_{s,n})) \\ &= X_{s,n,0} \setminus (X_{s,n,1} \setminus d^2_{n,k(s,n,n}(X_{s,n})) \\ &\vdots \\ &= X_{s,n,0} \setminus (X_{s,n,1} \setminus (\dots (X_{s,n,n-1} \setminus d^n_{n,k(s,n,n}(X_{s,n})) \dots)) \\ &= X_{s,n,0} \setminus (X_{s,n,1} \setminus (\dots (X_{s,n,n-1} \setminus X_{s,n,n}) \dots)), \end{aligned}$$

where in the last step we used 5. Thus, the $X_{s,n}$ are also uniquely determined. And since $X_s \in \mathcal{P}(fin(A))$, we have

$$X_s = \bigcup_{n\in\omega} X_{s,n}.$$

Thus it, and consequently $X$, are uniquely determined and we find $F$ to be injective. This shows us that

$$|\mathcal{P}(\text{fin}(A))^\omega| = \left(2^{\text{fin}(\mathfrak{m})}\right)^{\aleph_0} \leq |\mathcal{P}(\text{fin}(A))| = 2^{\text{fin}(\mathfrak{m})}.$$

□

Hence, we have proven that $2^{\text{fin}(\mathfrak{m})} \leq \left(2^{\text{fin}(\mathfrak{m})}\right)^{\aleph_0}$ and $2^{\text{fin}(\mathfrak{m})} \geq \left(2^{\text{fin}(\mathfrak{m})}\right)^{\aleph_0}$. Finally, we have the desired result by the Cantor-Bernstein Theorem 0.0.1, namely

$$2^{\text{fin}(\mathfrak{m})} = (2^{\text{fin}(\mathfrak{m})})^{\aleph_0}.$$

□

**Theorem 0.0.11.** *If* $\mathfrak{m}$ *is an infinite cardinal, then*

$$2^{\aleph_0} \cdot 2^{2^{\mathfrak{m}}} = 2^{2^{\mathfrak{m}}}.$$

*In particular, we get*

$$2^{2^{\mathfrak{m}}} + 2^{2^{\mathfrak{m}}} = 2^{2^{\mathfrak{m}}}.$$

*Proof.* In order to prove equality we have to show that $2^{2^{\mathfrak{m}}} \leq 2^{\aleph_0} \cdot 2^{2^{\mathfrak{m}}}$ and $2^{2^{\mathfrak{m}}} \geq 2^{\aleph_0} \cdot 2^{2^{\mathfrak{m}}}$.

Claim: $2^{2^{\mathfrak{m}}} \leq 2^{\aleph_0} \cdot 2^{2^{\mathfrak{m}}}$
Proof: this inequality is clearly true. □

Claim: $2^{2^{\mathfrak{m}}} \geq 2^{\aleph_0} \cdot 2^{2^{\mathfrak{m}}}$
Proof: Let $A$ be a set of cardinality $\mathfrak{m}$, $\text{inf}(A) := \mathcal{P}(A)\backslash\text{fin}(A)$ and $\text{inf}(\mathfrak{m}) := |\text{inf}(A)|$. Then by Fact (0.0.5)

$$\begin{aligned}2^{\mathfrak{m}} &= |\mathcal{P}(A)| = |\text{fin}(A)| + |\mathcal{P}(A)| - |\text{fin}(A)| = |\text{fin}(A)| + |\mathcal{P}(A)\backslash\text{fin}(A)| \\ &= \text{fin}(\mathfrak{m}) + |\mathcal{P}(A)\backslash\text{fin}(A)| = \text{fin}(\mathfrak{m}) + |\text{inf}(A)| = \text{fin}(\mathfrak{m}) + \text{inf}(\mathfrak{m}).\end{aligned}$$

Hence,

$$2^{2^{\mathfrak{m}}} = 2^{\text{fin}(\mathfrak{m})+\text{inf}(\mathfrak{m})} = 2^{\text{fin}(\mathfrak{m})} \cdot 2^{\text{inf}(\mathfrak{m})}.$$

Furthermore, Läuchli's Lemma (0.0.7) and Fact (0.0.6) yield

$$2^{2^{\mathfrak{m}}} = 2^{\text{fin}(\mathfrak{m})} \cdot 2^{\text{inf}(\mathfrak{m})} = \left(2^{\text{fin}(\mathfrak{m})}\right)^2 \cdot 2^{\text{inf}(\mathfrak{m})} = 2^{\text{fin}(\mathfrak{m})} \cdot \left(2^{\text{fin}(\mathfrak{m})} \cdot 2^{\text{inf}(\mathfrak{m})}\right) = 2^{\text{fin}(\mathfrak{m})} \cdot 2^{2^{\mathfrak{m}}} \geq 2^{\aleph_0} \cdot 2^{2^{\mathfrak{m}}}.$$

□

Thus, by the Cantor-Bernstein Theorem (0.0.1)

$$2^{\aleph_0} \cdot 2^{2^{\mathfrak{m}}} = 2^{2^{\mathfrak{m}}}.$$

□

# Bibliography

[1] Halbeisen, Lorenz J.. Combinatorial set theory : with a gentle introduction to forcing - 2nd ed. 2017 - Cham : Springer International Publishing, 2017. (Springer monographs in mathematics).