Part I: d. Ghebressilasie

Ordered Mostowski Models

This week we take a look at another permutation model called the ordered Mostowski model. We will see that in such a model, the axiom of choice does not hold and we will use its properties to show some rather surprizing cardinal relations.

Construction

Let $(A, \leq^M)$ be a countable, dense total order with no end points (meaning $(A, \leq^M) \cong (\mathbb{Q}, \leq)$ where $\mathbb{Q}$ is equipped with its natural order). The elements of A are our atoms.

Reminder: An atom is a set with no members but it's distinct from the empty set

Let $\mathcal{G} := \{\pi : A \leftrightarrow A \mid \pi \text{ bijective}; \forall a, b \in A \ (a \leq^M b \Leftrightarrow \pi a \leq^M \pi b)\}$ be the group of order-preserving permutations on A

Define $I := \text{fin}(A)$ the set of all finite subsets of A.

It is easy to see that I is a normal ideal since:

(a) the empty set is finite, so $\emptyset \in I$, (b) if $E \in I$ then all its subsets are also finite, (c) the union of two finite subsets is finite, (d) all $\pi \in \mathcal{G}$ are bijections, so $\forall E \in I \ (|E| = |\pi E|)$, hence πE is finite, and (e) singletons are finite, so $\forall a \in A \ (\{a\} \in I)$.

Then we construct a filter $\mathcal{F}$ as follows:

$\mathcal{F} := \{K \trianglelefteq \mathcal{G} \mid \exists E \in I \ (\text{fix}_{\mathcal{G}}(E) \trianglelefteq K)\}$ meaning we use elements $E \in I$ of our ideal to make generators $\text{fix}_{\mathcal{G}}(E)$ of $\mathcal{F}$

One can easily check that $\mathcal{F}$ is a normal filter $\mathcal{G}$.

Our model, denoted $\mathcal{V}_M$, is a subclass of $\mathcal{M}$

Reminder: $\mathcal{M}$ is the cumulative hierarchy of sets defined as follows: Take a model Ω of ZF and define

$$M_0 := A_0 \quad ; \quad M_{\alpha+1} := \mathcal{P}(M_\alpha) \quad ;$$

$$M_\alpha := \bigcup_{\beta \in \alpha} M_\beta \quad \text{for } \alpha \text{ a limit ordinal}$$

$$\text{and finally } \mathcal{M} := \bigcup_{\alpha \in \Omega} M_\alpha$$

Now $\mathcal{V}_M$ consists of those sets $x \in \mathcal{M}$ which are hereditarily symmetric w.r.t. $\mathcal{F}$. This means

$$\forall x \in \mathcal{V}_M \; (\mathrm{sym}_{\mathcal{G}}(x) := \{ \pi \in \mathcal{G} \mid \pi x = x \} \in \mathcal{F})$$

meaning it's symmetric and every set in its transitive closure is symmetric.

Properties

A basic property of the Mostowski model is that the order on the atoms is carried over into $\mathcal{V}_M$

Lemma 8.10 $R_< := \{ \langle a, b \rangle \mid a <^M b \} \subseteq A \times A$ belongs to $\mathcal{V}_M$

Proof Since all $\pi \in \mathcal{G}$ are order preserving, we have $\forall \pi \in \mathcal{G} \; \forall a, b \in A \; (a <^M b \Rightarrow \pi a <^M \pi b)$ which implies $\langle a, b \rangle \in R_< \iff \langle \pi a, \pi b \rangle \in R_<$ In particular, we have $\forall \pi \in \mathcal{G} \; (\pi R_< = R_<)$ or equivalently: $\mathrm{sym}_{\mathcal{G}}(R_<) = \mathcal{G} \in \mathcal{F}$

So $R_<$ is symmetric. Furthermore, $\forall \langle a, b \rangle \in R_<$ $\langle a, b \rangle = \{ a, \{ a, b \} \}$, thus the transitive closure of $R_<$ is $A \cup \{ \{ a, b \} \in A \times A \mid a <^M b \} \cup R_<$ (we just added all the necessary sets to make it transitive).

It is easily checked that sets of the form $\{a, \{a,b\}\}$ as well as $\{a,b\}$ are symmetric, so $R_<$ is hereditarily symmetric. $\Rightarrow R_< \in \mathcal{V}_M$ $\dashv$

Reminder: $x \in \mathcal{M}$ is symmetric w.r.t. $\mathfrak{I}$ iff

$\exists E \in I \ (\mathrm{fix}_{\mathcal{G}}(E) := \{\pi \in \mathcal{G} \mid \forall a \in E \ (\pi a = a)\} \subseteq \mathrm{sym}_{\mathcal{G}}(x))$

We call E a **support** of x. Note that $I = \mathrm{fin}(A)$ implies that the support of each $x \in \mathcal{V}$ is finite.

The next lemma shows that each set $x \in \mathcal{V}_M$ has a unique **least support**, which is the support E of x with the property $\forall F \in I \ (F \subsetneq E \Rightarrow F$ is not support of $x)$.

Lemma 8.11

(a) $\forall x \in \mathcal{V}_M \ (\forall E_1, E_2 \in I$ supports of $x \ (E := E_1 \cap E_2$ is a support of $x))$

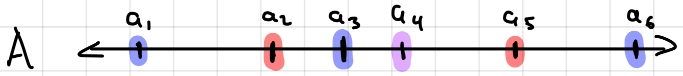
(b) Every $x \in \mathcal{V}_M$ has a least support

(c) The class of pairs (x, E) with $x \in \mathcal{V}_M$ and E is the least support of x is symmetric

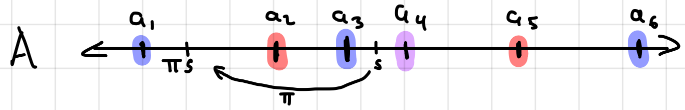
Proof

(a) Let $x \in \mathcal{V}_M$ be a set and $E_1, E_2 \in I$ be two supports of x. Define $E := E_1 \cap E_2$. It is now the goal to show that $\mathrm{fix}_{\mathcal{G}}(E) \subseteq \mathrm{sym}_{\mathcal{G}}(x)$. First note that a permutation $\pi \in \mathrm{fix}_{\mathcal{G}}(E)$ can be composed using finitely many permutations from $\mathrm{fix}_{\mathcal{G}}(E_1)$ and $\mathrm{fix}_{\mathcal{G}}(E_2)$. To see this more clearly, consider the following example:

Say $E_1 = \{a_1, a_3, a_4, a_6\}$, $E_2 = \{a_2, a_4, a_5\}$, thus $E = \{a_4\}$.
Then we get this picture where E_1 is blue, E_2 is red, and E is purple.



Now a permutation $\pi \in \mathrm{fix}(E)$ has to fix a_4, but on the left and right of a_4, it can squish and stretch the line even past the other points of E_1 and E_2 (remember that π is order preserving). Say a point $s \in (a_3, a_4)$ (for $a, b \in A$, (a,b) denotes the intervall $\{c \in A \mid a <^M c <^M b\}$) is pushed to $\pi s \in (a_1, a_2)$.



s has to be pushed past a_3 and a_2. This means we can use a permutation $q \in \mathrm{fix}_g(E_2)$ and push s past a_3 to $qs \in (a_2, a_3)$. From there we pick it up with a permutation $p \in \mathrm{fix}_g(E_1)$ and bring it past a_2 to $pqs \in (a_1, a_2)$. Since $E_1 \setminus E_2$ and $E_2 \setminus E_1$ are finite, there are only finitely many permutations needed to go past all the 'roadblocks'.

So we can write $\pi = p_n q_n p_{n-1} q_{n-1} \cdots p_1 q_1$ where
$\forall i \in \{1, \dots, n\} \, (p_i \in \mathrm{fix}_g(E_1) \wedge q_i \in \mathrm{fix}_g(E_2))$
But since $\mathrm{fix}_g(E_1) \cup \mathrm{fix}_g(E_2) \subseteq \mathrm{sym}_g(X)$ we immediatly get

$$\pi X = p_n q_n p_{n-1} q_{n-1} \cdots p_1 q_1 X = X, \quad \text{so} \quad \pi \in \mathrm{sym}_g(X)$$

(b) To see that each $x \in \mathcal{V}_M$ has a least support, consider an arbitrary support E_0 of x. Now intersect it with every support of x. Since E_0 is finite, this is a finite intersection and thus (a) implies that $\bigcap_{\substack{E \in I \\ E \text{ is support of } x}} E$ is a support of x.

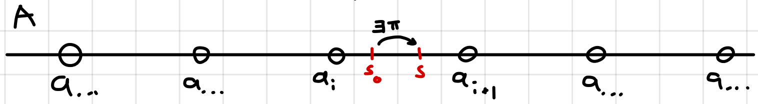
(c) $\forall x \in \mathcal{V}_M$ and $\forall \pi \in \mathcal{G}$, we have $\text{fix}_{\mathcal{G}}(\pi E) = \pi\, \text{fix}(E)\, \pi^{-1}$ and $\text{sym}_{\mathcal{G}}(\pi x) = \pi\, \text{sym}_{\mathcal{G}}(x) \pi^{-1}$ where E is the least support of x. This also implies that πE is a support of πx, so (x, E) is symmetric. $\dashv$

The next goal is to give a comprehensive description of sets $S \subseteq A$ with support $E \in I$

Lemma 8.12

$\forall E \in I \; (|E| = n \Rightarrow |\{S \subseteq A \mid S \in \mathcal{V}_M \text{ and } E \text{ is support of } S\}| = 2^{2n+1})$

Proof Let $E = \{a_1, \dots, a_n\}$ s.t. $a_1 <^M \dots <^M a_n$ and let S be a set with support E. Consider the following illustration :



Note that if $\exists s_0 \in (a_i, a_{i+1})$ then $\forall s \in (a_i, a_{i+1}) \exists \pi \in \text{fix}_{\mathcal{G}}(E) \; (\pi s_0 = s)$

In other words, if there is one $s_0 \in S$ between a_i and a_{i+1}, then the whole interval (a_i, a_{i+1}) has to be in S because for any $s \in (a_i, a_{i+1})$ there is a permutation π in $\text{fix}_{\mathcal{G}}(E)$ with $\pi s_0 = s$. Since E is support of S, we have to have $\pi S = S$, thus either $(a_i, a_{i+1}) \subseteq S$ or $S \cap (a_i, a_{i+1}) = \emptyset$.

The same is true for the intervals (∞, a_1) and (a_n, ∞).

So for S we have either $\forall i \in \{1, \dots, n\} \; (a_i \in S \vee a_i \notin S)$ and $\forall i \in \{1, \dots, n-1\} \; ((a_i, a_{i+1}) \subseteq S \vee (a_i, a_{i+1}) \cap S = \emptyset)$ and $(\infty, a_1) \in S \vee (\infty, a_1) \cap S = \emptyset$ and $(a_n, \infty) \in S \vee (a_n, \infty) \cap S = \emptyset$

In total, that makes $2^n \cdot 2^{n-1} \cdot 2 \cdot 2 = 2^{2n+1}$ possibilities. $\dashv$

We will make use of this description of sets in $\mathcal{P}(A)$ later on!

The next result will show that the axiom of choice fails in $\mathcal{V}_M$

Lemma 8.13 Let $\mathfrak{m} := |A|$. Then $\mathcal{V}_M \models \aleph_0 \not\leq 2^{\mathfrak{m}}$

Proof Assume towards a contradiction that $\aleph_0 \leq 2^{\mathfrak{m}}$, i.e. there exists an injective map $f : \omega \hookrightarrow \mathcal{P}(A)$. Consider the map as its graph $f = \{\langle n, f(n)\rangle \mid n \in \omega\}$. Now, $f \in \mathcal{V}_M$ means it has a support $E \in I = \mathrm{fin}(A)$ so $\forall \pi \in \mathrm{fix}_{\mathcal{G}}(E)\ (\pi f = f)$. But $\pi n = n\ \forall n \in \omega$ and f is injective, so we have to have $\pi f(n) = f(n)\ \forall n \in \omega$. This would mean that E supports $f(n)$ for all infinitely many n. But in the previous Lemma we've seen that each $E \in I$ only supports finitely many sets $\lightning$ $\dashv$

Note: We've just shown that $\mathcal{P}(A)$ is Dedekind finite even though it's infinite which is provably false in any model of ZFC, so the axiom of choice cannot hold in $\mathcal{V}_M$.

Reminder: Because of the **Jech-Sochor Embedding Theorem** we can embed $\hat{\mathcal{V}} \subseteq \mathcal{V}_M$ into a model of ZF. This gives us the existence of a model of ZF in which we have an infinite set $\mathfrak{m}$ s.t. $\aleph_0 \not\leq 2^{\mathfrak{m}}$. This implies the axiom of choice is independent of ZF. The following two results will also make use of the Jech-Sochor Embedding to make claims about ZF.

After establishing the ordered Mostowski Model we now are going to take a look at two cardinal relationships in $\mathcal{V}_M$. For the next part let $\mathfrak{m}$ denote the cardinality of A in $\mathcal{V}_M$.

The first Proposition is:

Proposition 8.14

Let A be the set of atoms of the ordered Motstowski model. Then in $\mathcal{V}_M$ there is a surjection from $\mathrm{fin}(A)$ onto $\mathcal{P}(A)$. Thus, it is consistent with ZF that there are infinite cardinals $\mathfrak{m}$ such that $2^{\mathfrak{m}} \leq^* \mathrm{fin}(\mathfrak{m})$

To give a little more context before we jump into the proof of Prop. 8.14 we want to look at some resuls we saw in previous presentations and connect them.

Theorem 5.21

If $\mathfrak{m}$ is an infinite cardinal, then $\mathrm{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$.

Fact 5.8

If $\mathfrak{m} \leq^* \mathfrak{n}$, then $2^{\mathfrak{m}} \leq 2^{\mathfrak{n}}$

In contrast to Th. 5.21 we can show with Fact 5.8 and Prop. 8.14 that $2^{2^m} \leq 2^{\mathrm{fin}(m)}$. Now we consider the

Cantor Bernstein Theorem 3.14

Let A and B be any sets. If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$

Finally we can conclude that $\mathcal{V}_M^{\sim}$ is a model of $2^{2^m} = 2^{\mathrm{fin}(m)}$.

Now let's actually prove the Proposition

<u>Proof:</u>

Goal: Construct surjective function

$$g: \mathrm{fin}(A) \twoheadrightarrow \mathcal{P}(A)$$

We already saw that for every finite set $E \subseteq A$, we can give a complete description of the subsets of A with support E.
We are using that by defining an ordering of the subsets of A sharing a given finite support.
So let's fix a support $E = \{a_1 <^M \dots <^M a_n\} \in \mathrm{fin}(A)$ and lets define

$I_0 = \{a \in A : a <^M a_1\}$

$I_n = \{a \in A : a_n <^M a\}$

$I_i = \{a \in A : a_i <^M a <^M a_{i+1}\}$ for $i \in \{1, \dots, n-1\}$

Note that these I_i describe the intervalls that were mentioned in the proof of Lemma 8.12.

For every function $x \in {}^{2n+1}2$ we assign a set $S_x \in \mathcal{P}(A)$ by

$$S_x = \bigcup_{x(2i)=1} I_i \cup \{a_i : x(2i-1) = 1\}$$

To understand this a little better we have to take a closer look at the function x. $x \in {}^{2n+1}2$ therefore we can find a connection between $x(i) \in \{0,1\}$ and every a_i $(i \in \{1,\dots,n\})$ and every intervall I_i $(i \in \{0,\dots,n\})$. Regarding the set S_x we see that:

$$x(2i) = \begin{cases} 1 & \rightarrow \quad I_i \subseteq S_x \\ 0 & \rightarrow \quad I_i \cap S_x = \emptyset \end{cases}$$

$$x(2i-1) = \begin{cases} 1 & \longrightarrow \quad a_i \in S_x \\ 0 & \longrightarrow \quad a_i \notin S_x \end{cases}$$

So for every $x \in {}^{2n+1}2$, E is a support of S_x.

In addition, for every $S_0 \subseteq A$ with support E there is a $x_0 \in {}^{2n+1}2$ such that $S_0 = S_{x_0}$.

This is because we don't have any constraints on x we can create 2^{2n+1}

diffrent functions x, all with support E and with Lemma 8.12 we get that there are 2^{2n+1} diffrent sets with support E. Therefore we find a set $S_{x_0} = S_0$ and hence, we find x_0.

We now consider the set ${}^{2n+2}2$.
Let $\leq_\ell$ be the lexicographic ordering on ${}^{2n+2}2$, that means $\varphi \leq_\ell \varphi'$ if $\exists k \in \{0, \dots, 2n+1\}$ s.t. $\varphi(k) < \varphi'(k)$ and $\varphi(i) = \varphi'(i)$ for all $i < k$.

Example:

φ_1 : 0 1 1 0 0 1 1 0 ...
φ_2 : 0 1 0 0 1 0 1 0 ...
φ_3 : 1 1 0 1 1 0 0 1 ...

$\varphi_2 < \varphi_1 < \varphi_3$

Now we define $\overline{\varphi}$: Let $\varphi \in {}^{2n+2}2$, then we define $\overline{\varphi}(i) = 1 - \varphi(i) \quad \forall i \in \{0, \dots, 2n+1\}$. In other words in $\overline{\varphi}$ we turn all zeros into ones and vice versa.

Example:

φ : 0 1 1 0 1 0 1 1 1 0 ...

$\overline{\varphi}$: 1 0 0 1 0 1 0 0 0 1 ...

Furthermore we define the function $\mu : {}^{2n+2}2 \longrightarrow {}^{2n+2}2$ via

$$\mu(\varphi) = \begin{cases} \varphi & \text{if } \varphi \leq_\ell \overline{\varphi} \\ \overline{\varphi} & \text{otherwise} \end{cases}$$

So μ takes the function f as input and outputs f if $f(0)=0$ and $\bar{f}$ if $f(0)=1$.

Back to the set ${}^{2n+1}2$.
For $x \in {}^{2n+1}2$ we define $x^+ := x \cup \{\langle 2n+1, 0\rangle\}$.
Now x^+ is an element of ${}^{2n+2}2$ (we "added" 0).
We define the ordering $\leq_n$ on ${}^{2n+1}2$ via

$$x_0 \leq_n x_1 \quad :\Longleftrightarrow \quad \mu(x_0^+) <_\ell \mu(x_1^+)$$

Ordering $x \in {}^{2n+1}2$ via the μ-function is going to be essential for the last part of the proof but comes with the problem that $\mu(x) = \mu(\bar{x})$. We solve this by switching over to ${}^{2n+2}2$ and taking x^+ instead of x.

$x : 01001\ldots \rightarrow \mu(x) : 01001\ldots$
$\bar{x} : 10110\ldots \rightarrow \mu(\bar{x}) : 01001\ldots$ $\qquad \mu(x) = \mu(\bar{x})$ ↯

$x : 01001\ldots \rightarrow \mu(x^+) : 01001\ldots 0$
$x : 10110\ldots \rightarrow \mu(\bar{x}^+) : 01001\ldots 1$ $\qquad \mu(x^+) \neq \mu(\bar{x}^+)$

Now we are ready to define a surjection from $\mathrm{fin}(A)$ onto $\mathcal{P}(A)$.

$$g: \begin{array}{l} \mathrm{fin}(A) \longrightarrow \mathcal{P}(A) \\ E \longmapsto S_{x^*_{|E|}} \end{array}$$

where $|E| = n$ and x_n^* denotes the n^{th} function off all the functions $x \in {}^{2n+1}2$ with respect to the ordering $<_n$.
We noted before that for every set $S_0 \in \mathcal{P}(A)$ with support E we find $x_0 \in {}^{2n+1}2$ such that $S_0 = S_{x_0}$. We will finish the proof by showing that $S_0 = S_{x_0} = S_{x^*_{|E|}}$
We do that by showing that for the least support E_0 of S_0, $|E_0| = m$, we can define x_{S_0} and check that x_{S_0} is the n^{th} function of all the functions in the ordering with respect to $<_m$, where $n \geq m$. In a second step we extend E_0 to a finite set E such that $|E| = n$ then $x_{S_0} = x_n^* = x^*_{|E|}$.
So let S_0 be any set in $\mathcal{P}(A)$ and $E_0 = \{a_1, \dots, a_m\}$ is the least support of S_0. We set $x_0 = x_{S_0}$ and have to check where in the ordering with respect to $<_m$ x_{S_0} lies.
Note that $x_{S_0}(0) = x_{S_0}(1) = x_{S_0}(2) = 1$ or $x_{S_0}(0) = x_{S_0}(1) = x_{S_0}(2) = 0$ are not possible otherwise $E \setminus \{a_1\}$ is a support of S_0.

Case 1: $x_{s_0}(0) = 1$

If $x_{s_0}(0) = 1$ then either $x_{s_0} = (1, 0, *, *, \ldots)$ or $x_{s_0} = (1, 1, 0, *, *, \ldots)$ where $* \in \{0, 1\}$.

Wlog we look at $x_{s_0} = (1, 1, 0, *, *, \ldots)$ since its going to be lower in the ordering than $x_{s_0} = (1, 0, *, *, \ldots)$.

$x_{s_0}^{+} = (1, 1, 0, *, *, \ldots, 0)$

$\mu(x_{s_0}^{+}) = (0, 0, 1, *, *, \ldots, 1)$

So even if all the unknown values would be zero all the functions of the form $(0, 0, 0, *, *, \ldots)$ would be lower in the ordering.

Those are $\sum_{i=0}^{2m-2} 2^i \geq m$

Hence $n \geq m$.

Case 2: $x_{s_0}(0) = 0$

With the same reasoning as above we look at $x_{s_0} = (0, 1, *, *, \ldots)$ and $\mu(x_{s_0}^{+}) = (0, 1, *, *, \ldots, 0)$

And again there are $\sum_{i=0}^{2m-1} 2^i$ functions that are lower in the ordering. So again $n \geq m$.

All that is left to do is extend E_0 to a finite set E such that $|E|=n$ and x_{S_0} is still the n^{th} function in the new ordering with respect to $<_n$.
This we can do by simply adding Atoms smaller than a_1 with respect to $<^M$ to the set E_0.
All the new functions will be ordered higher than x_{S_0} because they incorporate at least one of the new atoms.

Hence, $|E| = n$ and $S_0 = S_{x^*_{|E|}}$ as we wanted.

∎

Proposition 8.15

Let m denote the cardinality of the set of Atoms of the ordered Motowski model. Then, for each $n \in \omega$ we have:

$$\mathcal{N}_M \models n \cdot \mathrm{fin}(m) < 2^m < \aleph_0 \cdot \mathrm{fin}(m)$$

Proof:

We only give a proof sketch and do that in four steps.

$\mathcal{N}_M \models n \cdot \mathrm{fin}(\mathfrak{m}) < 2^{\mathfrak{m}} < \aleph_0 \cdot \mathrm{fin}(\mathfrak{m})$

$2^{\mathfrak{m}} \leq \aleph_0 \cdot \mathrm{fin}(\mathfrak{m})$:

For $S \subseteq A$ let E be the least support of S and $|E| = n$. Let $k \in \omega$ be such that $S = S_{x_k}$, where x_k denotes the k^{th} function of ${}^{2n+1}2$ with respect to the ordering $<_n$ from above.

We define the injection g:

$$g: \begin{array}{ccc} \mathcal{P}(A) & \longrightarrow & \omega \times \mathrm{fin}(A) \\ S & \longmapsto & (k, E) \end{array}$$

This is a modified version of the function we used in the proof of Proposition 8.14.

$2^{\mathfrak{m}} \neq \aleph_0 \cdot \mathrm{fin}(\mathfrak{m})$: By contradiction

Assume there is an injective function $f: \omega \times \mathrm{fin}(A) \hookrightarrow \mathcal{P}(A)$. For every $n \in \omega$ we define E_n as the least support of the set $f(\langle n, \emptyset \rangle)$. Now we get $E_0, E_1, \ldots$ and with those sets we could construct a injective mapping from ω to $\mathcal{P}(A)$ which is a contradiction to Lemma 8.13.

$n \cdot fin(m) \leq 2^m$:

For every $j \in n$ and $E \in fin(A)$ large enough we can define $S_{j,E}$ as the j^{th} set which has E as its least support. E has to be large enough to be the least support of more than n sets.

Now we can define the injection

$$g: \begin{array}{ccc} n \times fin(m) & \longrightarrow & \mathcal{P}(A) \\ (j, E) & \longmapsto & S_{j,E} \end{array}$$

For $E \in fin(A)$ which are not large enough to allow such an encoding, we use another encoding with a large enough auxiliary set E_0. How exactly this is done is left as an exercise.

$n \cdot fin(m) \neq 2^m$: By contradiction

Assume there is an injective function $f: \mathcal{P}(A) \hookrightarrow n \times fin(m)$. Let $k \in \omega$ be such that $2^{2k+1} > n \cdot 2^k$ and let $E_0 \subseteq A$ be a finite set with $|E_0| = k$.

Lemma 8.12 tells us that there are 2^{2k+1} subsets of A with support E_0. Let's denote those subsets by $S_0, S_1, \ldots$.

Since $n \cdot 2^k < 2^{2k+1}$ and $|\operatorname{fin}(E_0)| \leq 2^k$ there exist δ_j $(j \in \{1, \dots, 2^{2k+1}\})$ with $f(\delta_j) \notin n \times \operatorname{fin}(E_0)$. Hence, there exist a smallest δ_j with this property, denoted as δ_i.
We set $f(\delta_i) = \langle m, F_0 \rangle$ for some $m \in n$ and $F_0 \in \operatorname{fin}(A)$. Since $F_0 \not\subseteq E_0$ we get that $|F_0 \cup E_0| > |E_0|$. We can proceed with $E_1 = F_0 \cup E_0$ and get by repeating that argument the sets $E_0, E_1, E_2, \dots$ and with these sets we are able to find an injection from ω to $\mathcal{P}(A)$ which is a contradiction to Lemma 8.13.

$\blacksquare$