Seminar B9

Axiom of Choice

The Ramseyan Partition Principle Revisited

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Introduction

In B5 multiple choice principles have been presented. In the process, we were able to prove that the Ramseyan Partition principle (RPP) implies the Chain anti-Chain Principle (CaCP).
In the last two seminars we have been introduced to set theory with atoms (ZFA) and have encountered several models of it, the so-called *permutation models*.
The main goal of this seminar is to show that the reverse implication of (RPP) $\implies$ (CaCP) is not provable in ZFA, i.e. we want to prove the following proposition:

Proposition 8.26. *In* ZFA*,* CaCP *does not imply* RPP*.*

Before outlining the proof, we recall the choice principles in question and the necessary notations and definitions.

Let X be an arbitrary set and let $(P, \leq)$ be a partially ordered set.

By $[X]^2$ we denote the set of all 2-element subsets of X.

Chain: Non-empty subset $C \subseteq P$ that is linearly ordered.

Anti-chain: Non-empty subset $A \subseteq P$ such that all elements of A are pairwise incomparable.

Choice Principles:

- **Ramseyan Partition Principle (RPP):**
  If X is an infinite set and $[X]^2$ is 2-coloured, then there is an infinite subset Y of X such that $[Y]^2$ is monochromatic.
- **Chain anti-Chain Principle (CaCP):**
  Every infinite partially ordered set contains an infinite chain or an infinite anti-chain.

The proof of Proposition 8.26 is carried out with the following steps:

(1) In the first step, we construct a permutation model, for which we will be able to prove that RPP fails and CaCP holds. This model will be denoted by $\boldsymbol{\mathcal{V}}_T$.

(2) The second step consists of proving that RPP does not hold in $\boldsymbol{\mathcal{V}}_T$.

(3) In the third and ultimate step, we show that CaCP holds in our permutation model $\boldsymbol{\mathcal{V}}_T$.

Combining (2) and (3) then yields the statement of the proposition.

(1) Construction of the permutation model $\boldsymbol{\mathcal{V}}_T$

A permutation model is a model of ZFA constructed using a group of permutations of the atoms. First, we recall the basic idea for the construction of a permutation model.

We start by defining a set of atoms A. Then suppose that $\mathcal{G}$ is a group of permutations of A. A normal filter $\mathscr{F}$ of $\mathcal{G}$ is a collection of subgroups of $\mathcal{G}$ satisfying

- $\mathcal{G} \in \mathscr{F}$
- Any subgroup containing an element of $\mathscr{F}$ is in $\mathscr{F}$
- The intersection of two elements of $\mathscr{F}$ is in $\mathscr{F}$.
- For any $\pi \in \mathcal{G}$ and $H \in \mathscr{F}$, we have $\pi H \pi^{-1} \in \mathscr{F}$.
- The subgroup fixing any element of A is in $\mathscr{F}$.

An element x of a model $\boldsymbol{\mathcal{M}}$ of ZFA is called symmetric if its symmetric group

$$\operatorname{sym}_{\mathcal{G}}(x) := \{\pi \in \mathcal{G} : \pi x = x\}$$

belongs to $\mathscr{F}$. The element x is called hereditarily symmetric if x and all elements of its transitive closure are symmetric. The permutation model $\boldsymbol{\mathcal{V}}$ then consists of all hereditarily symmetric elements and it is a transitive model of ZFA.
Often, we consider the normal filter $\mathscr{F}$ of $\mathcal{G}$ that is generated by the subgroups

$$\operatorname{fix}_{\mathcal{G}}(E) := \{\pi \in \mathcal{G} : \pi a = a \text{ for all } a \in E\}$$

for $E \in I$, where I is some normal ideal.

We now construct the permutation model with the purpose of proving Proposition 8.26.

- **Set of Atoms** A**:**
  Consider an arbitrary infinite set I and for each $i \in I$ we set $A_i := \{a_i, b_i\}$ such that $A_i \cap A_j = \varnothing$ for distinct $i, j \in I$. Then we define our set of atoms as

  $$A = \bigcup \{A_i : i \in I\}.$$

  So simply put, our set of atoms is an infinite set of pairwise distinct 2-element sets.

- **Group of Permutations** $\mathcal{G}$**:**
  We let $\mathcal{G}$ be the group of permutations of A which move finitely many atoms, such that for every $\pi \in \mathcal{G}$ and each $i \in I$ we have $\pi A_i = A_j$ for some $j \in I$.

- **Normal Filter** $\mathscr{F}$**:**
  We consider the filter generated by I_{fin}, where I_{fin} denotes the set of all finite subsets of A. It can be easily checked that I_{fin} is a normal ideal. Indeed, for all subsets $E, F \subset A$ we have

  1. $\varnothing \in I_{\text{fin}}$, since the empty set is finite.
  2. $E \in I_{\text{fin}} \wedge F \in I_{\text{fin}} \implies F \in I_{\text{fin}}$, as $|F| \leq |E| < \infty$
  3. $E \in I_{\text{fin}} \wedge F \in I_{\text{fin}} \implies E \cup F \in I_{\text{fin}}$, since $|E \cup F| \leq |E| + |F| < \infty$
  4. $\pi \in \mathcal{G} \wedge E \in I_{\text{fin}} \implies \pi E \in I_{\text{fin}}$, because all $\pi \in \mathcal{G}$ are bijections and hence $|E| = |\pi E|$
  5. $\forall a \in A : \{a\} \in I_{\text{fin}}$

  Hence the filter $\mathscr{F}$ derived from I_{fin} is a normal filter.

The corresponding permutation model then cosists of all hereditarily symmetric elements w.r.t $\mathscr{F}$ and we denote it by $\boldsymbol{\mathcal{V}}_T$. Moreover, we note that, similarly as in the basic Fraenkel model, x is in $\boldsymbol{\mathcal{V}}_T$ if and only if x is symmetric and each $y \in x$ belongs to $\boldsymbol{\mathcal{V}}_T$ too.

Remark. The permutation model we consider here is essentially the second Fraenkel model, which was presented in B7. The difference is, that our set of atoms is an amorphous set (i.e. an infinite set which is not the disjoint union of two infinite subsets) of pairs instead of a countable set of pairs.

(2) RPP fails in $\boldsymbol{\mathcal{V}}_T$

In order to show that RPP does not hold in $\boldsymbol{\mathcal{V}}_T$ we need the following simple fact.

Fact 8.17. *Let $E \in I_{\text{fin}}$. Then each $S \subseteq A$ with support E is either finite or co-finite. Furthermore, if S is finite, then $S \subseteq E$ and if S is co-finite, then $(A \setminus S) \subseteq E$.*

Proof. Let $S \subseteq A$ with support E, i.e. E satisfies

$$\{\pi \in \mathcal{G} : \pi a = a \text{ for all } a \in E\} =: \text{fix}_{\mathcal{G}}(E) \subseteq \text{sym}_{\mathcal{G}}(S) := \{\pi \in \mathcal{G} : \pi a = a \text{ for all } a \in S\}.$$

Let $\pi \in \text{fix}_{\mathcal{G}}(E)$ and $a \in A$. We then have that $\pi a \in S$ if and only if $a \in S$. If S contains no elements of $A \setminus E$, then it is clear that $S \subseteq E$. Suppose now that S contains an element a_0 of $A \setminus E$. By the previous observation we see that $\pi a_0 \in S$ and since $\pi a_0 \neq a_0$, a permutation in $\text{fix}_{\mathcal{G}}(E)$ can send a_0 to any other element of $A \setminus E$. Thus S contains all the elements of $A \setminus E$, which yields that $A \setminus S \subseteq E$. □

With this we are now able to prove:

Lemma 8.18. RPP *fails in $\boldsymbol{\mathcal{V}}_T$.*

Proof. The proof amounts to showing that there exists a 2-colouring of $[A]^2$ such that for no infinite subset $Y \subseteq A$ belonging to $\boldsymbol{\mathcal{V}}_T$, $[Y]^2$ is monochromatic.
For this, we define the 2-colouring $\pi : [A]^2 \to 2$ by

$$\pi\{x, y\} = \begin{cases} 0 & \text{if } \{x, y\} = A_i \text{ for some } i \in I \\ 1 & \text{otherwise} \end{cases}$$

Now suppose that Y is an infinite subset of A, such that $[Y]^2$ is monochromatic. We distinguish two cases.
Case 1: Suppose that $\pi\{x, y\} = 0$ for all $\{x, y\} \in [Y]^2$. This is possible if and only if $Y = A_{i_0}$ for some $i_0 \in I$, because otherwise we can simply choose two elements x, y belonging to distinct A_i's and then $\pi\{x, y\} = 1$. We thus have that $|Y| = 2$, which then contradicts the fact that we chose Y to be infinite. ↯
Case 2: Now assume that $\pi\{x, y\} = 1$ for all $\{x, y\} \in [Y]^2$. By definition of π we have that $|Y \cap A_i| \leq 1$ for all $i \in I$. Due to the assumed infiniteness of Y, this yields that $|Y \cap A_i| = 1$ for infinitely many $i \in I$ and hence $A \setminus Y$ is infinite too. So Y is neither finite nor co-finite and therefore does not belong to $\boldsymbol{\mathcal{V}}_T$ by the previous fact.
The only sets $Y \subseteq A$ belonging to $\boldsymbol{\mathcal{V}}_T$, such that $[Y]^2$ is monochromatic, are finite. Hence, RPP fails in $\boldsymbol{\mathcal{V}}_T$. □

(3) CaCP holds in $\boldsymbol{\mathcal{V}}_T$

Before showing that CaCP does hold in $\boldsymbol{\mathcal{V}}_T$, it is helpful to prove a few properties of $\boldsymbol{\mathcal{V}}_T$. For this we need the notion of a closed set.

Definition. For a finite set $E \in \text{fin}(A)$, we say that E is *closed* if, for all $i \in I$,

$$A_i \cap E \neq \varnothing \to A_i \subseteq E.$$

We are able to prove that every set in $\boldsymbol{\mathcal{V}}_T$ has a unique least closed support.

Fact 8.19. *If E_x and E'_x are two closed finite supports of some $x \in \boldsymbol{\mathcal{V}}_T$, then $E_x \cap E'_x$ is also a closed finite support of x. Furthermore, for every $x \in \boldsymbol{\mathcal{V}}_T$ there exists a least closed finite support of x.*

Proof. Let E_x and E'_x be two closed finite supports of $x \in \boldsymbol{\mathcal{V}}_T$ and define $E := E_x \cap E'_x$. It is easily shown that the intersection of two closed sets is closed. If $A_i \cap (E_x \cap E'_x) \neq \varnothing$, for some $i \in I$, then $A_i \cap E_x \neq \varnothing$ as well as $A_i \cap E'_x \neq \varnothing$. Therefore $A_i \subseteq E_x$ and $A_i \subseteq E'_x$ which implies that $A_i \subseteq E_x \cap E'_x$, and therefore proves the desired closedness of the intersection.
So it remains to show that the intersection is indeed a support of x, i.e. we need to show that $\text{fix}_{\mathcal{G}}(E) \subseteq \text{sym}_{\mathcal{G}}(x)$. Observe that $\text{fix}_{\mathcal{G}}(E)$ is generated by the permutations in $\text{fix}_{\mathcal{G}}(E_x)$ and $\text{fix}_{\mathcal{G}}(E'_x)$. A permutation that is composed of elements of $\text{fix}_{\mathcal{G}}(E_x)$ and $\text{fix}_{\mathcal{G}}(E'_x)$ clearly leaves E fixed. To see that every in $\text{fix}_{\mathcal{G}}(E)$ can be written as such a composition, note that, since both E_x and E'_x are assumed to be finite, $E_x \setminus E'_x$ as well as $E'_x \setminus E_x$ are finite. Our group of permutations $\mathcal{G}$ consists of exactly those permutations of A which move just finitely many elements of A. Hence, we can find a permutation in $\text{fix}_{\mathcal{G}}(E_x)$ that moves the elements of $E'_x \setminus E_x$ and conversely a permutation in $\text{fix}_{\mathcal{G}}(E'_x)$ that moves the elements of $E_x \setminus E'_x$. So for every $\pi \in \text{fix}_{\mathcal{G}}(E)$ we can find $\rho_1, \ldots, \rho_n \in \text{fix}_{\mathcal{G}}(E_x)$ and $\sigma_1, \ldots, \sigma_n \in \text{fix}_{\mathcal{G}}(E'_x)$ such that

$$\pi = \rho_n \sigma_n \cdots \rho_1 \sigma_1.$$

Since $\text{fix}_{\mathcal{G}}(E'_x) \cup \text{fix}_{\mathcal{G}}(E_x) \subseteq \text{sym}_{\mathcal{G}}(x)$, we immediately obtain that

$$\pi x = \rho_n \sigma_n \cdots \rho_1 \sigma_1 x = \rho_n \sigma_n \cdots \rho_1 x = \cdots = \rho_n x = x,$$

i.e. $\pi \in \text{sym}_{\mathcal{G}}(x)$, as desired. □

Lemma 8.20. *Let $(P, \leq)$ be a partially ordered set in $\boldsymbol{\mathcal{V}}_T$ with support $E \in \text{fin}(A)$. Then for each $p \in P$, the set $\mathscr{O}_E(p) := \{\pi p : \pi \in \text{fix}_{\mathcal{G}}(E)\}$ is an anti-chain in P.*

Proof. We assume by contradiction that for some $p \in P$ we can find two elements q_0 and q_1 in $\mathscr{O}_E(p)$ that are comparable. Without loss of generality we can assume that $q_0 < q_1$. Now, by definition of $\mathscr{O}_E(p)$, we can find $\pi_0, \pi_1 \in \text{fix}_{\mathcal{G}}(E)$ such that $\pi_0 p = q_0$ and $\pi_1 p = q_1$.
For $\tau := \pi_1^{-1}\pi_0$ we obtain that

$$\tau p = \pi_1^{-1}\pi_0 p = \pi_1^{-1} q_0 < \pi_1^{-1} q_1 = p.$$

Since $\tau \in \mathcal{G}$ only moves finitely many elements of A, there exists a $k \in \omega$ such that $\tau^k = \iota$ where ι denotes the identity permutation. We then get the following chain of inequalities

$$p = \tau^k p < \tau^{k-1} p < \cdots < \tau p < p,$$

i.e. $p < p$, which is clearly a contradiction. Hence the elements of $\mathscr{O}_E(p)$ are not comparable and therefore form an anti-chain in P. □

From now on we excessively use the notion of a well-order, so it seems logical to briefly recall it here:

Definition. A linear ordering $\leq$ on a set A is a *well-ordering* on A if every non-empty subset $S \subseteq A$ has a $\leq$-minimal element, i.e., there exists an $x_0 \in S$ such that for each $y \in S$ we have $x_0 \leq y$.
If there exists a well-ordering on a set A, we say that A is *well-orderable.*

We obtain the next result as a corollary of Lemma 8.20.

Lemma 8.21. *Every partially ordered set $(P, \leq)$ in $\boldsymbol{\mathcal{V}}_T$ can be written as a well-orderable union of anti-chains.*

Proof. Suppose that the partially ordered set $(P, \leq)$ has support $E \in \text{fin}(A)$. We have that

$$\mathscr{O} := \{\mathscr{O}_E(p) : p \in P\}$$

is a partition of P and in Lemma 8.20, we have seen that each element of $\mathscr{O}$ is an anti-chain in P. Now recall from B7 that $\boldsymbol{\mathcal{M}}$ is a transitive model of ZFA, in which the Axiom of Choice (AC) holds. Since we constructed the permutation model $\boldsymbol{\mathcal{V}}_T$ as a subclass of $\boldsymbol{\mathcal{M}}$, every set can be well-ordered in $\boldsymbol{\mathcal{M}}$, due to the equivalence of AC and the well-ordering theorem. In particular, P is indeed a well-orderable union of anti-chains. □

Fact 8.22. *Let $x \in \boldsymbol{\mathcal{V}}_T$ be a set with support $E \in \text{fin}(A)$. If E is a support for each $z \in x$, then x can be well-ordered in $\boldsymbol{\mathcal{V}}_T$.*

Proof. We have constructed $\boldsymbol{\mathcal{V}}_T$ as a subclass of $\boldsymbol{\mathcal{M}}$, where $\boldsymbol{\mathcal{M}}$ is a model of ZFA in which AC holds. Thus, by the well-ordering theorem, we can well-order the set x in $\boldsymbol{\mathcal{M}}$, i.e. there exists a bijection $f \in \boldsymbol{\mathcal{M}}$ such that $f : \alpha \to x$ for some ordinal $\alpha \in \Omega$.
We claim that $\text{fix}_{\mathcal{G}}(E) \subseteq \text{sym}_{\mathcal{G}}(f)$. From this, it follows that $f \in \boldsymbol{\mathcal{V}}_T$ and hence x can be well-ordered in $\boldsymbol{\mathcal{V}}_T$.
To prove the claim, consider any $\pi \in \text{fix}_{\mathcal{G}}(E)$ and some $\langle \beta, z \rangle \in f$. Now β is an ordinal and therefore belongs to the kernel by definition. Furthermore, since $z \in x$, E is also a support of z and because $\pi \in \text{fix}_{\mathcal{G}}(E)$ we have $\pi z = z$. This yields that

$$\pi(\langle \beta, z \rangle) = \langle \pi\beta, \pi z \rangle = \langle \beta, z \rangle.$$

We derive that $\pi f = f$, as $\langle \beta, z \rangle$ was arbitrary. Now, by arbitrariness of $\pi \in \text{fix}_{\mathcal{G}}(E)$, we obtain that $\text{fix}_{\mathcal{G}}(E) \subseteq \text{sym}_{\mathcal{G}}(f)$. This proves the claim and thereby the assertion. □

Remark. Note that the only property of $\boldsymbol{\mathcal{V}}_T$ we used in the proof of the previous statement, is that $\boldsymbol{\mathcal{V}}_T$ is a subclass of $\boldsymbol{\mathcal{M}}$. We deduce that this fact therefore holds for any permutation model.

The next result will be crucial for the proof of the fact that CaCP does hold in $\boldsymbol{\mathcal{V}}_T$.

Lemma 8.23. *Let $x \in \boldsymbol{\mathcal{V}}_T$ be a set which cannot be well-ordered. Then x has an infinite subset $y \in \boldsymbol{\mathcal{V}}_T$ which has a partition into sets of cardinality at most two. Moreover, the infinite set y cannot be partitioned into two infinite subsets. In particular, y cannot be well-ordered.*

Remark. Infinite sets that cannot be partitioned into two infinite subsets are called *amorphous.* Lemma 8.23 can therefore be reformulated as:
A set $x \in \boldsymbol{\mathcal{V}}_T$ that cannot be well-ordered, has an amorphous subset y.

We need one final lemma before we can conclude the proof of the main result of this step.

Lemma 8.24. *The union of a well-orderable family of well-orderable sets in $\boldsymbol{\mathcal{V}}_T$ is well-orderable.*

Proof. Let $\gamma \in \Omega$ be an ordinal and let $\{w_\alpha : \alpha \in \gamma\}$ be a family of well-orderable sets w_α. We can assume without loss of generality that the w_α are disjoint for $\alpha \in \gamma$. We define the set

$$x := \bigcup\{w_\alpha : \alpha \in \gamma\}.$$

By contradiction, we assume that x cannot be well-ordered. Then by Lemma 8.23, there exists an amorphous subset $y \subseteq x$. In particular, y cannot be well-ordered. Note that a subset of a well-ordered set is itself well-ordered. Hence, for each $\alpha \in \gamma$, the set $y \cap w_\alpha$ is well-orderable as a subset of the well-orderable set w_α. Since it is also subset of y, $y \cap w_\alpha$ must be finite.
On the other hand y is infinite and γ is an ordinal and thus we can partition y into two infinite subsets, which contradicts the properties of y. □

Now, we are able to conclude the whole proof with the next result.

Lemma 8.25. CaCP *holds in $\boldsymbol{\mathcal{V}}_T$.*

Proof. We now assume towards a contradiction that CaCP does not hold in $\boldsymbol{\mathcal{V}}_T$. That is, we suppose that there exists an infinite partially ordered set that contains neither an infinite chain nor an infinite anti-chain. We denote this set by $(P, \leq)$. We have seen in Lemma 8.21 that P can be written as a well-orderable union of anti-chains. Now, by our assumption, each of these anti-chains have to be finite and therefore can be well-ordered. So we have that P is the union of a well-orderable family of well-orderable sets. Lemma 8.24 thus implies that P itself is well-orderable.
Therefore, for any colouring $\pi : [P]^2 \to 2$, there is always an infinite set $Y \subseteq P$ such that $[Y]^2$ is monochromatic, i.e. P satisfies RPP. Since RPP implies CaCP, we can find an infinite chain or an infinite anti-chain in P, which contradicts our assumptions on P and thereby concludes the proof. □

From Lemma 8.18 and 8.25 we deduce that we have constructed a permutation model $\boldsymbol{\mathcal{V}}_T$ such that CaCP holds, but RPP does not. This is exactly what we intended to prove.

Proposition 8.26. *In* ZFA, CaCP *does not imply* RPP.