KRULL IMPLIES ZORN

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Some time ago, W. Krull observed that by the Axiom of Choice, every commutative ring with $1 \neq 0$ has a maximal ideal. Dana Scott [**2**] asked whether the converse holds: If every commutative ring with $1 \neq 0$ has a maximal ideal, then the Axiom of Choice is true. The answer is Yes. In fact the following stronger statement is true.

THEOREM. *In Zermelo-Fraenkel set theory, the statement " Every unique factorisation domain has a maximal ideal " implies the Axiom of Choice.*

We begin the proof by paraphrasing the Axiom of Choice. By a *tree* we mean a partially ordered set $(T, \leqslant)$ such that for every $t \in T$, the set $\hat{t} = \{r \in T : r \leqslant t\}$ is linearly ordered. A *branch* in the tree is a maximal linearly ordered subset. Two elements r, t of T are said to be *comparable* if either $r \leqslant t$ or $t \leqslant r$.

Let *Tree* be the statement: Every tree has a branch.

LEMMA 1. *Tree is equivalent to the Axiom of Choice.*

Proof. Clearly Choice implies Tree. For the converse, assume Tree; we show that every set can be well-ordered. Let A be any set, and let T be the set of injective maps $f : \alpha \to A$ with α an ordinal. (This is a set, by Hartogs' Theorem.) Put $f \leqslant g$ if and only if g extends f. Then $(T, \leqslant)$ is a tree, which must have a branch B, and $\bigcup B$ is an injective map $h : \beta \to A$ for some ordinal β. But h is also onto A, or we could extend h within T, contradicting maximality.

Now let $(T, \leqslant)$ be a tree. We construct a ring $R(T, \leqslant)$ as follows. Let F be the field of rationals, and form the polynomial ring $F[T]$ with the elements $t \in T$ as indeterminates. This ring $F[T]$ is a unique factorisation domain. (No Choice is needed here.) If $G \subseteq T$, then $GF[T]$ is a prime ideal of $F[T]$.

Let L be the set of linearly ordered subsets of T, and define

$$S = F[T] - \bigcup_{G \in L} GF[T].$$

Then S is the complement of a union of prime ideals, so S is multiplicatively closed. Inverting S, we put

$$R(T, \leqslant) = S^{-1}F[T].$$

Every element c of $R(T, \leqslant)$ is of the form x/s with $x \in F[T]$ and $s \in S$; if common factors are cancelled, x and s are unique up to factors in F. The element c is invertible in $R(T, \leqslant)$ if and only if for all $t \in T$, $x \notin \hat{t}F[T]$. The ring $R(T, \leqslant)$ is a unique

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factorisation domain. (Still no Choice is needed.) Readers of Section 9 of Gordon and Robson [1] will see where this construction comes from.

Until further notice, put $R = R(T, \leqslant)$, and suppose that R has a maximal ideal M. Let c be any element of M; express c as x/s, so that x and s have no common non-scalar factors. Then x can be written as $q_1m_1 + \ldots + q_nm_n$ with $q_1, \ldots, q_n$ non-zero elements of F and $m_1, \ldots, m_n$ distinct monomials over T. Since c is not invertible, there is at least one finite linearly ordered set $A \subseteq T$ such that (1) each monomial m_i has a factor in A, and (2) each element of A occurs as a factor of some m_i. The set A is not necessarily unique, but clearly there are at most finitely many choices for it, say $A_1, \ldots, A_k$. Put $E(c) = \{\max A_i : 1 \leqslant i \leqslant k\}$. If $t \in E(c)$, then $c \in \hat{t}R$. If $c = 0$, then A is empty. If $c \neq 0$, and d is any other element of M which involves the monomials $m_1, \ldots, m_n$ (and perhaps others), then for every $r \in E(d)$ there is $t \in E(c)$ such that $t \leqslant r$.

We define D to be the set of those $t \in T$ such that for every non-zero $c \in M$, there is $r \in E(c)$ which is comparable with t.

LEMMA 2. $D \subseteq M$.

Proof. Let $t \in D$. If $t \notin M$, then since M is maximal, there are elements $a \in R$ and $c \in M$ such that $at + c = 1$. Since $\{t\} \in L$, t is not invertible, and so c is non-zero. Hence, by definition of D, there is an $r \in E(c)$ which is comparable with t, so $c \in \hat{r}R$. If $r \leqslant t$, then c, $t \in \hat{t}R$, and so $1 \in \hat{t}R$; if $t \leqslant r$, then c, $t \in \hat{r}R$ and so $1 \in \hat{r}R$. In both cases we have a contradiction to the definition of R.

LEMMA 3. *The set D is a linearly ordered initial segment of T.*

Proof. If t, $w \in D$ then by Lemma 2, $t + w \in M$ and hence $t + w$ is not invertible. It follows from the construction of R that t and w are comparable. Suppose that $v \leqslant t$ in T, and c is a non-zero element of M. Then there is $r \in E(c)$ which is comparable with t. If $t \leqslant r$, then $v \leqslant r$. If $r \leqslant t$, then r and v are comparable since $\hat{t}$ is linearly ordered. Hence $v \in D$.

LEMMA 4. $M \subseteq DR$.

Proof. Suppose there is an element $c \in M - DR$; we can assume without loss that c is a non-zero element of $F[T]$. No element t of D is in $E(c)$; for otherwise $c \in \hat{t}R \subseteq DR$, using Lemma 3. Let $t_1, \ldots, t_k$ be the distinct elements of $E(c)$. Since none of these is in D, there is, for each $i (1 \leqslant i \leqslant k)$, a non-zero element b_i of M such that for all $r \in E(b_i)$, r is incomparable with t_i. Without loss we can suppose that $b_i \in F[T]$. Now F is an infinite field, so that we can choose scalars $q_1, \ldots, q_k \in F$ in such a way that in

$$x = c + q_1b_1 + \ldots + q_kb_k$$

no monomial which occurs with non-zero coefficient in c or some b_i vanishes in x. Suppose that $w \in E(x)$. Then (by the choice of the numbers q_i) there is some t_j such that $t_j \leqslant w$. But (for the same reason) there is some $r \in E(b_j)$ such that $r \leqslant w$. Hence t_j and r are comparable, which contradicts the choice of b_j.

Now we prove the theorem.

Assume that every unique factorisation domain has a maximal ideal; we prove Tree. Let $(T, \leqslant)$ be a tree, and let R be $R(T, \leqslant)$. By assumption, R has a maximal ideal M. By Lemmas 2 and 4, $M = DR$ where D is defined as above. If D is not a branch of T, then by Lemma 3 there is $t \in T$ such that $t > r$ for all $r \in D$. Then $M = DR \subset \hat{t}R$, contradicting the maximality of M. Hence $(T, \leqslant)$ has a branch D as required.

References

1. R. Gordon and J. C. Robson, " Krull dimension ", *Mem. Amer. Math. Soc.*, 133 (1973).
2. D. S. Scott, " Prime ideal theorems for rings, lattices and Boolean algebras ", *Bull. Amer. Math. Soc.*, 60 (1954), 390.

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