Calculus: From Tangents and Areas to Derivatives and Integrals

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5.1 Introduction

The invention of calculus is one of the great intellectual and technical achievements of civilization. Calculus has served for three centuries as the principal quantitative tool for the investigation of scientific problems. It has given mathematical expression to such fundamental concepts as velocity, acceleration, and continuity, and to aspects of the infinitely large and infinitely small—notions that have formed the basis for much mathematical and philosophical speculation since ancient times. Modern physics and technology would be impossible without calculus. The most important equations of mechanics, astronomy, and the physical sciences in general are differential and integral equations—outgrowths of the calculus of the seventeenth century. Other major branches of mathematics derived from calculus are real analysis, complex analysis, and calculus of variations. Calculus is also fundamental in probability, topology, Lie group theory, and aspects of algebra, geometry, and number theory. In fact, mathematics as we know it today would be inconceivable without the ideas of calculus.

Isaac Newton and Gottfried Wilhelm Leibniz independently invented calculus during the last third of the seventeenth century. But their work was neither the beginning of the story nor its end. Practically all of the prominent mathematicians of Europe around 1650 could solve many of the problems in which elementary calculus is now used—but providing their procedures with rigorous foundations required two more centuries.

The infinitely small and the infinitely large—in one form or another—are essential in calculus. In fact, they are among the features which most distinguish that branch of mathematics from others. They have appeared throughout the history of calculus in various guises: infinitesimals, indivisibles, differentials, "evanescent" quantities, moments, infinitely large and infinitely small magnitudes, infinite sums, and power series. Also they have been fundamental at both the technical and conceptual levels—as underlying tools of the subject and as its foundational underpinnings. We will give examples of these manifestations of the infinite in the earlier evolution of calculus (seventeenth and eighteenth centuries).

5.2 Seventeenth-Century Predecessors of Newton and Leibniz

The Renaissance (ca. 1400–1600) saw a flowering and vigorous development of the visual arts, literature, music, the sciences, and—not least—mathematics. It witnessed the decisive triumph of positional decimal arithmetic, the introduction of algebraic symbolism, the solution by radicals of the cubic and quartic, the free use if not full understanding of irrational numbers, the introduction of complex numbers, the rebirth of trigonometry, the establishment of a relationship between mathematics and the arts through perspective drawing, and a revolution in astronomy, later to prove of great significance for mathematics. A number of these developments were necessary prerequisites for the rise of calculus, as was the invention of analytic geometry by René Descartes and by Pierre de Fermat in the early decades of the seventeenth century (see ▶ Chapter 3).

The Renaissance also saw the full recovery and serious study of the mathematical works of the Greeks, especially Archimedes' masterpieces. His calculations of areas, volumes, and centers of gravity were an inspiration to many mathematicians of that period. Some went beyond Archimedes in attempting systematic calculations of the centers of gravity of solids. But they used the classical "method of exhaustion" of the Greeks, which was conducive neither to the discovery of results nor to the development of algorithms. The temper of the times was such that most mathematicians were far more interested in results than in proofs; rigor, declared Bonaventura Cavalieri in the 1630s, "is the concern of philosophy and not of geometry [mathematics]" [10, p. 383]. To obtain results, mathematicians devised new methods for the solution of calculus-type problems. These were based on geometric, algebraic, and arithmetic ideas, often in interplay. We give two examples.

Cavalieri

A major tool for the investigation of calculus problems was the notion of an *indivisible*. This idea—in the form, for example, of an area as composed of a sum of infinitely many parallel lines, regarded as atomistic—was embodied in Greek physical theory and was also part of medieval scientific thought. Mathematicians of the seventeenth century fashioned indivisibles into a powerful tool for the investigation of area and volume problems.

Indivisibles were used in calculus by Galileo and others in the early seventeenth century, but it was Cavalieri who, in his influential *Geometry of Indivisibles* of 1635, shaped a vague concept into a useful technique for the determination of areas and volumes. His strategy was to consider a geometric figure to be composed of an infinite number of indivisibles *of lower dimension*. Thus a surface consists of an infinite number of equally spaced parallel lines, and a solid of an infinite number of equally spaced parallel planes. The procedure for finding the area (or volume) of a figure is to compare it to a second figure of equal height (or width), whose area (or volume) is known, by setting up a one-to-one correspondence between the indivisible elements of the two figures and using "Cavalieri's Principle": if the corresponding indivisible elements are always in a given ratio, then the areas (or volumes) of the two figures are in the *same* ratio. For example, it is easy to show that the ordinates of the ellipse $x^2/a^2+y^2/b^2=1$ are to the corresponding ordinates of the circle $x^2+y^2=a^2$ in the ratio b:a (see Figure 5.1), hence the area of the ellipse = (b/a) × the area of the circle = πab.

Fermat

Fermat was the first to tackle systematically the problem of tangents. In the 1630s he devised a method for finding tangents to any polynomial curve. The following example illustrates his approach.

Suppose we wish to find the tangent to the parabola $y=x^2$ at some point (x, x^2) on it. Let x + e be a point on the x-axis and let s denote the "subtangent" to the curve at the point (x, x^2) (see Figure 5.2). Similarity of triangles yields $x^2/s=k/(s+e)$. Fermat notes that k is "adequal" to $(x+e)^2$, presumably meaning "as nearly equal as possible", although he does not say so. Writing this as $k\cong(x+e)^2$, we get $x^2/s\cong(x+e)^2/(s+e)$. Solving for s we have $s\cong ex^2/[(x+e)^2-x^2]=ex^2/e(2x+e)=x^2/(2x+e)$. It follows that $x^2/s\cong 2x+e$. Note that x^2/s is the slope of the tangent to the parabola at (x, x^2). Fermat now "deletes" the e and claims that the slope of the tangent is 2x.

Fermat's method was severely criticized by some of his contemporaries, notably Descartes. They objected to his introduction and subsequent suppression of the "mysterious e". Dividing by e meant regarding it as not zero—but discarding e implied that it *was* zero. This is inadmissible, they rightly claimed. But Fermat's mysterious e embodied a crucial idea: the giving of a "small" increment to a variable. And it cried out for the limit concept, which was formally introduced only about two hundred years later. Fermat, however, considered his method to be purely *algebraic*.

The above examples give us a glimpse of the near-century of vigorous investigations in calculus prior to the work of Newton and Leibniz. Mathematicians plunged boldly into almost virgin territory—the mathematical infinite—where a more critical age might have feared to tread. They produced a multitude of powerful, if nonrigorous, infinitesimal techniques for the solution of area, volume, and tangent problems. What, then, was left for Leibniz and Newton to do?

Bonaventura Cavalieri (1598–1647)

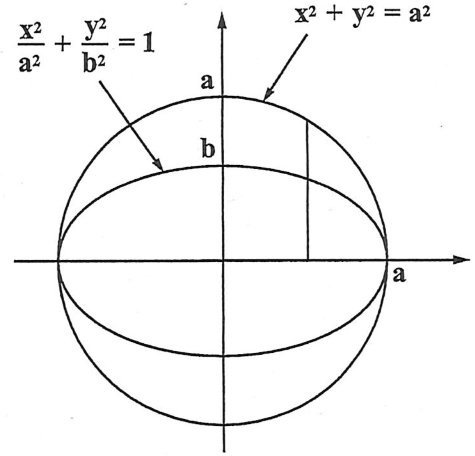


Figure 5.1 Area of an ellipse

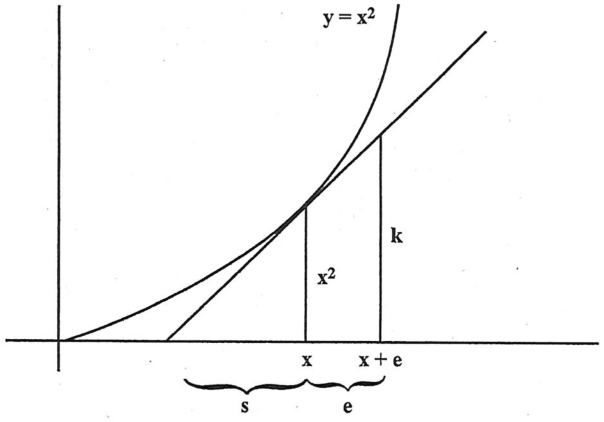


Figure 5.2 Finding the tangent to a parabola

5.3 Newton and Leibniz: The Inventors of Calculus

In the first two thirds of the seventeenth century mathematicians solved calculus-type problems, but they lacked a general framework in which to place them. This was provided by Newton and Leibniz. Specifically, they

a. invented the general concepts of derivative and integral—though not in the form we see them today. For example, it is one thing to compute areas of curvilinear figures and volumes of solids using ad hoc methods, but quite another to recognize that such problems can be subsumed under a single concept, namely the integral.
b. recognized differentiation and integration as inverse operations. Although several mathematicians before Newton and Leibniz noted the relation between tangent and area problems, mainly in specific cases, the clear and explicit recognition, in its complete generality, of what we now call the Fundamental Theorem of Calculus belongs to Newton and Leibniz.
c. devised a notation and developed algorithms to make calculus a powerful computational instrument.
d. extended the range of applicability of the methods of calculus. While in the past those methods were applied mainly to polynomials, often only of low degree, they were now applicable to "all" functions, algebraic and transcendental.

And now to some examples of the calculus as developed by Newton and by Leibniz. We should note that theirs is a calculus of *variables*—which Newton calls "fluents"—and equations relating these variables; it is *not* a calculus of *functions*. The notion of function as an explicit mathematical concept arose only in the early eighteenth century.

Newton

Newton considered a curve to be "the locus of the intersection of two moving lines, one vertical and the other horizontal. The x and y coordinates of the moving points are then functions of the time t, specifying the locations of the vertical and horizontal lines respectively" [4, p. 193]. Newton's basic concept is that of a "fluxion", denoted by $\dot{x}$; it is the instantaneous rate of change (instantaneous velocity) of the fluent x—in our notation, dx/dt. The instantaneous velocity is not defined, but is taken as intuitively understood. Newton aims rather to show how to *compute* $\dot{x}$.

Isaac Newton (1642–1727)

The following is an example of Newton's computation of the tangent to a curve with equation $x^3 - ax^2 + axy - y^3 = 0$ at an arbitrary point (x, y) on the curve. He lets o be an infinitesimal period of time. Then $\dot{x}o$ and $\dot{y}o$ are infinitesimal increments in x and y, respectively. (For, we have distance = velocity × time = $\dot{x}o$ or $\dot{y}o$, assuming with Newton that the instantaneous velocities $\dot{x}$ and $\dot{y}$ of the point (x, y) moving along the curve remain constant throughout the infinitely small time interval o.) Newton calls $\dot{x}o$ and $\dot{y}o$ *moments*, a "moment" of a fluent being the amount by which it increases in an infinitesimal time period. An infinitesimal was not formally defined, but was understood to be an "infinitely small" quantity, less than any finite quantity but not zero. Thus, $(x+\dot{x}o, y+\dot{y}o)$ is a point on the curve infinitesimally close to (x, y). In Newton's words: "Soe y[t] if y[e] described lines [coordinates] bee x and y, in one moment, they will bee $x+\dot{x}o$ and $y+\dot{y}o$ in y[e] next" [4, p. 193]. Substituting $(x+\dot{x}o, y+\dot{y}o)$ into the original equation and simplifying by deleting $x^3 - ax^2 + axy - y^3$ (which equals zero) and dividing by o, we get:

$$3x^2\dot{x} - 2ax\dot{x} + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} + 3x\dot{x}^2o - a\dot{x}^2o + a\dot{x}\dot{y}o - 3\dot{y}y^2o + \dot{x}^3o^2 - \dot{y}^3o^2 = 0.$$

Newton now discards the terms involving o, noting that they are "infinitely lesse" than the remaining terms. This yields an equation relating x and y, namely

$$3x^2\dot{x} - 2ax\dot{x} + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} = 0.$$

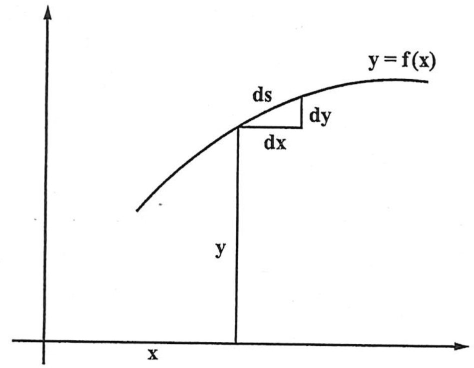
From this relationship we can get the slope of the tangent to the given curve at any point (x, y):

$$\dot{y}/\dot{x} = \frac{3x^2 - 2ax + ay}{3y^2 - ax}.$$

This procedure is quite general, Newton notes, and it enables him to obtain the slope of the tangent to *any* algebraic curve.

The problem of what to make of the "o's"—the "ghosts of departed quantities" [4, p. 294]—remained, according to Bishop George Berkeley, who launched a famous critique. Are they zero? Finite quantities? Infinitely small? Newton's dilemma was not unlike Fermat's a half-century earlier.

Figure 5.3 Leibniz' characteristic triangle



Leibniz

Leibniz' ideas on calculus evolved gradually, and like Newton, he wrote several versions, giving expression to his ripening thoughts. Central to all of them is the concept of "differential", although that notion had different meanings for him at different times.

Leibniz viewed a "curve" as a polygon with infinitely many sides, each of infinitesimal length. (Recall that the Greeks conceived a circle in just that way.) With such a curve is associated an infinite (discrete) sequence of abscissas $x_1, x_2, x_3, \ldots$, and an infinite sequence of ordinates $y_1, y_2, y_3, \ldots$, where (x_i, y_i) are the coordinates of the points of the curve.

The difference between two successive values of x is called the "differential" of x and is denoted by dx; similarly for dy. The differential dx is a fixed nonzero quantity, infinitely small in comparison with x—in effect, an infinitesimal. There is a *sequence* of differentials associated with the curve, namely the sequence of differences $x_i - x_{i-1}$ associated with the abscissas $x_1, x_2, x_3, \ldots$ of the curve [4, pp. 258, 261].

The sides of the polygon constituting the curve are denoted by ds—again, there are infinitely many such infinitesimal ds's. This gives rise to Leibniz' famous "characteristic triangle" with infinitesimal sides dx, dy, ds satisfying the relation $(ds)^2 = (dx)^2 + (dy)^2$ (see Figure 5.3). The side ds of the curve (polygon) is taken as coincident with the tangent to the curve (at the point x). Leibniz put it thus [9, pp. 234–235]:

> We have only to keep in mind that to find a *tangent* means to draw a line that connects two points of the curve at an infinitely small distance, or the continued side of a polygon with an infinite number of angles, which for us takes the place of the *curve*. This infinitely small distance can always be expressed by a known differential like ds.

The slope of the tangent to the curve at the point (x, y) is thus dy/dx—an actual *quotient* of differentials, which Leibniz calls the "differential quotient" (Figure 5.3).

Here are two further examples of his calculus. To discover and "prove" the product rule for differentials, he proceeds as follows:

$d(xy) = (x + dx)(y + dy) - xy = xy + xdy + ydx + (dx)(dy) - xy = xdy + ydx$. He omits (dx)(dy), noting that it is "infinitely small in comparison with the rest" [4, p. 255].

As a second example, Leibniz finds the tangent at a point (x, y) to the conic $x^2 + 2xy = 5$: Replacing x and y by x + dx and y + dy, respectively, and noting that (x + dx, y + dy) is a point on the conic "infinitely close" to (x, y), we get

$$(x + dx)^2 + 2(x + dx)(y + dy) = 5 = x^2 + 2xy.$$

Gottfried Wilhelm Leibniz (1646–1716)

Simplifying, and discarding (dx)(dy) and $(dx)^2$, which are assumed to be negligible in comparison with dx and dy, yields $2xdx + 2xdy + 2ydx = 0$. Dividing by dx and solving for dy/dx gives $dy/dx = (-x - y)/x$. This is of course what *we* would get by writing $x^2 + 2xy = 5$ as $y = (5 - x^2)/2x$ and differentiating this *functional* relation. (Recall that Leibniz' calculus predates the emergence of the function concept.)

We see in these examples how Leibniz' choice of a felicitous notation enabled him to arrive very quickly at reasonable convictions, if not rigorous proofs, of important results. His symbolic notation served not only to *prove* results but also greatly facilitated their *discovery*.

5.4 The Eighteenth Century: Euler

Brilliant as the accomplishments of Newton and Leibniz were, their respective versions of calculus consisted largely of loosely connected methods and problems, and were not easily accessible to the mathematical public, such as that was. The first systematic introduction to the Leibnizian differential calculus was given in 1696 by Guillaume de L'Hospital in his text *The Analysis of the Infinitely Small, for the Understanding of Curved Lines.* Calculus was further developed during the early decades of the eighteenth century, especially by the Bernoulli brothers Jakob and Johann. Several books appeared during this period, but the subject lacked focus. The main contemporary concern of calculus was with the geometry of curves—tangents, areas, volumes, and lengths of arcs (cf. the title of L'Hospital's text). Of course Newton and Leibniz introduced an algebraic apparatus, but its motivation and the problems to which it was applied were geometric or physical, having to do with curves. In particular, this was (as we already noted) a calculus of variables related by equations rather than a calculus of functions.

A fundamental conceptual breakthrough was achieved by Euler around the mid-eighteenth century. This was to make the concept of *function* the centerpiece of calculus. Thus calculus is not about curves, asserted Euler, but about functions. The derivative and the integral are not merely abstractions of the notions of tangent or instantaneous velocity on the one hand and of area or volume on the other—they are the basic concepts of calculus, to be investigated in their own right. But mathematicians of the eighteenth century did not readily embrace this centrality of functions, especially since variables seemed to serve them well.

Power series played a fundamental role in the calculus of the seventeenth and eighteenth centuries, especially in Newton's and Euler's. They were viewed as infinite polynomials with little, if any, concern for convergence. The following is an example of Euler's derivation of the power-series expansion of sin*x*, employing infinitesimal tools with great artistry [4, p. 235]:

Use the binomial theorem to expand the left-hand side of the identity $(\cos z + i\sin z)^n = \cos(nz) + i\sin(nz)$, and equate the imaginary part to $\sin(nz)$. We then get:

$$\sin(nz) = n(\cos z)^{n-1}(\sin z) - [n(n-1)(n-2)/3!](\cos z)^{n-3}(\sin z)^3 + [n(n-1)(n-2)(n-3)(n-4)/5!](\cos z)^{n-5}(\sin z)^5 - \ldots \quad (5.1)$$

Now let n be an infinitely large integer and z an infinitely small number (Euler sees no need to explain what these are). Then

$$\cos z = 1, \sin z = z, n(n-1)(n-2) = n^3, n(n-1)(n-2)(n-3)(n-4) = n^5 \ldots$$

(again no explanation from Euler, although of course we can surmise what he had in mind). Equation 5.1 now becomes

$$\sin(nz) = nz - (n^3z^3)/3! + (n^5z^5)/5! - \ldots$$

Let now nz=x. Euler claims that x is finite since n is infinitely large and z infinitely small. This finally yields the power-series expansion of the sine function:.

$\sin x = x - x^3/3! + x^5/5! - \ldots$. It takes one's breath away!

This formal, algebraic style of analysis, used so brilliantly by Euler and practiced by most eighteenth-century mathematicians, is astonishing. It accepted as articles of faith that what is true for convergent series is true for divergent series, what is true for finite quantities is true for infinitely large and infinitely small quantities, and what is true for polynomials is true for power series. Mathematicians put their trust in such broad principles because for the most part they yielded correct results.

5.5 A Look Ahead: Foundations

Mathematicians of the seventeenth and eighteenth centuries realized that the subject they were creating was not on firm ground. For example, Newton affirmed of his fluxions that they were "rather briefly explained than narrowly demonstrated" [4, p. 201]. Leibniz said of his differentials that "it will be sufficient simply to make use of them as a tool that has advantages for the purpose of calculation" [4, p. 265]. The Berlin Academy offered a prize in 1784, hoping that "it can be explained how so many true theorems have been deduced from a contradictory supposition [namely, the existence of infinitesimals]" [6, p. 41]. Lagrange made an elaborate—but essentially misguided—response to this challenge, although his work could be justified in the contemporary setting.

In the late eighteenth and early nineteenth centuries, the work of Lagrange, Joseph Fourier, and others forced mathematicians to confront the lack of rigor in calculus. Here is Niels-Henrik Abel on the subject [11, p. 973]:

> » Divergent series [employed by Newton, Euler, and others] are the invention of the devil. By using them, one may draw any conclusion he pleases, and that is why these series have produced so many fallacies and so many paradoxes.

Starting in 1821 and continuing for about half a century, a series of mathematicians, including Augustin-Louis Cauchy, Bernard Bolzano, Richard Dedekind, and Karl Weierstrass, supplied calculus with foundations, essentially as we have them today. The main features of their work were:

I. The emergence of the notion of limit as the underlying concept of calculus.
II. The recognition of the important role played—in definitions and proofs—by inequalities.
III. The acknowledgement that the validity of results in calculus must take into account questions of the domain of definition of a function. (In the eighteenth century a theorem of calculus was usually regarded as universally true by virtue of the *formal* correctness of the underlying algebra.)
IV. The realization that for a logical foundation of calculus one must have a clear understanding of the nature of the real number system, and that this understanding should be based on an *arithmetic* rather than a geometric conception of the continuum of real numbers.

The work on foundations of calculus did away "for good" with infinitesimals—used by Cauchy and his predecessors for over two centuries (two millennia, if we consider the Greek contributions). In 1960, infinitesimals were actually brought back to life, as genuine and rigorously defined mathematical objects, in the "nonstandard analysis" conceived by the mathematical logician Abraham Robinson—but that is another story!

Augustin-Louis Cauchy (1789–1857)

Problems and Projects

1. Describe some of Pascal's, Roberval's, or Wallis' work in calculus.
2. Discuss the priority dispute between Newton and Leibniz concerning the invention of calculus.
3. Write a short essay on Archimedes' *Method*.
4. Discuss Euler's use of power series.
5. Describe the essential elements in Lagrange's algebraic approach to calculus.
6. Discuss Bishop George Berkeley's critique of Newton's calculus.
7. Write an essay on the "Arithmetization of Analysis". See [1, 4, 8, 11].
8. Discuss some of the errors in calculus in the late eighteenth and early nineteenth centuries resulting from the lack of proper foundations. See [1, 5, 8].
9. Write a brief essay on the basic ideas of nonstandard analysis. See [2–4, 7].

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