and specified at due length at the end of Section 2.2.2 in terms of the disciplinary matrix and its elements.

A group of 'laws' may be treated in connection with Crowe's own main example, the history of vector analysis. 'New mathematical concepts frequently come forth . . . against the efforts of the mathematicians who create them' ('Law' 1). This formulation might be elegant, but it is misleading. Take Hamilton's invention of quaternions. Hamilton certainly did not struggle against quaternions, and quaternions did not come forth by themselves. Hamilton's work was guided by the exemplar of complex numbers, and one of the elements of the disciplinary matrix of his time was that multiplication was a commutative operation. Only after a long period of strenuous work did Hamilton abandon commutativity and find quaternions (Crowe 1967*a*). The problem Hamilton attacked was normal mathematics (Crowe 1967*a*, p. 12), but it could not be solved in a normal way; it grew into an anomaly. For a man like Hamilton who stubbornly stuck to the problem for many years, it is highly probable that in course of time he would try lines of thought that deviated more and more from the usual ways. Thus to find a solution like quaternions presupposes much time and effort, necessary for removing the restrictions imposed by accepted beliefs and concepts. This, I think, is what Crowe calls the struggle against new concepts.

'Many new mathematical concepts . . . meet forceful resistance after their appearance and achieve acceptance only after an extended period of time' ('Law' 2). The mathematical community, like the creator of a new concept, only reluctantly abandons some of its accepted beliefs and concepts. This could be seen functionally. A light-handed use of new concepts that break with many implicit restrictions and beliefs would endanger the very basis of the communication of the community. Furthermore, most mathematicians are concerned solely with normal mathematics and take no pains to understand and appreciate new and peculiar concepts and theories. Thus it takes a long time for an unusual concept like that of quaternions to be accepted, and sometimes inventions are overlooked completely.

'The fame of the creator of a new mathematical concept has a powerful . . . role in the acceptance of that . . . concept, at least if the new concept breaks with traditions' ('Law' 6). Reputation has been considered as functional in many ways by sociologists of science. The main point is that reputation ensures the disciplinary competence of a member of the mathematical community. As to the cited 'law', this plainly means that for simple reasons of economy the members of the mathematical community are more willing to spend time on the non-normal work of a famous mathematician than on that of an outsider. This is even more so when, as in the case of Hamilton and Grassman, the work of the outsider even looks strange and outsiderish.

'New mathematical creations frequently arise within . . . contexts far larger than the preserved contents of these creations . . .' ('Law' 7). A new concept or

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theory as it is worked out by an outsider or a mathematician who has over the course of a long period of work drifted away from the normal paths and has connected different ideas in unusual ways will very probably be framed in a peculiar, personal way. Furthermore, mathematicians who break with tradition and violate accepted belief tend to put something else in its place. They will either interpret their creations in a way to make them as compatible as possible with the contemporary disciplinary matrix, or they will draw up—as justification—a philosophy of their own, which helps them to keep their professional identity, and which will be closely connected with the new concept or theory. The same process will—in other forms—take place collectively in the process of acceptance by the mathematical community. Many of the peculiarities will be done away with, but some piece of what is not the 'content' (again, what is it?) may become a common possession.

'Multiple independent discoveries of mathematical concepts are the rule, not the exception' ('Law' 8). Crowe points to his 'Laws' 2 and 7 for a partial explanation. I can understand this only in the sense that the extended period of acceptance of new concepts gives room for independent discoveries, but this is a minor point as to the explanation of multiples. I do not pretend to be able to come forth with a general explanation for multiples in the history of mathematics; each one is different and each is multifactored. But the fact that multiple discoveries are frequent is a strong point in arguing that the interaction of the mathematicians in their community is the vital basis of the development of mathematics. The fact that discoveries are 'in the air' can only be rationally explained by the contemporary disciplinary matrix, the combination of certain elements of which, like exemplars, concepts, problems, and values, plus the existence of anomalies and, maybe, extra-mathematical influences, make possible the rise of certain new concepts. Again, the history of vector analysis is an example. Crowe (1967*a*, pp. 48, 248) speaks of a 'trend' or a 'movement' which evolved from different traditions. A comparative study considering the relation of each attempt to the contemporary background ought to throw more light on the causes of this 'trend'. To end this discussion of the 'laws' connected with the history of vector analysis, I should like to add that Crowe's book is—in the light of the viewpoints put forward in this chapter—an excellent piece of history of mathematics.

'Although the demands of logic, consistency, and rigour have at times urged the rejection of some concepts now accepted, the usefulness of these concepts has repeatedly forced mathematicians to accept and to tolerate them . . .' ('Law' 3). This regularity in the history of mathematics should be explained in terms of the basic beliefs and values of the mathematical community. I have argued above that fruitfulness is a value of higher priority than rigour. Furthermore, there is the generally implicit belief that mathematics is concerned with the solution of problems. Even if at times it might seem that the mathematical community, or part of it, is aiming at the construction of nice

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theories, in the case of Crowe's example, the imaginary numbers, mathematics was aimed at problem-solutions and those impossible numbers were an important solution. Questions of foundations and of ontology are left to the philosophers or to philosophically minded mathematicians, who are tolerated because they are willing to confront uncomfortable questions that do not concern the true mathematics. What the true mathematics is varies through history, and can be seen only as a consensus of the mathematical community that cannot be sharply delimited.

I have in Section 2.3 given an interpretative idea about the emergence of rigour, to which Crowe's 'Law' 4 alludes. Crowe states that the rigour of textbook presentation is frequently a late acquisition, rather forced upon than sought by the pioneers in the field. It may be added in explanation that before the textbook presentation of a subject there is frequently a period of normal mathematics in the field. In this period attention is given to the details and especially to those points which do not accord with the standards of the mathematical community, and thus rigour is forced upon the presentation of the subject.

NOTES

1. Foundation crises have been treated by J. Thiel (1972). He starts with a basically social definition of 'crisis', seems to forget about it, and ends up in a systematic, philosophical discussion. The book is historically unsatisfying. S. Bochner (1963), in a paper on Kuhn's book, even holds the mathematical foundation crises to be revolutions, without giving much evidence to the point. The considerable main point of his paper is the role of the mathematical paradigm in physics.
2. J. Höppner and myself, in a course on the foundation crises of mathematics given in Hamburg in 1973, have tried to apply the concepts of Kuhn and Thiel. The discussion showed both to be inadequate as generalization of the historical facts and as a guideline for historical inquiry.
3. In an exploratory study of the social features of mathematical problem-solving, C. S. Fisher (1972–3) has given an interesting description of the contemporary mathematical community which he describes as quite diffuse.
4. Possible doubts as to the correctness of this story in the case of Kummer do not weaken the argument (Edwards 1975).
5. This example comes from my research on the prehistory of lattice theory.
6. H. J. M. Bos remarked on this statement that it shows the intrinsic difference between science and mathematics: science uses symbolic generalizations *of something*; mathematics studies symbolic generalizations themselves—they are the concepts.

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3

Appendix (1992): revolutions reconsidered

HERBERT MEHRTENS

Thomas Kuhn's terms 'paradigm' and 'scientific community' have made their way into the standard vocabulary of science. 'Revolutions' appear to be somewhat outmoded, and the 'disciplinary matrix' with its elements is no longer called by this name, but is nevertheless commonly in use in analyses of the sciences and their history. The debate on the relative importance of 'internal' and 'external' factors in the development of science has faded away, and various forms of constructivism and contextualism now dominate advanced historical and theoretical work.

The history of mathematics, however, appears to be somewhat behind the trends. And although there is a substantial body of literature on the social history of mathematics, no integrative history of mathematical knowledge and mathematical practices inside or outside academia has been achieved. The complaint I hear once in a while from historians of mathematics, that their field is too isolated from and too little recognized by colleagues in the history of natural sciences, marks a problem, not of the inaccessibility of mathematics, but rather of the inability of its historians to relate to issues of interest in general history of science. This, to me, is one of the reasons for reconsidering revolutions in mathematics.

Re-reading my paper of 1976, I am quite in agreement with my younger self, although in the meantime I have read and learnt, changed terminology and aspects. The 'community', for example, is a friendly term for a phenomenon which needs a more sober and realistic consideration. Pierre Bourdieu (1975) preferred the term 'scientific field' in his analysis of the symbolic capitalism of science. And sociological systems theory has provided tools for analysis which do not presuppose belief in the values scientists advertise. These 'communities' are social systems structured (not only) functionally and by internal and external power relations. With such an approach, the analysis of, for instance, mathematics in Nazi Germany can escape the perils of legitimatory constructions (Mehrtens 1987, 1988, 1990*b*; Maass 1988).

'Revolution', with its strong political meaning, is a metaphor when applied to science. As such it may be used in historical writing on mathematics.

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Whether the use is adequate or not is a matter of style and of conceptual and historical precision. 'Revolution' means the overthrow of a dominating and pervasive power-structure, and is usually used in a positive sense by the protagonists of the event or their heirs. Used as a metaphor in the history of mathematics, this may apply to dominating traditions, as in the example of Cambridge I gave in the 1976 paper. I would see no problems with the word as long as it is carefully used as a metaphor, that is, in a predominantly stylistical sense. It should be observed, however, that political terms change their meaning and their fields of associations. Revolutions are no longer what they once were; they are not so easily combined with the adjectives 'glorious' or 'great' any more. Maybe this is the main reason why the term appears to be outmoded for the sciences.

Such changing interconnections of meaning lead me to the question of why we should revisit the question of 'revolutionary' developments in mathematics. If there are turns in its history that are construed as so fundamental that they might be called 'revolutionary', these are certainly not just 'in' mathematics. They show mathematics in context and connect it with society, culture, economy, the natural sciences, technology, and so on.

The positive sound of 'revolution' indicates that the term is a value-laden construct. Political revolutions in their construction are connected with dates, governments, and leaders. Scientific revolutions can be constructed in a similar way, marked by names like Galileo, Newton, and Einstein, and by the dates of publications. But this is mainly hero-worship, a somewhat mythical, retrospective construction marking the foundation of a new and better era in science. In such constructions the cultural and political context is usually blanked out or taken in a very specific way as a value-adding mark. If there is to be a serious analytic use of the metaphor, then it should aim at the structures of power and legitimacy before and after the event. In mathematics and the sciences one has to press the metaphor hard to make it work in this sense.

Gaston Bachelard (1938) and Michel Foucault (1969) have used the term 'epistemological rupture' for such fundamental changes. Such a rupture need not be dated—it may be of wide diachronical and synchronical extension. Non-Euclidean geometry, to take the standard example of a 'revolution' in mathematics, was constructed in the 1830s, acknowledged in the 1860s, but questioned until far into the twentieth century. And it was not alone: the rise of symbolic or modern algebra is a parallel development, part of the same rupture in the consciousness of mathematics.[1] Individually and collectively, mathematicians were working to overcome the obstacle implied in Euclidean geometry, the rejection and maybe even fear of a possible multiplicity of geometrical worlds. The obstacle exists as long as geometry is taken to be not a construction but a representation of something, and as such 'true'. Thus there is no definite 'end' to the rupture, although one might interpret the dominance

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of the 'paradigm' of multiple geometrical constructions of immanent truth as the end of the rupture. In this case one should be aware that the self-understanding of the discipline with this paradigm relates itself to the obstacle it has overcome, and it constructs the 'revolution' as part of its historical identity.

As to the context of this rupture, is the fact that the artistic modernism of the latter part of the nineteenth century related to non-Euclidean geometry and to the 'fourth dimension' in its way towards abstraction (Henderson 1983) a parallel development, an effect of the developments in mathematics, or part of a fundamental cultural change? This, I am afraid, is a question that must be rejected, like that of whether Euclidean geometry or non-Euclidean geometry is true. There are obvious historical interconnections between geometry and algebra, mathematics and art, but there is no single definite answer to that question. The interpretation is not in history but in the historian, dependent on the historian's decisions on what to write about, for whom, and with which messages. There may be a good history of the 'revolution in geometry' ignoring algebra, art, and philosophy, given, however, that the author is aware of his decision for ignorance and is able to decide methodically which interpretations and explanations hold water within the limits he is setting. A similar argument would hold for a book with, say, the title *The cultural rupture of the nineteenth century: Truth and representation in the arts, the sciences, and philosophy*. Its author would have to omit the finer structures of the change in geometrical work, and should be aware of this in adopting a methodology.

History is a constructive art and science. We construct the revolutions and ruptures, trying to be scientific by a methodology tying the construction to the remnants of the past and renouncing the conscious production of historical fiction. In the eighteenth century fiction was still possible in historiography, for example, in writing fictitious speeches for historical actors. This change in historiography is, by the way, another candidate for rupture analysis connectible to mathematics as one of the ways of linguistic construction of objectivity. In the history of science, and especially of mathematics, we also have to be aware that we are constructing and writing from the present state of the science. We are well beyond the rupture that established non-Euclidean geometries as a legitimate and important piece of mathematics. We cannot return to a mental state of innocence. We have to be aware of this phenomenon, the *histoire recurrente*, to be able to avoid the implicit teleology and the presentism of traditional history of science (Fichant and Pêcheux 1969; Mehrtens 1990*b*, Section 6.3.3.).

All this admitted, let us return to the 'revolutionary' ruptures. We are looking for *epistemological* shifts; that is, we consider fundamental restructurings of scientific ways of knowing. To be able to mark the obvious difference between mathematics and other fields of knowledge and knowledge production, we need some conception of the specificities of mathematical

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knowledge. Before we can even start with the 'revolutions', we arrive at the infamous question, 'What is *in* mathematics?' It is exactly this question I posed in 1976 about the little word 'in' Michael Crowe used. An answer is only possible if we are aware that it is historical and not universal. The question 'What is . . .' asked about a field of knowledge and knowledge production can be answered only by the self-understanding of the field or by its (possibly deviating) interpretation.

If we talk in chemistry about phlogiston and oxygen and in physics about classical and relativistic mechanics, we might use the term 'revolution' and think of it as the overthrow of beliefs in fundamental truths about the physical world. The 'about' poses the problem if we come to mathematics. Brian Rotman, in his semiotical analysis (1988, p. 34), writes that mathematics 'is "about"—in so far as this locution makes sense—itself. The entire discourse refers to, is "true" about, nothing other than its own signs.' And Davis and Hersh (1981, p. 406) write about 'true facts about imaginary objects'. To be brief and apodictic, mathematics is the construction of sign-systems of a specific kind. These systems work with sign–token combinations signifying the rules for their own use. The rules are of a (grammatically) imperative character. Loosely speaking, mathematical sign combinations encapsulate orders about the use of the very signs that represent these orders (Mehrtens 1990*b*, Chap. 6.3). This semiotic approach is the best I know, because it starts from the simple observation of what mathematicians do and what they have left on paper as results of their work. Clearly, there are hosts of thorny questions to be answered about the epistemology and the practices of mathematics from a semiological point of view.[2] Much needs to be done in this respect. But with this approach we can do away with the unnecessary questions of the 'in' and the 'about'. And we can take the 'truth' and 'meaning' of the mathematical sign-systems historically as the (self-)interpretation of mathematical practices and knowledge. Although historically very useful, this is a modern construction not adequate for the self-understanding of, say, seventeenth-century mathematics. We cannot escape this dilemma, but the struggle between historical adequacy and a presentist historical perspective is not a vicious circle but a productive, epistemic one.

Epistemological ruptures in mathematics relate to the question of what it is that we know (and how we know it) when we know mathematics. Thus they are in the self-understanding of mathematics, not 'in' the sign-systems, but definitely 'in' mathematics understood as a practice of knowledge production and management. Taurinus, in his struggle with the alternative geometry, wrote: 'If the third system (i.e. hyperbolic geometry) were the true one, there would be absolutely no Euclidean geometry.'[3] And Gottlob Frege wrote:

> Nobody can serve two masters. One cannot serve truth and untruth at the same time. If Euclidean geometry is true, then Non-Euclidean geometry is false, and if Non-Euclidean geometry is true, then Euclidean geometry is false. (Frege 1969, p. 183)

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Both authors presuppose what has been the dominating understanding of geometry up to the nineteenth century, its unicity: one world, one geometry, one truth. The modern position in the self-understanding of mathematics is Hilbert's: truth and existence in mathematics are equivalent to consistency. Mathematics can construct multiple symbolic universes with multiple immanent truths.

This is not a 'discovery' about mathematics and truth, but rather a new construction of something called 'truth' within mathematics which has nothing to do any more with truth in any sense relating to representation and objectivity. Mathematical theories are constructions of signs and rules, that indeed present 'true' facts about imaginary objects signified by signs on paper or on blackboards.

The spectacular controversies in the latter half of the nineteenth century and in the first half of the twentieth revolve about this problem of the self-understanding of mathematics, of how we know and what we know in mathematics. This is indeed an epistemological rupture. The change is located in the relation of the producers of knowledge to their product. What does my product, my mathematical theory or theorem, mean, and what is my meaning as the possessor or producer of this knowledge? Is it a gift from the gods, a sign of the real order of the cosmos? Is it a tool of universal reason shared with the divine intellect? Is it our free creation? Is it a perfect imaginary universe or is it an imperfect tool resting on arbitrary assumptions? These are questions connected with the self-construction of mathematics which is also always the self-construction of the value and meaning of the mathematician. I have written elsewhere at length on these aspects of the rise of mathematical modernism and the controversies accompanying it (Mehrtens 1990*b*).

In the first step of mathematics into its status as a science in ancient Greece we may locate the first rupture, when mathematical truth was constituted as something that could be established as such. Earlier mathematics just worked when used, and that was it. Now it could be shown in itself to be working, and that was called truth. We may locate a further rupture in the Renaissance, when autonomous reason was established. And maybe, since Gödel and the introduction of the computer we are now working on a new problem, involved in a new rupture that might lead away from the imaginary unity of mathematics. Hilbert's quest for the universal establishment of mathematical truths in a metamathematics failed. But mathematics still works as it did before the Greeks, and the paradigm of the absolute proof. Today we find that, in electronic computation, very large numbers and possible errors in the recognition of signs play their role, posing new problems that do not look like mathematical problems but might turn out to be problems fundamental to the self-understanding of, say, 'post-modern' mathematics (Davis 1985; Knuth 1976). Since the very concept of 'proof' comes into question, one might even speculate about the end of Euclidean mathematics and the return to a

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manifold system of mathematical practices, whose 'truths' depend on the sign systems working as they are supposed to work. Some such practices might retain the habit of 'rigorous' proofs, but they would no longer dominate the self-understanding of mathematical knowledge-production and knowledge-handling. The 'post-modern' proof and truth would be that with a more-or-less well-determined factor of probability (Specker 1988).

The epistemological self-construction of mathematics is a historical phenomenon. It is a prime factor in the production of new knowledge, as the history of the rupture in geometry shows. It is also a prime factor in the historical understanding of mathematics; we historians should know this, and risk speculations like that on post-modernism once in a while to question our self-understanding and to train our constructive minds for historiographical work. Further, and I return to the point of departure, the self-construction of mathematics and of the mathematician belong to the general social constructions of meaning and order in cultural practices.

Again, the rupture in geometry shows the point. Foucault (1966, Chap. 8) locates the beginning of literature in the modern sense of the term at the opening of the nineteenth century. At this time the understanding and handling of language diverged into formalization on the one hand and interpretation on the other. Gauss, who decidedly spanned the divide between the old order and the new in geometry, turned interpretation into a tool within mathematics. Give the 'impossible numbers' a mathematical interpretation, and they have their mathematical legitimacy. They become 'natural', however, only to the mathematician; the decisive connection is not with some kind of imagery but with a well-established mathematical field of objects. Similarly, Lobachevsky (1898, p. 24) stated that the new geometry, even if it is not realized in nature, can be realized in our imagination and opens manifold possibilities of applying geometry in analysis and vice versa. The British debate on symbolic algebra centred around the use of uninterpreted signs. In the outcome, in algebra and in geometry, there was nothing left to be interpreted about the meaning of signs and theories in images or terms taken from outside mathematics. Mathematicians turned formalization with their artificial sign-language into the centre of their productive work; their internal 'interpretations' became mathematical models or simply new theories, themselves part of formalization. Interpretation became a matter for other cultural fields like literature and history. With this divergence, the question of truth, meaning, and representation in general came to be posed in a new way. It was treated by Cézanne and Picasso, by Mach and Einstein, by Baudelaire and Nietzsche. In many cases one can point to concrete historical interactions. The cultural uses of the new geometries give ample illustrations of the interrelations of these shifts in the general construction of meaning and order. Mathematics is always part of the social system of cultural production of signs and meanings. Its epistemological ruptures may well be analysed by

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concentrating on the work of mathematicians, but they also show us more clearly that mathematics is an integral part of intellectual history. Its isolation and that of its historians is a part of their self-construction and self-understanding. It can be otherwise and, if so, certainly fruitful for a meaningful historiography that is not only presentistic and antiquarian but also futuristic.

NOTES

1. 'It is remarkable too that at the very period in history when significant steps where taken to release geometry from its Euclidean shackles, a similar movement was taking place, quite independently, to rescue algebra from arithmetic' (Dubbey 1977, p. 302; see also Nový 1973, Chap. 6).
2. 'Semiology' is a term coined by Saussure (1916) for the study of signs in their social use and meaning (see also Kristeva 1977).
3. My translation, cited according to Imre Toth, who gives the most convincing interpretation of the rise of non-Euclidean geometry (Toth 1980).

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4

Conceptual revolutions and the history of mathematics: two studies in the growth of knowledge (1984)*

JOSEPH DAUBEN

> In most sciences one generation tears down what another has built, and what one has established another undoes. In mathematics alone each generation builds a new storey to the old structure.
>
> Hermann Hankel

> Je le vois, mais je ne le crois pas.
>
> Georg Cantor

> Transformation, by presenting each anterior concept, theory, law, or principle as the *occasion* of an innovation, focuses attention on the *cause*, the possible reason why only one of the many scientists to whom the scientific idea was known produced the transformation in question.
>
> I. Bernard Cohen

It has often been argued that revolutions do not occur in the history of mathematics and that, unlike the other sciences, mathematics accumulates positive knowledge without revolutionizing or rejecting its past.[1] But there are certain critical moments, even in mathematics, that suggest that revolutions do occur—that new orders are brought about and eventually serve to supplant an older mathematics. Although there are many important examples of such innovation in the history of mathematics, two are particularly instructive: the discovery by the ancient Greeks of incommensurable magnitudes, and the creation of transfinite set theory by Georg Cantor in the nineteenth century. Both examples are as different in character as they are separated in time, and yet each provides a clear instance of a major transformation in mathematical thought. The Greeks' discovery of incommensurable magnitudes brought about changes that were no less significant than the revolutionary transformation mathematics experienced in the twentieth century as a result of Georg

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Cantor's set theory. Taking each of these as marking important transitional periods in mathematics, this essay is an attempt to investigate the character of such transformations.

Recently there has been considerable interest in the growth of mathematics, the nature of that growth, and its relation to the development of knowledge generally. In autumn 1974, at the fiftieth anniversary meeting of the History of Science Society, an entire session was devoted to the historiography of mathematics and to the relationship between the growth of mathematical knowledge and the patterns described in Thomas S. Kuhn's book *The structure of scientific revolutions* (1962, second edition; enlarged, 1970*a*). Naturally, the question of revolutions arose, and with it the problem of whether revolutions occur at all in the history of mathematics. When invited to consider the example of Cantorian set theory, I took the opportunity to suggest that revolutions did indeed occur in mathematics, although the example of transfinite set theory seemed to imply that Cantor's revolutionary work did not fit the framework of Professor Kuhn's model of anomaly–crisis–revolution.[2] Nor is there, perhaps, any reason to expect that a purely logicodeductive discipline like mathematics should undergo the same sort of transformations, or revolutions, as the natural sciences.

Similar interest in the nature of mathematical knowledge and its growth was evidenced at the Workshop on the Evolution of Modern Mathematics held at the American Academy of Arts and Sciences in Boston, 7–9 August 1974. Of all the participants at the workshop, no one questioned the phenomenon of revolutions in mathematics so directly as did Professor Michael Crowe of the University of Notre Dame. In a short paper prepared for the workshop and subsequently published in *Historia Mathematica*, he concluded emphatically with his tenth 'law' that 'revolutions never occur in mathematics'.[3] My intention here, however, is to argue that revolutions can and *do* occur in the history of mathematics, and that the Greeks' discovery of incommensurable magnitudes and Georg Cantor's creation of transfinite set theory are especially appropriate examples of such revolutionary transformations.

4.1. REVOLUTIONS AND THE HISTORY OF MATHEMATICS

Whether one can discern revolutions in any discipline depends upon what one means by the term 'revolution'. In insisting that revolutions never occur in mathematics, Professor Crowe explains that his reason for asserting this 'law' depends on his own definition of revolutions. As he puts it, 'My denial of their existence is based on a somewhat restricted definition of "revolution" which in my view entails the specification that a previously accepted entity *within* mathematics proper be rejected' (Crowe 1975, p. 470). Having said this, however, he is willing to admit that non-Euclidean geometry, for example, 'did

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lead to a revolutionary change in views as to the nature of mathematics, but not within mathematics itself' (Crowe 1975, p. 470).

Certainly one can question the definition Professor Crowe adopts for 'revolution'. It is unnecessarily restrictive, and in the case of mathematics it defines revolutions in such a way that they are inherently impossible within his conceptual framework. Nevertheless, revolutionary moments have been identified, not only by historians but by mathematicians as well. Rather than dictate the meaning of revolution, there is no reason not to allow its use in legitimately describing certain penetrating changes in the evolution of mathematics. However, before challenging further the assertion that revolutions never occur in the history of mathematics, it will be helpful to consider briefly the meaning of revolution as a historical concept. Here we are fortunate in having a recent study by Professor Cohen to guide us. In fact, what follows is a very brief résumé of results owing largely to Professor Cohen's research on the subject of revolutions.[4]

The concept of revolution first made its appearance with reference to scientific and political events in the eighteenth century, although with considerable confusion and ambiguity as to the meaning of the term in such contexts. In general, the word was regarded in the eighteenth century as indicating a breach of continuity, a change of great magnitude, even though the old astronomical sense of revolution as a cyclical phenomenon persisted as well. But, following the French Revolution, the new meaning gained currency, and thereafter revolution commonly came to imply a radical change or departure from traditional or acceptable modes of thought. Revolutions, then, may be visualized as a series of discontinuities of such magnitude as to constitute definite breaks with the past. After such episodes, one might say that there is no returning to an older order.

Bernard de Fontenelle may well have been the first author to apply the word 'revolution' to the history of mathematics, specifically to its evolution in the seventeenth century. In his *Éléments de la géométrie de l'infini* (1727), he was thinking of the infinitesimal calculus of Newton and Leibniz.[5] What Fontenelle perceived was a change of so great an order as to have altered completely the state of mathematics. In fact, Fontenelle went so far as to pinpoint the date at which this revolution had gathered such force that its effect was unmistakable. In his eulogy of the mathematician Rolle, published in the *Histoire de l'Académie Royale des Sciences* of 1719, Fontenelle referred to the work of the Marquis de l'Hôpital, his *Analyse des infiniment petits* (first published in 1696, with later editions in 1715, 1720, and 1768), as follows:

In those days the book of the Marquis de l'Hôpital had appeared, and almost all the mathematicians began to turn to the side of the new geometry of the infinite, until then hardly known at all. The surpassing universality of its methods, the elegant brevity of its demonstrations, the finesse and directness of the most difficult solutions, its singular

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and unprecedented novelty, it all embellishes the spirit and has created, in the world of geometry, an unmistakable revolution.[6]

Clearly this revolution was qualitative, as all revolutions must be. It was a revolution that Fontenelle perceived in terms of character and magnitude, without invoking any displacement principle—any rejection of earlier mathematics—before the revolutionary nature of the new geometry of the infinite could be proclaimed. For Fontenelle, Euclid's geometry had been surpassed in a radical way by the new geometry in the form of the calculus, and this was undeniably revolutionary.

Traditionally, then, revolutions have been those episodes of history in which the authority of an older, accepted system has been undermined and a new, better authority appears in its stead. Such revolutions represent breaches in continuity, and are of such degree, as Fontenelle says, that they are unmistakable even to the casual observer. Fontenelle has aided us, in fact, by emphasizing the discovery of the calculus as one such event—and he even takes the work of l'Hôpital as the identifying marker, much as Newton's *Principia* of 1687 marked the scientific revolution in physics or the Glorious Revolution of the following year marked England's political revolution from the Stuart monarchy. The monarchy, we know, persisted, but under very different terms.

In much the same sense, revolutions have occurred in mathematics. However, because of the special nature of mathematics, it is not always the case that an older order is refuted or turned out. Although it may persist, the old order nevertheless does so under different terms, in radically altered or expanded contexts. Moreover, it is often clear that the new ideas would never have been permitted within a strictly construed interpretation of the old mathematics, even if the new mathematics finds it possible to accommodate the old discoveries in a compatible or consistent fashion. Often, many of the theorems and discoveries of the older mathematics are relegated to a significantly lesser position as a result of a conceptual revolution that brings an entirely new theory or mathematical discipline to the fore. This was certainly how Fontenelle regarded the calculus. Similarly, it is also possible to interpret the discovery of incommensurable magnitudes in Antiquity as the occasion for the first great transformation in mathematics, namely, its transformation from a mathematics of discrete numbers and their ratios to a new theory of proportions as presented in Book V of Euclid's *Elements*.

4.2. THE PYTHAGOREAN DISCOVERY OF INCOMMENSURABLE MAGNITUDES

Aristotle reports the Pythagorean doctrine that all things were numbers, and surmises that this view doubtless originated in several sorts of empirical

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observation.[7] For example, in terms of Pythagorean music theory the study of harmony had revealed the striking mathematical constancies of proportionality. When the ratios of string lengths or flute columns were compared, the harmonics produced by other, but proportionally similar lengths, were the same. The Pythagoreans also knew that any triangle with sides of length 3, 4, 5, whatever unit might be taken, was a *right* triangle. This too supported their belief that ratios of whole numbers reflected certain invariant and universal properties. In addition, Pythagorean astronomy linked such terrestrial harmonies with the motions of the planets, where the numerical harmony, or cyclic regularity of the daily, monthly, or yearly revolutions, was as striking as the musical harmonies the planets were believed to create as they moved in their eternal cycles. All these invariants gave substance to the Pythagorean doctrine that numbers—the whole numbers—and their ratios were responsible for the hidden structure of all nature. As Aristotle comments:

> The so-called Pythagoreans, having begun to do mathematical research and having made great progress in it, were led by these studies to assume that the principles used in mathematics apply to all existing things . . . they were more than ever disposed to say that the elements of all existing things are found in numbers.[8]

But what were these numbers? For the early Pythagoreans, Aristotle indicates that they were apparently something like physical 'monads'. In the *Metaphysics*, for example, one passage offers the following elaboration: '[The Pythagoreans] compose all heaven of numbers (ἐξ ἀριθμῶν), not of numbers in the purely arithmetical sense, though, but assuming that monads have size.'[9]

Thus the Pythagoreans apparrently came to regard the numbers themselves as providing the structure and form of the material universe, their ratios determining the shapes and harmonies of all symmetrical things. The Pythagoreans gave the word λόγοι to the groups of numbers determining the character of a given object, and later the meaning of this word was extended, as we shall see, from that of 'word' to 'ratio'.[10]

This sort of arithmology found its realization in the Pythagoreans' quest to associate numbers with all things, and to determine the internal properties, ratios, and relations between numbers themselves. Thus the number of stones needed to outline the figure of a man or a horse was taken by the Pythagorean Eurytus as the 'number' for man or horse.[11] The essence of such things was expressed by a particular number. Moreover, some Pythagoreans sought to establish the number for justice, or for marriage. Others distinguished numbers that were perfect (the tetractys, for example, $1+2+3+4=10$), amicable, or friendly. Figurate numbers, including pentagonal and solid numbers, were also subjects of great interest.[12] It is against this background of Pythagorean numerology, in which the λόγος of all things was thought to be an invariant principle of the universe, expressible in terms of whole numbers

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and their ratios, that the discovery of incommensurable magnitudes must be viewed. The Pythagoreans' arithmology would doubtless have provided sufficient incentive for their search for the hidden numbers, the prevailing logos governing the most important objects of their mysticism, for example the pentagon or the golden section. It is also possible that the discovery was made in less rarefied contexts, through study of the simplest of right triangles, the isosceles right triangle.

Exactly when incommensurable magnitudes were first discovered is not particularly relevant for the argument here.[13] Similarly, the details of the initial discovery are also of secondary importance, and we can dispense with the dilemma of whether the discovery was first made in the context that Aristotle reports it, by studying the ratio of the length of a square's edge with its diagonal, or whether, as has been argued by K. von Fritz (1945) and by S. Heller (1958), that Hippasus found incommensurability in considering the construction of the regular pentagon.[14] What concerns us is the discovery and its subsequent effect. Philosophically, it would certainly have represented a crisis for the Pythagoreans.[15] Having been tempted by the seductive harmony of generalization, some Pythagoreans had carried too far their universal principle that all things were numbers. The complete generalization was inadmissible, and this realization was a major blow to Pythagorean thought, if not to Greek mathematics. In fact, a scholium to Book X of Euclid's *Elements* reflects the gravity of the discovery of incommensurable magnitudes in the well-known fable of the shipwreck and the drowning of Hippasus:

> It is well known that the man who first made public the theory of irrationals perished in a shipwreck in order that the inexpressible and unimaginable [Καὶ ἄλογον Καὶ ἀνεἰδεον] should ever remain veiled . . . and so the guilty man, who fortuitously touched on and revealed this aspect of living things, was taken to the place where he began and there is forever beaten by the waves.[16]

What deserves attention here are the words 'inexpressible' and 'unimaginable'. It is difficult, if not impossible, for us to appreciate how hard it must have been to conceive of something one could not determine or name—the inconceivable—and this was exactly the name given to the diagonal: ἄλογον. This reflects the double meaning of the word *logos* as *word*, as the 'utterable' or 'nameable', and now the irrational, the *alogon*, as the 'unspeakable', the 'unnameable'. In this context, it is easy to understand the commentary: 'Such fear had these men of the theory of irrationals, for it was literally the discovery of the "unthinkable".'[17]

Ultimately, however, the Greeks regarded the discovery not as a crisis but as a great advance. Whether or not discovery of incommensurable magnitudes precipitated a crisis in Greek mathematics, and, if so, whether it affected only the foundations of mathematics rather than the mathematics itself, the significant issue concerns the *response* mathematicians were forced to make

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once the existence of incommensurable magnitudes had been divulged and was a matter of general knowledge.[18]

What ultimate effect did this discovery have on the content and nature of Greek mathematics? Above all, the theories of proportion advanced by Theaetetus and Eudoxus in the early fourth century BC (390–350 BC) served to reverse the emphasis of earlier mathematics. Consider, for example, the statement of Archytas (an early Pythagorean and teacher of Eudoxus), who was emphatic that arithmetic was superior to geometry for supplying satisfactory proofs.[19] After the discovery of incommensurable magnitudes, such a statement would be virtually impossible to justify. In fact, the opposite was closer to the truth, as the subsequent development of Greek geometric algebra demonstrates.

Basically, the transformation from a simple theory of commensurable proportions (where geometry and arithmetic might be regarded as coextensive) to a new theory embracing incommensurable magnitudes (for which arithmetic was inadequate) centres on the contributions of Theaetetus and Eudoxus. However, we know from Plato's *Theaetetus* that a major step toward the better understanding of the irrational was taken by Theaetetus's teacher, Theodorus, who established the incommensurability of certain magnitudes up to (but not including) $\sqrt{17}$ by means of geometric constructions. Although Theodorus's achievements were limited owing to his lack of a sufficiently developed arithmetic theory, some historians have argued that he began to develop a metric geometry capable of handling arithmetic properties in much the form of propositions in Book II of Euclid's *Elements*.[20]

Following his teacher Theodorus, Theaetetus became interested in the general properties of incommensurables and produced the classification that so impressed Socrates in Plato's dialogue (*Theaetetus*, 147C–148B). Also, Theaetetus realized that, to treat incommensurables successfully, geometry had to embody more of the results of arithmetic theory, and so he sought to translate necessary algebraic results into geometric terms. Here he focused on the arithmetic properties of relative primes, using the process of determining greatest common factors by means of successive subtraction, or *anthyphairesis*.[21] This enabled Theaetetus to reformulate the theory of proportion to include certain incommensurable magnitudes that he classified as the *medial*, *binomial*, and *apotome*, and these were enough for the results in which he was interested. But Theaetetus apparently was not inspired to study the new theory of proportion itself—something his premature death certainly precluded.

Eudoxus, however, realized that the methods Theaetetus had brought to geometry from arithmetic for the purpose of studying incommensurables could actually provide the basis for an even more comprehensive theory of proportion. In studying the construction of the regular pentagon, dodeca-

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hedron, and icosahedron, Eudoxus seems to have realized that these, like segments divided into mean and extreme ratio, involved incommensurable magnitudes that were not included in the three classes treated by Theaetetus (Knorr 1975, pp. 286–8). Because of his interest in a formal, more comprehensive theory of proportions, he transformed Theaetetus's methods involving *anthyphairesis* by focusing on the theory of proportion itself and producing in large measure the theorems elaborated in Book V of Euclid's *Elements*, where the concept of equal multiples made it possible to develop a theory of proportion that was generally applicable to incommensurables. The advantages of the new Eudoxan theory were considerable, and comparison with Theaetetus's anthyphairetic approach made clear the differences. Aristotle, in fact, contrasted the two on several occasions, and noted the superiority of Eudoxus's formulation explicitly.[22]

Having produced a comprehensive theory of proportion, however, Eudoxus and his followers, perhaps chief among them Hermotimus of Colophon, were also interested in providing a systematic development of the new theory that eventually provided the basic framework for Euclid's Book V of the *Elements*, a book a scholiast tentatively attributes to Eudoxus.[23] In dealing with incommensurable magnitudes, 'unfamiliar and troublesome' concepts as Morris Kline (1972, p. 50) has described them, the need to formulate axioms and to deduce consequences one by one so that no mistakes might be made was of special importance. This emphasis, in fact, reflects Plato's interest in the dialectic certainty of mathematics and was epitomized in the great Euclidean synthesis, which sought to bring the full rigour of axiomatic argumentation to geometry. It was in this spirit that Eudoxus undertook to provide the precise logical basis for the incommensurable ratios, and in so doing, gave great momentum to the logical, axiomatic, *a priori* 'revolution' identified by Kant (1781–7) as the great transformation wrought upon mathematics by the Greeks (see aso Cohen 1976, pp. 283–4).

In concluding this brief summary of Greek mathematics and the transformation caused by the discovery of incommensurable magnitudes, several aspects of that transformation deserve particular emphasis. Primarily, two things were unacceptable after the discovery of incommensurables: (1) the Pythagorean interpretation of ratio, and (2) the coming into play of proofs they had given concerning commensurable magnitudes. A new theory was needed to accommodate irrational magnitudes—and this was provided by Theaetetus and Eudoxus. The less dramatic transformation of the definition of the number concept was a lengthier process, but over the course of centuries it eventually led to admission of irrational *numbers* as being as acceptable ontologically as natural numbers or fractions.[24]

Wholly apart from the slower, more subtle transformation of the number concept, however, was the dramatic, much quicker transformation of the character of Greek mathematics itself. Because Pythagorean arithmetic could

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not accommodate irrational magnitudes, geometric algebra (cumbersome though it was) developed in its stead. In the process, Greek mathematics was directly transformed into something more powerful, more general, more complete. Central to this transformation were auxiliary elements that reflected the transformation under way. A new interpretation of mathematics must have discarded as untenable the older Pythagorean doctrine that all things were number—there were now clearly things that did not have numbers in the Pythagorean sense of the word—and consequently their view of number was correspondingly inadequate. The older concept of number was severely limited, and in the realization of this inadequacy and the creation of a remedy to solve it came the revolution. New proofs replaced old ones.[25] Soon a new theory of proportion emerged, and as a result, after Eudoxus, no one could look at mathematics and think that it was the same as it had been for the Pythagoreans. Nor was it possible to assert that Eudoxus had merely added something to a theory that previously was perfectly all right. The lesson of the irrational was that everything was *not* all right. As a result of the new theory of proportion, the methods and content of Greek mathematics were vastly different, and comparison of Book V of Euclid with the Pythagorean books VII–IX (perhaps reflecting directly earlier arithmetics from the previous century) reveals the deep transformation that Eudoxus and his theory of proportion brought to Greek mathematics.[26] The old methods were supplanted, and eventually, although the same words, 'number' or 'proportion', might continue in use, their meaning, scope, and content would not be the same.

In fact, the transformation in conceptualization from irrational magnitudes to irrational numbers represented a revolution of its own in the number concept, although this was not a transformation accomplished by the Greeks. Nor was it an upheaval of a few years, as are most political revolutions, but a basic, fundamental change. Even if the evolution was relatively slow, this does not alter the ultimate effect of the transformation. The old concept of number, although the word was retained, was gone, and in its place, numbers included irrationals as well.

This transformation of the concept of number, however, entailed more than just extending the old concept of number by adding on the irrationals—the entire concept of number was inherently changed, transmuted as it were, from a world-view in which integers alone were numbers, to a view of number that was eventually related to the completeness of the entire system of real numbers.

In much the same way, Georg Cantor's creation of transfinite numbers in the nineteenth century transformed mathematics by enlarging its domain from finite to infinite numbers. Above all, the conceptual step from transfinite sets to transfinite numbers represents a shift that was in many ways the same as the shift from irrational magnitudes to irrational numbers. From the concrete to the abstract, the transformation in both cases revolutionized mathematics.

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4.3. GEORG CANTOR'S DEVELOPMENT OF TRANSFINITE SET THEORY

Born in St Petersburg (Leningrad) in 1845, Georg Cantor left Russia for Germany with his parents in 1856.[27] Following study at the *Gymnasium* in Wiesbaden, private schools in Frankfurt-am-Main and the Realschule in Darmstadt, he entered a *Höhere Gewerbeschule* (Trade School), also in Darmstadt, from which he graduated in 1862 with the endorsement that he was a 'very gifted and highly industrious pupil' (Fraenkel 1930, p. 192). But his interests in mathematics prompted him to go on to university, and with his parents' blessing he began his advanced studies in the autumn of that same year at the *Polytechnicum* in Zürich. Unfortunately, his first year there was interrupted early in 1863 by the sudden death of his father, although within the year he resumed his studies, at the university in Berlin. There he studied mathematics, physics, and philosophy, and was greatly influenced by three of the greatest mathematicians of the day: Kummer, Weierstrass, and Kronecker.

After the summer term of 1886, which he spent in Göttingen, Cantor returned to the University of Berlin from which he graduated in December with the distinction 'Magna cum laude' (Fraenkel 1930, p. 194). Following three years of local teaching and study as a member of the prestigious Schellbach seminar for teachers, Cantor left Berlin for Halle in 1869 to accept an appointment as a *Privatdozent* in the Department of Mathematics. There he came under the influence of one of his senior colleagues, Eduard Heine, who was just completing a study of trigonometric series. Heine urged Cantor to turn his talents to a particularly interesting but extremely difficult problem: that of establishing the uniqueness of the representations of arbitrary functions by means of trigonometric series.[28] Within the next three years Cantor published five papers on the subject. The most important of these was the last, published in 1872, in which he presented a remarkably general and innovative solution to the representation problem.

With impressive skill Cantor was able to show that any function represented by a trigonometric series was not only uniquely represented, but that in the interval of representation an infinite number of points could be excepted provided only that the set of exceptional points be distributed in a specific way.[29] The condition was limited to sets Cantor described as point sets of the *first species* (Dauben 1979, pp. 41–2). Given a set P, the collection of all limit points p in P defined its first derived set, P'. Similarly, P'' represented the second derived set of P, and contained all limit points of P'. Proceeding analogously, for any set P Cantor was able to generate an entire sequence of derived sets $P', P'', \ldots$. P was described as a point set of the first species if, for some index n, $P^n = \varnothing$.

As outlined in the paper of 1872, Cantor's elementery set-theoretic concepts could not break away into a new autonomy of their own. Though he

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had the basic idea of the transfinite numbers in the sequence of derived sets P', P'', . . . , P^{∞}, $P^{\infty+1}$, . . . , the basis for any articulate conceptual differentiation between P^n and P^{∞} was lacking. As yet, Cantor had no precise basis for defining the first transfinite number ∞ following all finite natural numbers n.[30] A general framework within which to establish the meaning and utility of the transfinite numbers was lacking. The only guide Cantor could offer was the vague condition that $P^n \neq \varnothing$ for all n, which separated sets of the first species from those of the second. Cantor could not begin to make meaningful progress until he had realized that there were further distinctions yet to be made in orders of magnitude between discrete and continuous sets. Until the close of 1873, Cantor did not even suspect the possibility of such differences.

In order to argue his uniqueness theorem of 1872, Cantor discovered that he needed to present a careful analysis of limit points and the elementary properties of derived sets, as well as a rigorous theory of irrational numbers.[31] It was the problem of carefully and precisely defining the irrational numbers that forced Cantor to face the topological complexities of the real line and to consider seriously the structure of derived sets of the first species.

After the success of his paper of 1872, it was a natural step to search for properties that would distinguish the continuum of real numbers from other infinite sets like the totality of rational or algebraic numbers. What Cantor soon established was something most mathematicians had assumed, but which no one had been able to formulate precisely: that there were more real numbers than natural, rational, or algebraic numbers (Cantor 1874). Cantor's discovery that the real numbers were non-denumerable was not in itself revolutionary, but it made possible the invention of new concepts and a radically new theory of the infinite. When coupled with the idea of one-to-one correspondences, it was possible to distinguish mathematically for the first time between different magnitudes, or powers, of infinity. In 1874 he was only able to identify denumerable and non-denumerable sets. But as his thinking advanced, he was eventually able to detach his theory from the specific examples of point sets, and in 1883 he was ready to publish his *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*, in which he presented a completely general theory of transfinite numbers.[32] It was in the *Grundlagen* that Cantor introduced the entire hierarchy of infinite number classes in terms of the order types of well-ordered sets. More than twelve years later, in his last major publication, the *Beiträge* of 1895 and 1897, he formulated the most radical and powerful of his new ideas, the entire succession of his transfinite cardinal numbers:[33]

$$\aleph_0, \aleph_1, \ldots .$$

Cantor's introduction of the actual infinite in the form of transfinite numbers was a radical departure from traditional mathematical practice, even dogma. This was especially true because mathematicians, philosophers, and

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theologians in general had repudiated the concept since the time of Aristotle.[34] Philosophers and mathematicians rejected completed infinities largely because of their alleged logical inconsistency. Theologians represented another tradition of opposition to the actual infinite, regarding it as a direct challenge to the unique and absolute infinite nature of God. Mathematicians, like philosophers, had been wary of the actual infinite because of the difficulties and paradoxes it seemed inevitably to introduce into the framework of mathematics. Gauss, in most authoritative terms expressed his opposition to the use of such infinities in mathematics in a celebrated letter to Heinrich Schumacher:

> But concerning your proof, I protest above all against the use of an infinite quantity [*Grösse*] as a *completed* one, which in mathematics is never allowed. The infinite is only a *façon de parler*, in which one properly speaks of limits.[35]

Cantor believed, on the contrary, that on the basis of rigorous, mathematical distinctions between the potential and the actual infinite, there was no reason to hold the old objections and that it was possible to overcome the objections of mathematicians like Gauss, philosophers like Aristotle, and theologians like Thomas Aquinas, and to do so in terms even they would find impossible to reject. In the process, Cantor was led to consider not only the epistemological problems his new transfinite numbers raised, but to formulate as well an accompanying metaphysics. In fact, he argued convincingly that the idea of the actual infinite was implicitly part of any view of the potential infinite and that the only reason mathematicians had avoided using the actual infinite was because they were unable to see how the well-known paradoxes of the infinite, celebrated from Zeno to Bolzano, could be understood and avoided. He argued that once the self-consistency of his transfinite numbers was recognized, they could not be refused a place alongside the other accepted but once disputed members of the mathematical family, including irrational and complex numbers (Cantor 1883, p. 182). In creating transfinite set theory, Cantor was making a significant contribution to the constellation of mathematical ideas.

Of central concern to Cantor's entire defence of transfinite set theory was the nature of mathematics and the question of what criteria determined the acceptability of mathematical concepts and arguments. He reinforced his support of transfinite set theory with a simple analysis of the familiar and accepted positive integers. Insofar as they were regarded as well defined in the mind, distinct and different from all other components of thought, they served in a connectional or relational sense, he said, to modify the substance of thought itself (Cantor 1883, p. 181). Cantor described this reality that the whole numbers consequently assumed as their intrasubjective or immanent reality. In contradistinction to the reality numbers could assume strictly in terms of mind, however, was the reality they could assume in terms of body,

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manifest in objects of the physical world. Cantor explained further that this second sort of reality arose from the use of numbers as expressions or images of processes in the world of natural phenomena. This aspect of the integers, be they finite or infinite, Cantor described as their transubjective or transient reality.[36]

Cantor specifically claimed the reality of both the physical and ideal aspects of his approach to the number concept. The dual realities, in fact, were always found in a joined sense, in so far as a concept possessing an immanent reality always possessed a transient reality as well. Cantor believed that to determine the connections between the two kinds of reality was one of the most difficult problems of metaphysics.

In emphasizing the intrasubjective nature of mathematics, Cantor concluded that it was possible to study only the immanent realities, without having to confirm or conform to any subjective content. As noted earlier, this set mathematics apart from all other sciences and gave it an independence from the physical world that provided great freedom for mathematicians in the creation of mathematical concepts. It was on these grounds that Cantor offered his now-famous dictum that the essence of mathematics is its freedom. As he put it in the *Grundlagen* (Cantor 1883, p. 182):

> Because of this extraordinary position which distinguishes mathematics from all other sciences, and which produces an explanation for the relatively free-and-easy way of pursuing it, it especially deserves the name of *free mathematics*, a designation which I, if I had the choice, would prefer to the now customary 'pure' mathematics.

Cantor was asserting the freedom within mathematics to allow the creation and application of new ideas on the basis of intellectual consistency alone. Mathematics was therefore absolutely free in its development and bound only to the requirement that its concepts permit no internal contradictions, but that they follow in definite relation to previously given definitions, axioms, and theorems. Mathematics, Cantor believed, was the one science that was justified in releasing itself from any metaphysical fetters. Its freedom, insisted Cantor, was its essence.

The detachment of mathematics from the constraints of an imposed structure embedded in the natural world frees it from the metaphysical problems inherent in any attempt to understand the ultimate status of the physical and life sciences. Mathematicians do not face the preoccupation of scientists who must try to make theory conform with some sort of given, external reality against which those theories may be tested, articulated, improved, revised, or rejected.[37] Mathematicians, if they worry at all, need do so only in terms of the internal consistency of their work. This effectively eliminates the possibility of later discrepancies. Thus the grounds do not seem present within mathematics for generating anomaly and crisis, or for displacing earlier theory with some incompatible new theory.

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One important consequence, in fact, of the insistence on self-consistency within mathematics is that its advance is necessarily cumulative. New theories cannot displace the old, just as the calculus did not displace geometry. Though revolutionary, the calculus was not an incompatible advance requiring subsequent generations to reject Euclid; nor did Cantor's transfinite mathematics require displacement and rejection of previously established work in analysis, or in any other part of mathematics.

Advances in mathematics, therefore, are generally compatible and consistent with previously established theory; they do not confront and challenge the correctness or validity of earlier achievements and theory, but augment, articulate, and generalize what has been accepted before. Cantor's work managed to transform or to influence large parts of modern mathematics without requiring the displacement or rejection of previous mathematics.

4.4. REVOLUTIONARY ADVANCE IN MATHEMATICS

Does this mean, then, that mathematics, because it represents a form of knowledge in which progress is genuinely cumulative, cannot experience periods of legitimate revolution? Surely not. To say that mathematics grows by the successive accumulation of knowledge, rather than by the displacement of discredited past theory by new theory, is not the same as to deny revolutionary advance. Cantor's proof of the non-denumerability of the real numbers, for example, led to the creation of the transfinite numbers. This was conceptually impossible within the bounds of traditional mathematics, yet in no way did it contradict or compromise finite mathematics. Cantor's work did not displace, but it *did* augment the capacity of previous theory in a way that was revolutionary, that would otherwise have been impossible. It was revolutionary in breaking the bonds and limitations of earlier analysis, just as imaginary and complex numbers carried mathematics to new levels of generality and made solutions possible that would otherwise have been impossible to formulate. Moreover, the extensive revision due to transfinite set theory of large parts of mathematics, involving the rewriting of textbooks and precipitating debates over foundations, are all results of what Thomas Kuhn has diagnosed as companions to revolutions.[38] And all these are reflected in the historical development of Cantorian set theory.

4.5. THE NATURE OF SCIENTIFIC RESOLUTION

I have deliberately juxtaposed the words 'revolution' and 'resolution' in order to emphasize what I take to be the nature of scientific advance reflected in the development of the history of mathematics—be it the Greek discovery of

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incommensurables and the concomitant creation of a theory of proportion to accommodate them, or Cantor's profound discovery of the non-denumerablity of the real numbers and his subsequent creation of transfinite numbers and the development of a general, transfinite set theory. Because mathematics is restricted only by the limits imposed by consistency, the inherent structure of logic determines the structure of mathematical evolution. I have already suggested the way in which that evolution is necessarily cumulative. As theory develops, it provides more complete, more powerful, more comprehensive problem-solutions, sometimes yielding entirely new and revolutionary theories in the process. But the fundamental character of such advance is embodied in the idea of resolution. Like the microscopist, moving from lower to higher levels of resolution, successive generations of mathematicians can claim to understand more, with a greater stockpile of results and increasingly refined techniques at their disposal. As mathematics becomes increasingly articulated, the process of resolution brings the areas of research and subjects for problem-solving into greater focus, until solutions are obtained or new approaches developed to extend the boundaries of mathematical knowledge. Discoveries accumulate, and some inevitably lead to revolutionary new theories uniting entire branches of study, producing new points of view, sometimes wholly new disciplines that would have been impossible to produce within the bounds of previous theory.

This is as true of the discovery of incommensurable magnitudes as it is of the advent of irritational, imaginary, and transfinite numbers, of the invention of the calculus, or the discovery of non-Euclidean geometries. None of these involved crisis or the rejection of earlier mathematics, although each represented a response to the failures and limitations of prevailing theory. New discoveries, particularly those of revolutionary import like those discussed here, provide new modes of thought within which more powerful and general results are possible than ever before. As Hermann Hankel (1871, p. 25) once wrote, 'In mathematics alone each generation builds a new storey to the old structure.' This is the most obvious sense in which I mean that the nature of scientific advance can be understood directly, in terms of the logic of argument and mathematics, as one of increasingly powerful resolution.

4.6. RESISTANCE TO CHANGE

One last feature of the evolution of mathematics may help to corroborate further the fact that it does experience revolutionary transformations, for resistance to new discoveries may be taken as a strong measure of their revolutionary quality. One form of this resistance was reflected in the Greeks' inability to conceive of anything as number except the integers—although eventually this prejudice was overcome, just as Cantor eventually overcame

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