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NON-STANDARD ANALYSIS

Revised Edition

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Our proof of the finiteness principle 2.5.1 involved the use of the axiom of choice including the use of Zorn's lemma. There are other proofs of 2.5.1 which employ the maximal ideal theorem for Boolean algebras instead. It follows that any mathematical theorem that will be proved in this book subsequently, and which does not involve the axiom of choice in other ways can be proved by means of the maximal ideal theorem for Boolean algebras alone. This is relevant since it is known that, in the appropriate axiomatic setting, the maximal ideal theorem for Boolean algebras is strictly weaker than the axiom of choice (HALPERN [1964]).

2.12 Remarks and references. The finiteness principle of the Lower Predicate Calculus is due to A. I. Malcev (MALCEV [1936]). It is closely related to the completeness theorem (GÖDEL [1930]) and its generalizations (for details, see ROBINSON [1963]). The name 'compactness theorem' was suggested by A. Tarski (TARSKI [1952]).

The notion of an ultrapower was introduced by J. Łoś (ŁOŚ [1955]). The theory of this and of related concepts has been developed further in several papers (CHANG and MOREL [1958], KOCHEN [1961], KEISLER [1962], FRAYNE *et al.* [1962]). The ultrapower construction overlaps with E. Hewitt's construction of hyperreal fields (HEWITT [1948], compare GILMAN and JERISON [1960]).

The extension of completeness (or compactness) arguments to higher order languages is due to L. Henkin (HENKIN [1950]).

CHAPTER III

DIFFERENTIAL AND INTEGRAL CALCULUS

3.1 Non-standard Arithmetic. Let $^*N$ be a higher order non-standard model of Arithmetic. Thus, $^*N$ is a mathematical structure which possesses the following properties:

(i) Every mathematical notion which is meaningful for the system of Natural Numbers is meaningful also for $^*N$. In particular, addition, multiplication, and order are defined for $^*N$.

(ii) Every mathematical statement which is meaningful and true for the system of Natural Numbers is meaningful and true also for $^*N$: provided that we interpret any reference to entities of any given type, e.g., sets, or relations, or functions, in $^*N$ not in terms of the totality of entities of that type, but in terms of a certain subset, called the set of *internal* entities of that type. For example, if the statement contains a phrase 'for all *sets* of numbers', we interpret this as 'for all *internal* sets of numbers'. Similarly the phrases 'there exists a *set* of numbers', 'there exists a *function*', as 'there exists an *internal* set', 'there exists an *internal function*'. However, all individuals of $^*N$ are internal: the phrase 'for all numbers' is interpreted in $^*N$ as 'for *all* individuals of $^*N$'.

(iii) The system of internal entities in $^*N$ has the following property. If S is an internal set of relations, then all elements of S are internal. More generally, if S is an internal n-ary relation, $n \geq 1$, and the n-tuple $(S_1, \ldots, S_n)$ satisfies (belongs to) S then $S_1, \ldots, S_n$ all are internal relations.

(iv) $^*N$ properly contains the system of natural numbers, N; there is an individual in $^*N$ which, according to the relation of order defined in N and $^*N$, is greater than all numbers of N.

See chapter II, in particular 2.10.3, for a more precise, formal, description of the properties of $^*N$. We may, and shall, suppose that $^*N$ is normal so that the constants e_τ, which denote the relations of identity in N for the various types τ, denote relations of identity also in $^*N$.

Let K be the set of all stratified sentences which are admissible and hold in N as defined formally in chapter II and informally in (ii) above. In K, let e denote the relation of identity for individuals, so that $e=e_0$, while q denotes the relation of order and s and p denote the relations of addition and multiplication, respectively. Thus, the sentences $[\Phi_{(0,0)}(e,a,b)]$, $[\Phi_{(0,0)}(q,a,b)]$, $[\Phi_{(0,0,0)}(s,a,b,c)]$ and $[\Phi_{(0,0,0)}(p,a,b,c)]$ have the conventional translations $a=b$, $a<b$, $a+b=c$, and $ab=c$, respectively.

From now on, we shall refer to *all* individuals of $^*N$ as natural numbers.

It is not difficult to see that the familiar laws of Arithmetic which are true for N are true also for $^*N$. For example, the commutative law of addition holds in N and this can be formulated as a sentence of K, so

3.1.1 $$(\forall x)(\forall y)(\forall z)[\Phi_{(0,0,0)}(s,x,y,z)\supset\Phi_{(0,0,0)}(s,y,x,z)].$$

Since any sentence of K holds also in $^*N$ it follows that $^*N$ satisfies the commutative law of addition.

There are many other elementary assertions which can be *transferred* in this way from N to $^*N$. For example, an argument which is quite similar to that just given shows that every number of $^*N$ is the sum of the squares of four numbers of $^*N$ (since this is known to be true for N).

At this point one might be tempted to conclude that all that can be said about $^*N$ is deducible in this fashion, rather trivially, from facts about N. However, this impression would be mistaken.

q denotes the relation of order in N. Thus, q denotes a relation in N which is irreflexive, transitive, and satisfies the law of trichotomy (i.e., for any two *distinct* numbers a and b in N the relation denoted by q holds either between a and b or between b and a). This can be expressed by a sentence of K and accordingly, is true also in $^*N$. Thus, q denotes a relation of order also in $^*N$. We write $a<b$, as usual, if q holds between numbers a and b in $^*N$. The relation of order $<$ in $^*N$ is compatible with the ordinary relation of order in N since, for any numbers a and b in N, $[\Phi_{(0,0)}(q,a,b)]$ holds in $^*N$ if and only if it holds in N. Thus, the numbers of N constitute an ordered subset of the ordered set of numbers of $^*N$.

Moreover, the following statements also are true for N are expressible as sentences of K.

'0 is smaller then all other numbers'.

'1 is smaller than all other numbers except 0'.

'2 is smaller than all other numbers except 0 and 1', and so on.

Indeed, sentences as required are provided by

3.1.2 $$(\forall x)[\Phi_{(0,0)}(e,0,x)\vee\Phi_{(0,0)}(q,0,x)]$$

3.1.3 $$(\forall x)[\Phi_{(0,0)}(e,0,x)\vee\Phi_{(0,0)}(e,1,x)\vee\Phi_{(0,0)}(q,1,x)]$$

3.1.4 $$(\forall x)[\Phi_{(0,0)}(e,0,x)\vee\Phi_{(0,0)}(e,1,x)\vee\Phi_{(0,0)}(e,2,x)\vee\Phi_{(0,0)}(q,2,x)].$$

and so on, respectively, where 0, 1, 2 have the obvious denotation. This shows that the ordinary natural numbers (numbers of N) constitute an initial segment of $^*N$. That is to say, any ordinary natural number (number of N) is smaller than every natural number (individual of $^*N$) that is not contained in N. From now on, we shall call all natural numbers that belong to N *finite*, while all other natural numbers will be called *infinite*. Thus any finite natural number is smaller than any infinite natural number. The finite natural numbers are the *standard* natural numbers according to a general classification introduced previously.

There is no smallest infinite number. For if a is infinite then $a\neq 0$, hence $a=b+1$ (the corresponding fact being true in N). But b cannot be finite, for then a would be finite. Hence, there exists an infinite numbers which is smaller than a.

The function $z=|x-y|$ – 'z is the absolute value of $x-y$' – is defined in $^*N$ as a function of two variables which is denoted in $^*N$ by the same ternary relation as the function $z=|x-y|$ in N. It follows that $|x-y|$ has the same basic properties in $^*N$ as in N. Thus, for any a and b in $^*N$, $|a-b|$ is equal to $a-b$ if b is smaller than or equal to a and $|a-b|$ is equal to $b-a$ if a is smaller than b.

We define a binary relation $\sim$ for the numbers of $^*N$, by the condition that $a\sim b$ if and only if $|a-b|$ is finite. Observe that, by the manner of its definition, the relation $\sim$ is not necessarily internal, in fact, we shall show presently that it is *external*.

It is not difficult to see that $\sim$ is an equivalence relation, i.e., that it is reflexive, symmetrical and transitive. Moreover, the equivalence classes determined by this relation are *intervals* of the ordered set of numbers of $^*N$. That is to say, if $a\sim b$ and $a<c<b$ then $a\sim c$ (and hence, $c\sim b$). It is not difficult to see that all finite numbers of $^*N$ (i.e., all numbers of N) constitute an equivalence class with respect to the relation $\sim$. Now let a be an infinite natural number, and let D_a be the equivalence class of a with respect to $\sim$. Then the numbers $a-1, a-2, \ldots, a-n, \ldots$, n a number in N, all

exist since a is infinite and the numbers $a, a\pm 1, a\pm 2, \ldots, a\pm n$, n a number in N, all belong to D_a since $|(a\pm n)-a|=n$ is finite. Moreover, the numbers just mentioned constitute the entire class D_a. Thus, D_a possesses the order type of the integers, which may be written as $\omega^*+\omega$ (ω^* the inverse order type to ω). Since the order type of the finite natural numbers is ω we then obtain for the order type of the numbers of $^*N$,

3.1.5 $$\alpha=\omega+(\omega^*+\omega)\theta\,.$$

In this formula, θ is the order type of the set of equivalence classes $\{D_a\}$, a infinite, which are ordered according to the rule that $D_a<D_b$ if $a<b$. It is not difficult to see that this definition is independent of our particular choice of a and b. Thus, suppose that $D_a=D_{a'}$, and $D_b=D_{b'}$ where $a'\sim a$ and $b'\sim b$ while $|a-b|$ is infinite. We wish to show that if $a<b$ then $a'<b$ and vice versa. Suppose on the contrary that $a<b$ but $b<a'$. (If $a<b$ then $a'=b$ is excluded by the fact that this would imply $a'\sim b$. At the same time, $a\sim a'$, and so we should draw the conclusion that $a\sim b$, contrary to assumption). Then $a<b<a'$ and so b belongs to the interval between a and a'. But we have already accepted the fact that the equivalence classes constitute intervals. Hence, $a<b<a'$ together with $a\sim a'$ implies $a\sim b$, contrary to the assumption that $|a-b|$ is infinite. We conclude that $a'<b$. Similarly, if $a<b$ then $b'<a$ would imply that a belongs to the interval from b' to b, which is again impossible. This proves that the definition of the ordering in $\{D_a\}$ is legitimate. Let its order type be θ. The definition of the product of two order types now leads directly to 3.1.5.

Of any two successive numbers in $^*N$, a and $a+1$, one at least (and at most) is divisible by 2 (i.e., even). For this statement holds it we write N for $^*N$ and can be formulated as a sentence of K. We conclude that it holds also in $^*N$. Thus, either $a=2a'$ for some number a' or else $a+1=2a'$. Accordingly, every equivalence class D_a contains (many) numbers that are divisible by 2. We shall make use of this fact in order to show that θ is dense without first or last elements. Indeed, let D_a be one of our equivalence classes, for infinite a. We are going to show that there exists an infinite number b such that $D_b<D_a$. As we have just seen, D_a contains even numbers, so we may assume that a is itself even, $a=2b$. b is infinite for if b were finite a would be finite. Also, $b<a$, for $b\geq 2b$ only for $b=0$ in N and this is a fact which can be denoted by a sentence of K and hence, is true also in $^*N$. Thus, either $D_b=D_a$ or $D_b<D_a$. But $D_b=D_a$ is possible

only if $|a-b|=b$ is finite, which is not the case. Hence, $D_b<D_a$. This shows that θ does not contain a first element. A similar argument shows that θ does not contain a last element. Now let $D_a<D_b$ where both a and b are infinite. As we have seen, we may suppose that both a and b are even without changing the equivalence classes under consideration. Thus, $a=2a'$, $b=2b'$. Let $c=a'+b'$, then we claim that $D_a<D_c<D_b$. Indeed, $2c-2a=a+b-2a=b-a$ exists and is infinite by assumption, so $c-a=b'-a'$ exists and is infinite. This proves $D_a<D_c$. Similarly, $2b-2c=2b-(c+b)=b-a$, so $b-c=b'-a'$ exists and is infinite. This shows that $D_c<D_b$. It follows that between any two elements of the ordered set $\{D_a\}$, a infinite, there is another element of the set. Hence, θ is dense. Summing up we have

3.1.6 Theorem. The order type of $^*N$ is of the form

$$\alpha=\omega+(\omega^*+\omega)\theta\,,$$

where θ is a dense order type without first or last element.

Observe that any prime number in N remains a prime number in $^*N$ according to the definition that a number $a>1$ is prime if its only divisors are 1 and a. It follows that the number of primes in $^*N$ is infinite. More strongly, it is true that *for every number $a\in{}^*N$ there exists a prime number p which is greater than a.* For the statement just written in it is true in N and can be expressed as a sentence of K. Accordingly, it holds in $^*N$.

It is natural to ask whether every natural number (in $^*N$) can be expressed as the product of a finite number of powers of primes. If we understand the word *finite* here in the absolute sense then this signifies that the primes that occur in the product are in one-to-one correspondence with the set of natural numbers which are smaller than a number $n\in N$. With this interpretation of the word *finite* the answer to our question is negative. For let $\{p_n\}$ be the sequence of primes in N in ascending order of magnitude, $p_0=2, p_1=3$, and so on. For every natural number n define

$$q_n=\prod_{k=0}^{n} p_n$$

so that $q_0=2, q_1=2\cdot 3=6, q_2=2\cdot 3\cdot 5=30$, and so on. Then $\{q_n\}$ may be regarded as a function which maps the finite natural numbers into the finite natural numbers and which, accordingly is given by a relation Q of type (0,0) in N. The corresponding relation $*Q$ in $*N$, i.e., the relation of $*N$ which is denoted by the same constant of K as Q, defines a function $\{r_n\}$ from the numbers of $*N$ into the numbers of $*N$, such that $r_n = q_n$ for all finite n. In N, q_n is divisible by the first power of all primes up to p_n, by no higher power of these primes, and by no other prime. The same can therefore be said of r_n within $*N$. Thus, for infinite n, the number of prime divisors of r_n is infinite.

On the other hand, as can be seen by formalization within K, it is still true that for every natural number a in $*N$ other than 0 there exists a natural number m and a sequence of natural numbers j_n, $n=0,1,\ldots,m$, such that a is divisible by $p_n^{j_n}$, $n=0,1,\ldots,m$, and not by any higher power of these primes, and not by any other prime.

The following results show that there exists external sets of natural numbers in $*N$.

3.1.7 THEOREM. The set of infinite natural numbers is an external set in $*N$.

PROOF. Let N_i be the set of infinite natural numbers. As we have seen, N_i does not possess a smallest element. On the other hand it is true in N that every non-empty set of natural numbers possesses a smallest element. This can be expressed as a sentence of K, so

3.1.8 $$(\forall x)[[(\exists y)\Phi_{(0)}(x,y)] \supset [(\exists y)[\Phi_{(0)}(x,y) \wedge [(\forall z)[\Phi_{(0)}(x,z) \supset [\Phi_{(0,0)}(e,y,z) \vee \Phi_{(0,0)}(q,y,z)]\ldots],$$

where we recall that $\Phi_{(0,0)}(q,y,z)$ stands for $y<z$. Since 3.1.8 holds also in $*N$, it follows that every non-empty *internal* set of natural numbers possesses a smallest element. This shows that N_i cannot be an internal set, proving the theorem.

3.1.9 THEOREM. The set of finite natural numbers is an *external* set in $*N$.

PROOF. the following sentence holds in N and belongs to K.

3.1.10 $$(\forall x)(\exists y)(\forall z)[[\neg\Phi_{(0)}(x,z)] \equiv \Phi_{(0)}(y,z)].$$

This sentence states that for every set of type (0) there exists a set which is

its complement, i.e., which consists of all individuals that are not contained in the former set. Since the sentence belongs to K it must be true also in $*N$ where we may recall that the interpretation of 'set' in $*N$ is 'internal set'. It follows that if the set of finite natural numbers were an internal set in $*N$ then the set of infinite natural numbers would be an internal set in $*N$. But we have seen already that the set of infinite natural numbers is not an internal set in $*N$. This proves 3.1.9.

Similarly, we may prove that the relation $\sim$, which was defined earlier in this section by the condition that $a \sim b$ if $|a-b|$ is finite, is an external relation in $*N$. For the sentence

$$(\forall x)(\exists y)(\forall z)[\Phi_{(0,0)}(x,0,z) \equiv \Phi_{(0)}(y,z)]$$

holds in N and $*N$. This shows that if the relation $\sim$ were internal, then the set of natural numbers z such that $0 \sim z$, i.e., the set of finite natural numbers would be internal, contrary to 3.1.9.

3.2 Non-standard Analysis. Let $*R$ be a higher order non-standard model of Analysis as explained in 2.10.4. Thus, $*R$ satisfies conditions (i) – (iv) specified at the beginning of section 3.1, where we have to replace 'Natural Numbers' everywhere by 'Real Numbers'. At the same time, we have to substitute R and $*R$ for N and $*N$, where R is the system (higher order structure) of real numbers, in the ordinary sense. Let K be the set of stratified sentences which are admissible and hold in R.

$*R$ constitutes an ordered field, for the fact that R constitutes an ordered field can be expressed by means of sentences of K. From now on, we shall refer to all individuals of $*R$ as real numbers, reserving the name of standard real numbers to the individuals of R. In order to simplify our notation we shall write $a \in R$ (or $a \in *R$) in order to indicate that a is a real number (a standard real number) although strictly speaking R and $*R$ are not sets of individuals.

$*R$ is non-archimedean since it contains numbers which are greater then all numbers of R, which constitutes a subfield of $*R$. Thus, for some number a in $*R$,

$$1<a, 1+1<a, \ldots, 1+1+\cdots+1<a, \ldots, \qquad \text{and so on}$$

where we add 1 any (finite) number of times to itself. It is now easy to appreciate the following facts and definitions about $*R$.

If we define the absolute value $|a|$ of a number $a \in {}^*R$ by $|a| = a$ if $a \geq 0$ and by $|a| = -a$ if $a < 0$ then $|x|$ may be regarded as a standard function in ${}^*R$ which is an extension of the corresponding standard function in R. A real number $a \in {}^*R$ will be called *finite* if there exists a standard number $m \in R$ such that $|a| \leq m$, while any other number of ${}^*R$ will be called *infinite*. The finite numbers constitute a ring in R, which will be denoted by M_0.

A number $a \in {}^*R$ will be called *infinitesimal* or *infinitely small* if $|a| < m$ for *all* positive numbers m in R. By this definition, 0 is infinitesimal. The set of infinitesimal numbers constitutes a ring M_1 in ${}^*R$. M_1 is a subring of M_0. More particularly, M_1 is an ideal in M_0. A number $r \in {}^*R$, $r \neq 0$, is infinitesimal if and only if r^{-1} is infinite. If $a-b$ is infinitesimal, then we say that b is *infinitely close* to a, and we write $a \simeq b$.

Consider the quotient ring M_0/M_1. We claim that it is isomorphic to the field of standard real numbers.

Let A be any equivalence class in M_0 modulo M_1. Then A cannot contain two *distinct* standard real numbers r_1 and r_2. For in that case, $|r_1 - r_2|$ would not be infinitesimal (not being smaller than the standard positive real number $m = |r_1 - r_2|$), which contradicts the assumption that $r_1 - r_2$ belongs to M_1.

On the other hand, A must contain at least one standard real number. For let a be an arbitrary element of A. If a is a standard real number then we have finished. If a is not a standard then we divide the standard real numbers into sets D_1 and D_2 where D_1 contains all standard real numbers that are smaller than a and D_2 contains all standard real numbers which are greater than a. Then D_2 is not empty since $a \leq |a| < m$ for some standard m and D_1 is not empty since $|a| < m$ implies $a \geq -|a| > -m$, for the same standard m. However, if $d_1 \in D_1$ and $d_2 \in D_2$, then $d_1 < d_2$ since $d_1 < a$ and $a < d_2$. Thus, the pair (D_1, D_2) constitutes a Dedekind cut. The set determines a standard real number r which is either the greatest element of D_1 or the smallest of D_2. We claim that in either case r belongs to A, since $a - r$ is infinitesimal.

Suppose first that r is the greatest element of D_1, so that $a - r < 0$. If $a - r$ is not infinitesimal then there exists a positive standard real number δ such that $a - r \geq \delta$. Hence $a > r + \frac{1}{2}\delta$, which shows that the standard real number $r + \frac{1}{2}\delta$ belongs to D_1. But this contradicts the assumption that r is the greatest element of D_1 and proves that $a - r$ is infinitesimal in this case.

Suppose next that r is the smallest element of D_2. Then $r - a < 0$ and, if $a - r$ is not infinitesimal, $r - a \geq \delta$ for some standard positive real number δ. Hence $r - \frac{1}{2}\delta > a$, so that $r - \frac{1}{2}\delta$ belongs to D_2. This contradicts the assumption that r is the smallest element of D_2 and completes the proof that $a - r$ is infinitesimal in all cases. It follows that the correspondence $A \to r$, under which every equivalence class in M_0 modulo M_1 is mapped on the standard real number contained in it, provides the required isomorphism between M_0/M_1 and the field of standard real numbers R.

For any finite real number a in ${}^*R$ we call the uniquely determined standard real number which is infinitely close to a (r in the above proof) the *standard part* of a, in symbols $r = \mathrm{st}(a)$ or, more briefly, $r = {}^0a$. Moreover, given *any* real number in ${}^*R$ we call the set of real numbers which are infinitely close to a, the *monad* of a, to be denoted by $\mu(a)$. Thus $\mu(a)$ is the equivalence class of a in ${}^*R$ modulo M_1, where both ${}^*R$ and M_1 are regarded as additive groups. The elements of M_0/M_1, are the monads which belong to M_0.

Let N be the higher order structure of the natural numbers, regarded as a partial structure of R in the manner explained in section 2.11, and let ${}^*N$ be the corresponding structure within ${}^*R$. If ${}^*R$ is an enlargement of R then theorem 2.11.3 shows that ${}^*N$ is an enlargement of N. If ${}^*R$ is only known to be a higher order non-standard model of Analysis then we can still show that all sentences of A that hold in N hold also in ${}^*N$. Moreover, we claim that under the relation of order in ${}^*R$ which is also the relation of order in ${}^*N$, there exist individuals in ${}^*N$ which are greater than all individuals of N. Indeed, let q be the constant of K which denotes the order relation, as before, and let v denote the sets of individuals in N and ${}^*N$. Then the sentence,

3.2.1 $$(\forall x)(\exists y)\left[\Phi_{(0,0)}(q,x,y) \wedge \Phi_{(0)}(v,y)\right]$$

holds in R since it states that for every real number x there exists a natural number y which is greater than x. Accordingly, 3.2.1 holds also in ${}^*R$. Now let a be a number of ${}^*R$ which is greater than all numbers of R. Such an a exists since ${}^*R$ is a non-standard model of Analysis. By 3.2.1, as interpreted in ${}^*R$, there exists a number b which is an individual of ${}^*N$ and which is greater than a and hence, is greater then all numbers of R and in particular, is greater than all individuals of N (which constitute a subset of the numbers of R). This shows that $^*N$ is a non-standard higher-order model of Arithmetic, so that the results of 3.1 are applicable.

We shall use the symbols N and $^*N$, without danger of confusion, also in order to denote the sets of individuals of the respective structures. In this sense N is the set of ordinary, or *standard* natural numbers while $^*N$ is the set of natural numbers in $^*R$. The numbers of N are the finite natural numbers, both in the sense of section 3.1 and in the sense introduced above for the real numbers, while the numbers of $^*N-N$ are the infinite natural numbers in both senses.

3.2.2 THEOREM. The set R is an external set in $^*R$.

For if R were internal in $^*R$, then the intersection of R and of $^*N$, which is N, would be an internal set in $^*R$ and hence in $^*N$. This contradicts 3.1.9.

We show by the same method –

3.2.3 THEOREM. The set of finite real numbers, M_0, is an external set in $^*R$. So is the set of infinite real numbers.

Next we prove –

3.2.4 THEOREM. Let a be any number in $^*R$. Then $\mu(a)$ (the set of numbers which are infinitely close to a) is an external set in $^*R$.

PROOF. Suppose that $\mu(a)$ is an internal set for some a. Then the set $\mu_0=\{b|b=c-a, c\epsilon\mu(a)\}$ also is internal in $^*R$. μ_0 consists of all numbers which are infinitely close to 0, i.e., it is the monad of 0. But if μ_0 were internal, then the set of reciprocals of elements of μ_0 other than 0 also would be internal. But this is precisely the set of infinite real numbers. This proves the theorem.

3.3 Convergence. Let $\{s_n\}$, $n=0,1,2,\ldots$, be an infinite sequence of real numbers in the ordinary sense. $\{s_n\}$ is represented in R by some binary relation F, which in turn is denoted by a constant f within the formal language Λ. Since F defines a function from the natural numbers into the real numbers, the following sentences hold in R.

3.3.1 $$(\forall x)[\Phi_{(0)}(v,x)\equiv[(\exists y)\,\Phi_{(0,0)}(f,x,y)]]$$

3.3.2 $$(\forall x)(\forall y)(\forall z)[[\Phi_{(0,0)}(f,x,y)\wedge \Phi_{(0,0)}(f,x,z)]\supset \Phi_{(0,0)}(e,y,z)],$$

where v denotes the set of natural numbers and e denotes the relation of identity for individuals, as before. The two sentences together state that F defines a real-valued function whose domain is the set of natural numbers. Reinterpreted in $^*R$, the same sentences state that the relation $^*F$ (which is denoted by f in Λ) defines a real-valued function in the sense of $^*R$ whose domain is the set of natural numbers $^*N$. We may regard this function also as a sequence, $\{s'_n\}$, where n varies over all natural numbers in $^*N$. Now any finite natural number n determines the corresponding $s=s_n$ uniquely so that $\Phi_{(0,0)}(f,n,s)$ in both R and $^*R$. It follows that $s_n=s'_n$ for all finite n. Thus, it cannot give rise to any misunderstanding if we denote the sequence determined by f in $^*R$ by $\{s_n\}$ rather than by $\{s'_n\}$. The sequence $\{s_n\}$ in R then is the restriction of the sequence $\{s_n\}$ in $^*R$ to the finite natural numbers n. We observe that this ambiguity in the case of the symbol $\{s_n\}$ conforms to common usage in the theory of functions where, for example, $\sin x$ may first be defined for real arguments x only, but the same symbol is retained after the definition is extended to complex values of the argument. If necessary, we shall use the phrases 'in R' or 'in $^*R$' in order to indicate whether the original sequence $\{s_n\}$ is intended, or its extension to $^*R$.

3.3.3 THEOREM. In order that a sequence $\{s_n\}$ in R be bounded in R, it is necessary and sufficient that the elements of $\{s_n\}$ in $^*R$ are all finite.

PROOF. Suppose that $\{s_n\}$ is bounded in R. Thus, there exists a standard real number m such that $|s_n|<m$ for all $n\epsilon N$. It follows that if $\{s_n\}$ is represented in R by a binary relation F, which in turn is denoted by a constant f in Λ, as above, then the sentence

3.3.4 $$(\forall x)(\forall y)(\forall z)[[\Phi_{(0,0)}(f,x,y)\wedge\Phi_{(0,0)}(a,y,z)]\supset\Phi_{(0,0)}(q,z,m)]$$

holds in R, where $\Phi_{(0,0)}(a,y,z)$ stands for 'z is the absolute value of y' and q and f have their previous meanings. But 3.3.4 belongs to K and so, holds also in $^*R$. Hence $|s_n|<m$ for all natural numbers n, finite or infinite. This proves that the condition of the theorem is necessary.

The condition of the theorem is also sufficient. For suppose that it is satisfied. Then the following sentence holds in $^*R$, for any positive infinite real number r.

3.3.5 $$(\forall x)(\forall y)(\forall z)[[\Phi_{(0,0)}(f,x,y)\wedge\Phi_{(0,0)}(a,y,z)]\supset\Phi_{(0,0)}(q,z,r)].$$

We conclude by existential quantification that the sentence

3.3.6 $X=(\exists w)(\forall x)(\forall y)(\forall z)$
$$[[\Phi_{(0,0)}(f,x,y)\wedge\Phi_{(0,0)}(a,y,z)]\supset\Phi_{(0,0)}(q,z,w)]$$

also holds in $^*R$. While 3.3.5 is not admissible in R since the constant r does not have any interpretation in R, the sentence X, which is given by 3.3.6 is admissible in R. Hence, if X did not hold in R, then $\neg X$ would hold in R, and would belong to K and thus, would hold also in $^*R$. This contradicts the conclusion just reached that X holds in $^*R$ and shows that X holds also in R. Accordingly, there exists a number m *in* R such that $|s_n|<m$ for all natural numbers in N. This completes the proof of 3.3.3.

In the proof we have made use of the general principle that if X is admissible in R and holds in $^*R$ then X holds also in R. For if not, then $\neg X$ is admissible and holds in R and hence, holds also in $^*R$, contrary to assumption. The same argument will be applied repeatedly in the sequel.

The realization that certain statements can be formulated within the language Λ and in terms of the vocabulary of K constitutes an important element in our proof procedure. However, it would be quite impracticable actually to write down the formalized sentences in each case. Generally speaking the *possibility* of such a formulation is of paramount importance while the details are likely to be irrelevant. Accordingly, it will usually be sufficient to *state* that the formalization in question is possible, without carrying it out in practice. On occasion, a semi-formal notation may be helpful. For example, in place of X in 3.3.6 we may write

$$(\exists w)(\forall x)(\forall y)(\forall z)[[y=s_x\wedge z=|y|]\supset z<w]$$

or even more briefly,

$$(\exists w)(\forall x)[|s_x|<w].$$

3.3.7 THEOREM. Let $\{s_n\}$ be a standard infinite sequence and let s be a standard real number. Then s is the limit of $\{s_n\}$ within R, $\lim_{n\to s}s_n=s$ in the classical sense if and only if $s_n\simeq s$ (i.e., s_n is infinitely close to s) for all infinite subscripts n.

PROOF. Suppose $\lim_{n\to\infty}s_n=s$ in the classical sense. Then we have to show that s_n-s is infinitesimal for all infinite n. That is to say, for any standard $\varepsilon>0$ and for any infinite natural number n we have to prove $|s_n-s|<\varepsilon$.

Now we know that for any given $\varepsilon>0$ in R there exists a natural number ν in N such that

$$|s_n-s|<\varepsilon \quad \text{for} \quad n>\nu, n\in N.$$

Thus, it is true in R that

3.3.8 $$(\forall x)[[x\in N\wedge x>\nu]\supset|s_x-s|<\varepsilon.$$

But 3.3.8 can be expressed as a sentence of K and hence, is true also in $^*R$. Since any *infinite* natural number is greater than ν we deduce that $|s_n-s|<\varepsilon$ for all infinite n. This shows that the condition of the theorem is necessary.

To see that the condition is also sufficient suppose that it is satisfied and let ε be any standard positive real number. Then for any infinite natural ν, $|s_n-s|<\varepsilon$ for $n>\nu$ since $|s_n-s|$ is then even infinitesimal. It follows that 3.3.8 holds in $^*R$. As it stands, 3.3.8 cannot now be formulated within the vocabulary of K since ν is not in R. However,

3.3.9 $$(\exists w)(\forall x)[[x\in N\wedge x>w]\supset|s_x-s|<\varepsilon]$$

no longer contains ν, and holds in $^*R$. A corresponding formal sentence can be expressed in the vocabulary of K. Accordingly, 3.3.9 holds also in R, for some natural number ν *in* R, $|s_n-s|<\varepsilon$ for $n>\nu$. This completes the proof of 3.3.7.

From 3.3.7 it is easy to deduce some standard theorems on limits in R. For example, if $\lim_{n\to\infty}s_n=s$, $\lim_{n\to\infty}t_n=t$, then $s_n\simeq s$, $t_n\simeq t$ for all infinite n, by 3.3.7. Hence, $s_n+t_n\simeq s+t$, $\lim_{n\to\infty}(s_n+t_n)=s+t$.

Let $\{s_n\}$ be a standard infinite sequence. A standard real number s is by definition a *limit point* of $\{s_n\}$ in R if for every standard $\varepsilon>0$ and for every finite natural number ν there exists a finite natural number $n>\nu$ such that $|s_n-s|<\varepsilon$. Thus, if s is a limit point of $\{s_n\}$ in R then the following semi-formalized statement is true in R.

3.3.10 $$(\forall x)(\forall y)[[x>0\wedge y\in N]\supset[(\exists z)[z\in N\wedge z>y\wedge|s_z-s|<x]]].$$

But 3.3.10 can be expressed in the vocabulary of K and, accordingly, holds also in $^*R$. Hence, if we take $x=\varepsilon$, where ε is positive and infinitesimal, and $y=v$, where v is an infinite natural number, then

3.3.11 $$(\exists z)\left[z\in N \wedge z>v \wedge |s_z-s|<\varepsilon\right]$$

holds in $^*R$. This shows that $|s_n-s|<\varepsilon$ holds in $^*R$ for some infinite n, and proves one half of the following theorem.

3.3.12 THEOREM. Let $\{s_n\}$ be a standard infinite sequence. Then the set of limit points of $\{s_n\}$ in R is given by $S'=\{{}^0s_n\}$ where n varies over the *infinite* natural numbers for which s_n is *finite* (and hence, possesses a standard part, ${}^0s_n$).

PROOF. We have just shown that if s is a limit point of $\{s_n\}$ then $s_n\simeq s$ for some infinite n and so $s={}^0s_n$, $s\in S'$. Conversely, if $s={}^0s_n$, where n is infinite, then $|s_n-s|$ is infinitesimal. Let $\varepsilon>0$, ε standard and let $v\in N$. Then there exists a natural number z (i.e., the number $z=n$) which is greater than v and such that $|s_z-s|<\varepsilon$. Thus, 3.3.11 is true in $^*R$ and hence, belongs to K and is true also in R. This shows that s is a limit point of $\{s_n\}$ in R and completes the proof of 3.3.12.

In particular, if the standard sequence $\{s_n\}$ is bounded in R, then all s_n are finite in $^*R$, by 3.3.3. It follows that the standard parts ${}^0s_n$ exist for all infinite n and hence, that the set S' is not empty. This proves

3.3.13 THEOREM (Bolzano–Weierstrass). A bounded infinite sequence possesses at least one limit point.

Let $\{s_{nm}\}$, $n=0,1,2,\ldots$, $m=0,1,2,\ldots$, be a double sequence in R. $\{s_{nm}\}$ may be regarded as a function defined on $N\times N$ and taking values in R. It is given by a three place relation, F, such that (n,m,s) satisfies F if and only if $s=s_{nm}$. The corresponding relation $^*F$ in $^*R$ defines a function from $^*N\times{}^*N$ into $^*R$, which coincides with $\{s_{nm}\}$ on $N\times N$. Accordingly, we denote it also by $\{s_{nm}\}$.

By definition, a standard number s is the (double) limit of $\{s_{nm}\}$ in R, in symbols

$$\lim_{\substack{n\to\infty\\ m\to\infty}} s_{nm}=s$$

if for any standard $\varepsilon>0$ there exists a finite natural number v such that $|s_{nm}-s|<\varepsilon$ for all finite natural n and m greater than v.

The method used in the proof of 3.3.7 leads to

3.3.14 THEOREM. Let $\{s_{nm}\}$ be a standard double sequence and let s be a standard real number. Then

$$\lim_{\substack{n\to\infty\\ m\to\infty}} s_{nm}=s$$

in R if and only if $s_{nm}\simeq s$ for all infinite n and m.

In particular, we note

3.3.15 COROLLARY. Let $\{s_{nm}\}$ be standard. Then

$$\lim_{\substack{n\to\infty\\ m\to\infty}} s_{nm}=0$$

in R if and only if s_{nm} is infinitesimal for all infinite n and m.

Cauchy's necessary and sufficient condition for convergence in the classical case (i.e., in R) states

3.3.16 THEOREM. A sequence $\{s_n\}$ converges if and only if

$$\lim_{\substack{n\to\infty\\ m\to\infty}} (s_n-s_m)=0\,.$$

By 3.3.15, this is equivalent to

3.3.17 THEOREM. A standard sequence $\{s_n\}$ converges (possesses a limit) in R if and only if $s_n\simeq s_m$ for all infinite n and m.

PROOF. of 3.3.17. Suppose that the standard sequence $\{s_n\}$ converges to the standard real number s in R. Then, for any infinite n and m, we have, by 3.3.7, $s_n\simeq s$ and $s_m\simeq s$. This shows that $s_n\simeq s_m$, as asserted, and proves that the condition of the theorem is necessary.

Suppose now that $s_n\simeq s_m$ for all infinite n and m, and let ω be any infinite natural number. We claim that s_ω is finite.

Suppose on the contrary that s_ω is infinite. Define a set of natural numbers A by

$$A=\{n\mid |s_\omega-s_n|<1\}\,.$$

A is an internal set. For it is true in R that 'for every natural number x there exists a set of natural numbers y such that a natural number z is contained in y if and only if $|s_x - s_z| < 1$'. The statement just given in quotation marks can be formulated as a sentence of K and, accordingly, is true also in $^*R$. Specializing the statement by taking ω for x, we obtain A for y.

A contains all infinite natural numbers for if n is infinite, $|s_\omega - s_n|$ is even infinitesimal. On the other hand, $|s_\omega| = |(s_\omega - s_n) + s_n| \leq |s_\omega - s_n| + |s_n|$ so that, for $s_n \in A$, $|s_\omega| \leq 1 + |s_n|$. But s_n is standard and hence finite for finite n while s_ω is infinite. This shows that A does not contain any finite N, A coincides with the set of infinite natural numbers. But this set is an external set (see 3.1.7) a contradiction which shows that s_ω is finite. This being the case, s_ω possesses a finite part $s = {}^0s_\omega$. Then $s \simeq s_\omega$ and more generally $s \simeq s_n$ for any infinite n since $s_\omega \simeq s_n$. This shows that s is the limit of $\{s_n\}$, by 3.3.7 and completes the proof of 3.3.17.

Observe that the proof of the last theorem is independent of the accepted ('standard') definition of the limit of a sequence. We may if we wish *define* $\lim_{n\to\infty} s_n = s$ by the condition that $s_n \simeq s$ for all infinite n and we may then proceed directly to the development of the theory of convergence of infinite sequences.

Arguments similar to some of those used in the proof of 3.3.17 lead to the following results which apply to internal (and not only to standard) infinite sequences.

3.3.18 THEOREM. Let $\{s_n\}$ be an internal sequence of real numbers such that $|s_n| \leq m$ for all finite n, where m is some fixed number in $^*R$. Then there exists an infinite natural number ν such that $|s_n| \leq m$ for all $n < \nu$.

PROOF. If $|s_n| \leq m$ for *all* n, finite or infinite, then we may take for ν any infinite natural number. If there is an n such that $|s_n| > m$, define a set of natural numbers A by

$$A = \{n \mid |s_n| > m\}.$$

Then A is a non-empty internal set of natural numbers. It therefore possesses a first element, which may serve as the ν of the theorem.

3.3.19 THEOREM. Let $\{s_n\}$ be an internal sequence of real numbers such that $|s_n| \leq m$ for all infinite n. Then there exists a finite natural number ν such that $|s_n| \leq m$ for all $n > \nu$.

Proof omitted.

3.3.20 THEOREM. Let $\{s_n\}$ be an internal sequence such that s_n is infinitesimal for all finite n. Then there exists an infinite natural number ν such that s_n is infinitesimal for all $n < \nu$.

PROOF. Consider the internal sequence $\{t_n\}$ where $t_n = ns_n$. For all finite n, $|t_n| = n|s_n|$ is infinitesimal and so $|t_n| \leq 1$. Hence, by 3.3.18, there exists an infinite ν such that $|t_n| \leq 1$ for all infinite $n < \nu$. For such n, $|s_n| \leq 1/n$, so that s_n is infinitesimal also for all infinite $n < \nu$. This proves the theorem.

3.3.21 THEOREM. Let $\{s_n\}$ be an internal infinite sequence which is finite for all infinite n. Then there exists a finite natural number ν such that s_n is finite for all $n > \nu$.

PROOF. It follows from the assumption of the theorem that the sequence $\{t_n\}$ which is given by $t_n = (1/n)s_n$, $n \geq 1$, satisfies $|t_n| \leq 1$ for all infinite n. It follows that $|t_n| \leq 1$ also for all finite $n > \nu$ where ν is a natural number, which exists by 3.3.19. For such n, $|s_n| \leq n$, s_n is finite.

3.4 Continuity and differentiation. Let $f(x)$ be a real-valued function of a real variable x in R which is defined in an open interval (a,b), a and b standard, $a < b$. On passing to $^*R$, $f(x)$ is extended to a function which is defined for all numbers x in $^*R$ such that $a < x < b$. By agreement we denote this function again by $f(x)$. (This is in fact the function which is given by a binary relation in $^*R$ that is denoted by the same constant in K as the binary relation in R that defined the original $f(x)$.)

3.4.1 THEOREM. In order that the standard real number c be the limit of $f(x)$ in R as x approaches b (from below), $\lim_{x\to b} f(x) = c$, it is necessary and sufficient that $f(x) \simeq c$ for all $x \simeq b$, $a < x < b$.

PROOF. Suppose that $\lim_{x\to b} f(x) = c$ in the ordinary sense. Let ε be any standard positive number. Then there exists a standard positive h such that $|f(x) - c| < \varepsilon$ for $|x - b| < h$, $a < x < b$. Now, if $x \simeq b$ then $|x - b|$ is actually infinitesimal, and so $|x-b|<h$ and $|f(x)-c|<\varepsilon$. Since ε is arbitrary positive in R this shows that $f(x)\simeq c$, the condition of the theorem is necessary.

The condition is also sufficient. For suppose that it is satisfied and choose any positive infinitesimal h. Then the following semi-formalized statement is true in $^*R$, for arbitrary standard positive ε.

3.4.2 $$(\forall x)\big[[|x-b|<h\wedge a<x<b]\supset|f(x)-c|<\varepsilon\big].$$

Thus, in $^*R$, it is true that

3.4.3 $$(\exists y)\big[y>0\wedge[(\forall x)[[|x-b|<y\wedge a<x<b]\supset|f(x)-c|<\varepsilon]]\big].$$

But 3.4.3 is admissible in R and, accordingly, must belong to K and must hold also in R. Since ε was standard positive but otherwise arbitrary we conclude directly from the interpretation of 3.4.3 in R that $\lim_{n\to b}f(x)=c$. This completes the proof of 3.4.1.

Corresponding results hold for the limit of $f(x)$ as x tends to a (from above) and for the limit of $f(x)$ as x tends to an interior point of the interval (a,b). We state our conclusion for the latter case as a theorem.

3.4.4 THEOREM. In order that the standard real number c be the limit of $f(x)$ in R as x approaches x_0, x_0 a standard point (number), $a<x_0<b$, it is necessary and sufficient that $f(x)\simeq f(x_0)$ for all $x\neq x_0$ such that $x\simeq x_0$.

The theorem still holds if $f(x)$ is defined in the interval (a,b) *except for* $x=x_0$.

With $f(x)$ as before, the classical definition of continuity at a point taken in conjunction with 3.4.4 leads immediately to

3.4.5 THEOREM. The function $f(x)$ is continuous at a standard point x_0, $a<x_0<b$, if and only if $f(x)\simeq f(x_0)$ for all $x\simeq x_0$.

This is equivalent to saying that $f(x)$ maps the monad of x_0 into the monad of $f(x_0)$.

Suppose now that $f(x)$ is defined also at b. Then 3.4.1 shows that $f(x)$ is continuous at b if and only if $f(x)\simeq f(b)$ for all $x\simeq x_0$, $x<b$, and a corresponding statement holds for a. Thus, in these cases also, the part of the monad of $x_0=b$ (or $x_0=a$) that belongs to the interval of definition of $f(x)$ is mapped into the monad of $f(b)$ (or of $f(a)$).

We are now in a position to prove the intermediate value theorem for continuous functions (Bolzano-Weierstrass).

3.4.6 THEOREM (Standard). Let $f(x)$ be a function which is defined and continuous in the closed interval $a\leq x\leq b$, and such that $f(a)<0$, $f(b)>0$. Then there exists a point c, $a<c<b$ such that $f(c)=0$.

The word 'standard' is added above in order to indicate that we are dealing with the standard situation, i.e., with the ordinary real numbers. We shall make the same insertion also on other occasions when the circumstances are not specified elsewhere in the text.

PROOF of 3.4.6 Let ω be an infinite natural number. Then the sequence $\{x_0=a,x_1,x_2,\dots,x_\omega=b\}$ which is defined by $x_j=a+(j/\omega)(b-a)$ is an internal sequence in $^*R$. Indeed, since it is true in R that for every natural number n, there exists the sequence of numbers $\{x_j\}$, $j=0,\dots,n$, $x_j=a+(j/\omega)(b-a)$, the same must be true in $^*R$. We note that $\{x_0,\dots,x_\omega\}$ is a finite sequence in the sense of $^*R$ although it contains an infinite number of elements in the absolute sense. Now $f(x_0)<0$ and $f(x_\omega)>0$. It follows that the set A of natural numbers j for which $f(x_j)\geq 0$ is not empty but excludes $j=0$. A is an internal set which possesses a first element $j=\mu$. The difference $x_\mu-x_{\mu-1}=(1/\omega)(b-a)$ is infinitesimal and so ${}^0x_\mu={}^0x_{\mu-1}$. We write ${}^0x_\mu={}^0x_{\mu-1}=c$. If $a<c<b$ then $f({}^0x_\mu)\simeq f(c)$ and $f({}^0x_{\mu-1})\simeq f(c)$ by 3.4.5. If $c=a$ or $c=b$ then these relations still hold by the remarks following 3.4.5. But $f({}^0x_{\mu-1})<0$; since $f(c)$ is infinitely close to $f({}^0x_{\mu-1})$ this implies that $f(c)\leq 0$. Similarly, $f({}^0x_\mu)\geq 0$ implies that $f(c)\geq 0$. Hence, $f(c)=0$. This shows that $c\neq a$, $c\neq b$, c is the required argument value for which $f(c)=0$. This completes the proof of 3.5.6.

Let the standard function $f(x)$ be defined in the open interval (a,b), a and b standard, $a<b$, and let x_0 be a standard point in the interior of (a,b). We say that $f(x)$ is bounded at x_0 in R if for some standard $h>0$ and $m>0$, $|f(x)|<m$ for all x such that $|x-x_0|<h$.

3.4.7 THEOREM. $f(x)$ is bounded at x_0 if and only if $f(x)$ is finite in the monad of x in $^*R$.

PROOF. If $f(x)$ is bounded at x_0 then '$|f(x)|<m$ for all x such that $|x-x_0|<h$' is true in R, for some standard $h>0$, $m>0$. The statement in quotation marks can be formulated within K and hence, holds also in $^*R$. Now, in $^*R$, $|x-x_0|<h$ for all x in the monad of x_0 since, for such x, $|x-x_0|$ is actually infinitesimal. Hence $|f(x)|<m$ for all x in the monad of x_0, the condition of the theorem is necessary.

Again, if the condition is satisfied, choose h infinitesimal and m positive infinite. Then it is true in $^*R$, that $|f(x)|<m$ for all x for which $|x-x_0|<h$ since all such x are in the monad of x_0. Thus the statement 'there exist $y>0$ and $z>0$ such that $|f(x)|<z$ for all x such that $|x-x_0|<y$' holds in $^*R$. But this sentence can be formulated within A, in the vocabulary of K and, accordingly, belongs to K and holds also in R. This shows that the condition of the theorem is also sufficient and completes its proof.

3.4.8 THEOREM. With $f(x)$ and x_0 as before, if $f(x)$ is continuous at x_0 then it is bounded at x_0 (in R).

PROOF. By 3.4.5, $f(x)\simeq f(x_0)$ for all x in the monad of x_0. Hence $f(x)$ is finite in the monad of x_0, $f(x)$ is bounded at x_0, by 3.4.7.

With $f(x)$ defined as before and $a<x_0<b$, where x_0 is standard, consider the function

$$k(x)=\frac{f(x)-f(x_0)}{x-x_0},$$

$k(x)$ is defined for all values of x in (a,b), both in R and in $^*R$, except for $x=x_0$. The derivative of $f(x)$ at x_0, $f'(x_0)$ is given, classically, by $\lim_{x\to x_0}k(x)$ whenever that limit exists. Hence, by 3.4.4,

3.4.9 THEOREM. In order that the standard number c be the derivative of $f(x)$ at x_0 it is necessary and sufficient that

$$\frac{f(x)-f(x_0)}{x-x_0}\simeq c$$

for all $x\neq x_0$ in the monad of x_0.

3.4.10 THEOREM. If $f(x)$ is differentiable (possesses a derivative) of x_0 in R then $f(x)$ is continuous at x_0 in R.

PROOF. If $f(x)$ is differentiable at x_0 then for all $x\neq x_0$ in the monad of x_0,

$f(x)-f(x_0)\simeq c(x-x_0)$ where c is the derivative of $f(x)$ at x_0. This shows that for such x, $f(x)-f(x_0)$ is infinitesimal, $f(x)$ is continuous at x_0, by 3.4.4.

The derivation of the addition rule for differentiation from 3.4.9 is trivial. Let us consider the product rule.

Let the standard function $g(x)$, like $f(x)$, be defined in the interval (a,b) and let $f(x)$ and $g(x)$ be differentiable at the standard point x_0, $a<x_0<b$, with derivatives $f'(x_0)$ and $g'(x_0)$, respectively. Putting $h(x)=f(x)g(x)$, we then have

$$\begin{aligned} h(x)-h(x_0)&=f(x)g(x)-f(x_0)g(x_0)\\ &=(f(x)-f(x_0))g(x)+f(x_0)(g(x)-g(x_0)). \end{aligned}$$

Dividing by $x-x_0$, we obtain

$$\frac{h(x)-h(x_0)}{x-x_0}=\frac{f(x)-f(x_0)}{x-x_0}g(x)+f(x_0)\frac{g(x)-g(x_0)}{x-x_0}.$$

Now take $x\neq x_0$ in the monad of x_0. Then $g(x)$ belongs to M_0 (the ring of finite numbers in $^*R$), by 3.4.7, 3.4.8, 3.4.10, and so does $f(x_0)$. Also,

$$\frac{f(x)-f(x_0)}{x-x_0}\simeq f'(x_0),\qquad \frac{g(x)-g(x_0)}{x-x_0}\simeq g'(x_0),\text{ by 3.4.9}.$$

Hence,

$$\frac{f(x)-f(x_0)}{x-x_0}g(x)+f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\simeq f'(x_0)g(x)+f(x_0)g'(x_0)$$

and

$$\frac{h(x)-h(x_0)}{x-x_0}\simeq f'(x_0)g(x)+f(x_0)g'(x_0)\simeq f'(x_0)g(x_0)+f(x_0)g'(x_0),$$

since $g(x)\simeq g(x_0)$, by 3.4.5 and 3.4.10. Referring to 3.4.9, we now see that the derivative of $h(x)$ at x_0 exists and is equal to

$$h'(x_0)=f'(x_0)g(x_0)+f(x_0)g'(x_0).$$

With $f(x)$ defined in an interval (a,b), and $a<x_0<b$, $f(x)$, a,b, and x_0 standard as before, we say that $f(x)$ has a maximum at x_0 (in R, and hence in $^*R$) if there exists a standard positive number h such that $f(x)\leq f(x_0)$ for all x for which $|x-x_0|<h$. One proves by methods used repeatedly already,

3.4.11 THEOREM. The function $f(x)$ possesses a maximum at x_0 if and only if $f(x) \leq f(x_0)$ for all $x \simeq x_0$.

There is a corresponding result for minima.

3.4.12 THEOREM. Suppose $f(x)$ has a maximum at the standard point x_0, $a < x_0 < b$, and is differentiable at that point. Then $f'(x_0) = 0$.

PROOF. Choose x_1 and x_2 in the monad of x_0 such that $x_1 < x_0 < x_2$. Then $f(x_1) - f(x_0) \leq 0$ and $f(x_2) - f(x_0) \leq 0$, by assumption and so

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \leq 0 \quad \text{and} \quad \frac{f(x_2) - f(x_0)}{x_2 - x_0} \geq 0.$$

But

$$f'(x_0) \simeq \frac{f(x_1) - f(x_0)}{x_1 - x_0} \simeq \frac{f(x_2) - f(x_0)}{x_2 - x_0}$$

and the only standard number which has in its monad both non-positive (≤ 0) and non-negative (≥ 0) numbers is 0. Hence $f'(x_0) = 0$ as asserted.

The same conclusion applies if $f(x)$ has a minimum at $x = x_0$.

3.4.13 THEOREM. Let the standard function $f(x)$ be defined and continuous in the closed interval $a \leq x \leq b$. Then $f(x)$ attains its absolute maximum at some standard point c, i.e., $f(c) \geq f(x)$ for all x in the closed interval (in R and hence, in $^*R$).

PROOF. Define the sequence $\{x_0 = a, x_1, x_2, \ldots, x_\omega = b\}$ by $a_j = a + (j/\omega)(b-a)$ as in the proof of 3.4.6. Consider the sequence of real numbers of $^*R$, $\{f(x_0) = f(a), f(x_1), f(x_2), \ldots, f(x_\omega) = f(b)\}$. It is not difficult to see that this is an internal sequence. Now it is a true statement about R that for every natural number n and for every sequence of real numbers $\{r_0, r_1, \ldots, r_n\}$ there is a natural number j such that $r_j \geq r_i$, $i = 0, 1, \ldots, n$. This is a statement which can be formulated within K and, accordingly, holds also in $^*R$. It follows that for some natural number j in $^*R$, $f(x_j) \geq f(x_i)$, $i = 0, 1, \ldots, \omega$. Put $c = {}^0x_j$, then we claim that $f(c) \geq f(x)$ for all standard x in the closed interval $a \leq x \leq b$.

Indeed, the length of the sub-interval defined by the sequence $\{x_0, x_1, \ldots, x_\omega\}$ is infinitesimal since it is equal to $(b-a)/\omega$. It follows

immediately that every standard number in the closed interval from a to b contains in its monad at least one (in fact, infinitely many) x_i. Now suppose that $f(c) < f(x)$ for some standard x in the interval. Choose an x_i such that $x = {}^0x_i$. Then $f(c) \simeq f(x_j)$ and $f(x) \simeq f(x_i)$. But $f(c)$ and $f(x)$ are standard numbers and so it will be seen that $f(c) < f(x)$ implies $f(c) + \eta_1 < f(x) - \eta_2$ for arbitrary infinitesimal η_1 and η_2. Hence, $f(x_j) < f(x_i)$. This contradicts the definition of x_j and proves the theorem.

Since the fact that $f(c) \geq f(x)$ for all x in the interval can be stated as a sentence of K it is true also in $^*R$. Similarly, there is a standard number at which $f(x)$ attains its minimum in the interval.

3.4.14 THEOREM (Standard, Rolle). Let the function $f(x)$ be defined and continuous in the closed interval $a \leq x \leq b$ and differentiable in the open interval $a < x < b$, $a < b$ and such that $f(a) = f(b) = 0$. Then $f'(c) = 0$ for some c in the interior of the interval.

The proof is standard. If $f(x) = 0$ throughout the interval then $f'(c) = 0$ throughout the interval, so the conclusion is true, trivially. If $f(x) > 0$ somewhere in the interval then $f(x)$ has a positive absolute maximum in the interval, at a point c, say, and since $f(c) > 0$ it follows that $a < c < b$. Then $f(x)$ has a maximum at c in the sense of 3.4.12, and so $f'(c) = 0$. If $f(x) < 0$ somewhere in the interval we obtain the same conclusion by considering the absolute minimum. This proves Rolle's theorem.

We may now prove the mean value theorem by the usual linear transformation.

3.5 Integration. In this section, we shall consider the Cauchy integral. Let $f(x)$ be a standard function which is continuous in the interval $a \leq x \leq b$, where a and b are standard and $a < b$. By a *fine partition* of the interval (a,b) we mean a sequence $\{x_0, x_1, \ldots, x_\omega\}$, where ω is a natural number in $^*N$, such that $x_0 = a < x_1 < x_2 < \ldots < x_\omega = b$ and $x_j - x_{j-1}$ is infinitesimal for $i = 1, 2, \ldots, \omega$. There are fine partitions which are internal, e.g., the sequence used in the proof of 3.4.6, which was defined by $x_j = a + (j/\omega)(b-a)$ where ω is an infinite natural number.

Let $\pi = \{x_0, x_1, \ldots, x_\omega\}$ be an internal fine partition and let $\xi = \{\xi_1, \xi_2, \ldots, \xi_\omega\}$ be an internal sequence such that $x_j \leq \xi_{j+1} \leq x_{j+1}$, $j = 0, 1, \ldots, \omega - 1$. There are such internal sequences, for example the sequence which is given by $\xi_j = x_{j-1}$, $j=1,2,\dots,\omega$ and also the sequence given by $\xi_j = x_j$, $j=1,2,\dots,\omega$. We define

3.5.1
$$S(\pi,\xi) = \sum_{j=1}^{\omega} f(\xi_j)(x_j - x_{j-1}).$$

$S(\pi,\xi)$ has a meaning in the sense that it is given by a functional which is the extension in $^*R$ of the sum $\Sigma_{j=1}^{n} f(\xi_j)(x_j - x_{j-1})$ for sequences $\{x_0, x_1, \dots, x_n\}$ and $\{\xi_1, \xi_2, \dots, \xi_n\}$ satisfying the appropriate conditions, i.e., $x_0 < x_1 < \dots < x_n$ and $x_j \le x_{j+1}$, $_i = 0,1,\dots,n-1$ in R.

3.5.2 THEOREM. For any internal fine partition, and for any ξ satisfying the appropriate conditions as detailed,

$$\int_a^b f(x)\,\mathrm{d}x = {}^0S(\pi,\xi).$$

PROOF. We have to show that $\int_a^b f(x)\,\mathrm{d}x \simeq S(\pi,\xi)$, i.e., that for any standard $\varepsilon > 0$

$$\left|\int_a^b f(x)\mathrm{d}x - S(\pi,\xi)\right| < \varepsilon.$$

But the following statement is known to be true in R.

'For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\int_a^b f(x)\,\mathrm{d}x - \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1})\right| < \varepsilon \quad \text{for any} \quad \{x_0, x_1, \dots, x_n\},$$

$\{\xi_1, \xi_2, \dots, \xi_n\}$, such that $x_0 = a < x_1 < \dots < x_n = b$, $x_j \le \xi_{j+1} \le x_{j+1}$, $j = 0,1,\dots,n-1$, *and such that* $|x_j - x_{j-1}| < \delta$ *for* $j = 1,2,\dots,n$'.

Now this statement can be formulated in the vocabulary of K and accordingly, belongs to K and holds also in $^*R$. But the condition $|x_j - x_{j-1}| < \delta$, $_i = 1,2,\dots,\omega$ is certainly satisfied by any fine partition since in this case the differences $x_j - x_{j-1}$ are actually infinitesimal. Hence

$$\left|\int_a^b f(x)\,\mathrm{d}x - \sum_{j=1}^{\omega} f(\xi_j)(x_j - x_{j-1})\right| < \varepsilon.$$

This proves the theorem.

The above argument relies on the standard theory of Cauchy integration. This is necessary if we wish to *identify* $^0S(\pi,\xi)$ with the familiar integral. Alternatively, we may *define* the integral in question as the standard part of 3.5.1. In order to justify this definition, we have to show first that the standard part in question exists, i.e., that $S(\pi,\xi)$ is finite and then, that all the $S(\pi,\xi)$ in question are infinitely close to one another.

In order carry out the first step, we observe that $f(x)$ is finite throughout the interval $a \le x \le b$ in $^*R$, by 3.4.7. (The proof of 3.4.7 is formulated for points x_0 such that $a < x_0 < b$ but can be adapted to $x_0 = a$, $x_0 = b$ by trivial changes.) Thus, for every infinite positive real number r, $|f(x)| < r$ for all x in the interval. Hence

$$|S(\pi,\xi)| = \left|\sum_{j=1}^{\omega} f(\xi_j)(x_j - x_{j-1})\right| \le \sum_{j=1}^{\omega} |f(\xi_j)|(x_j - x_{j-1})$$
$$< \sum_{j=1}^{\omega} r(x_j - x_{j-1}) = r(b-a).$$

Now let ω be any infinite positive number. Then $r = \omega/(b-a)$ is positive infinite and so $|S(\pi,\xi)| < \omega(b-a)^{-1}(b-a) = \omega$. Let A be the set of positive numbers p such that $|S(\pi,\xi)| < p$. As we have just shown, this set includes all infinite positive numbers. Accordingly, we conclude that $|S(\pi,\xi)|$ is finite and possesses a standard part. The second step in the argument depends on the fact that a refinement of a given fine partition, i.e., the inclusion of additional x_j, does not change the standard part of $S(\pi,\xi)$. We omit the details.

With $f(x)$ as before, define $F(x) = \int_a^x f(x)\,\mathrm{d}x$, $a \le x \le b$. $F(x)$ is, in the first instance, defined for x in R but, as usual, this definition is extended on passing to $^*R$. Now for any standard x_0, $a < x_0 < b$, and for any positive standard h such that the number $x_0 + h$ still belongs to the closed interval $a \le x \le b$, we have

$$F(x_0 + h) - F(x_0) = \int_{x_0}^{x_0+h} f(x)\,\mathrm{d}x.$$

We define a fine partition of the interval $x_0 \le x \le x_0 + h$ by $x_j = x_0 + (j/\omega)h$, $j = 0,1,\dots,\omega$ where ω is some infinite natural number, and we define a corresponding sequence $\xi=\{\xi_1,\ldots,\xi_\omega\}$ by $\xi_j=x_{j-1}$, $j=1,\ldots,\omega$. Then, by 3.5.1 and 3.5.2,

3.5.3 $$F(x_0+h)-F(x_0)={}^0\Big(\sum_{j=1}^{\omega} f(x_{j-1})(x_j-x_{j-1})\Big)$$
$$={}^0\Big(\frac{h}{\omega}\sum_{j=1}^{\omega} f(x_{j-1})\Big)=h\,{}^0\Big(\frac{1}{\omega}\sum_{j=1}^{\omega} f(x_{j-1})\Big).$$

Now since $f(x)$ is continuous in the closed interval $x_0\le x\le x_0+h$ it attains its maximum and minimum for that interval at certain points ξ', ξ''. Then

$$f(\xi')\le f(x_j)\le f(\xi''),\ j=1,\ldots,\omega, \qquad \text{and so}$$

$$f(\xi')\le\frac{1}{\omega}\sum_{j=1}^{\omega} f(x_{j-1})\le f(\xi''), \qquad \text{and hence}$$

$$f(\xi')\le{}^0\Big(\frac{1}{\omega}\sum_{j=1}^{\omega} f(x_{j-1})\Big)\le f(\xi''), \qquad \text{and hence, by 3.5.3}$$

3.5.4 $$f(\xi')\le\frac{F(x_0+h)-F(x_0)}{h}$$

and

3.5.5 $$\frac{F(x_0+h)-F(x_0)}{h}\le f(\xi'').$$

Now 3.5.4 and 3.5.5 can be formulated within K and, accordingly, hold also for infinitesimal $h\neq 0$. In this case, both ξ' and ξ'' are infinitely close to x_0 and so, by the continuity of $f(x)$,

$$f(\xi')\simeq f(\xi'')\simeq f(x_0).$$

Hence from 3.5.4 and 3.5.5,

3.5.6 $$\frac{F(x_0+h)-F(x_0)}{h}\simeq f(x_0)$$

for all positive infinitesimal h. A similar argument shows that 3.5.6 is true also for negative infinitesimal h. Hence, by 3.4.9,

3.5.7 THEOREM (Standard). Let $f(x)$ be continuous for $a\le x\le b$. Then $F(x)=\int_a^x f(x)\,\mathrm{d}x$ exists and $F'(x)=f(x)$ for all $a<x<b$.

This is the fundamental theorem of the Integral Calculus for continuous functions. For right and left derivatives respectively, the argument can be adapted so as to include $x=a$ and $x=b$ in the result.

Next, let us consider an example of an improper integral. Let the standard function $f(x)$ be defined and continuous for all $x\ge a$, a standard. Then the improper integral $\int_a^\infty f(x)\,\mathrm{d}x$ is defined classically as $\lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$, whenever that limit exists. Adapting the non-standard version of Cauchy's condition (Theorem 3.3.17) we prove without difficulty,

3.5.8 THEOREM. In order that the improper integral $\int_a^\infty f(x)\,\mathrm{d}x$ exist it is necessary and sufficient that $\int_\eta^\zeta f(x)\,\mathrm{d}x\simeq 0$ for all positive infinite η and ζ.

Observe that $\int_\eta^\zeta f(x)\,\mathrm{d}x$ exists, in $^*R$, for all infinite η and ζ since $\int_c^d f(x)\,\mathrm{d}x$ exists in R for all $c\ge a$, $d\ge a$. The integral $\int_\eta^\zeta$ possesses even for non-standard η and ζ the usual properties of an integral to the extent to which they can be formulated within our language. For instance,

$$\int_\zeta^\eta f\,\mathrm{d}x=-\int_\eta^\zeta f\,\mathrm{d}x \quad\text{and}\quad \int_\eta^\zeta f\,\mathrm{d}x+\int_\zeta^\lambda f\,\mathrm{d}x=\int_\eta^\lambda f\,\mathrm{d}x.$$

It is natural to ask whether the integral $\int_\eta^\zeta f(x)\,\mathrm{d}x$ can be approximated by a sum such as 3.5.1, even for infinite η and ζ. To answer this question, we shall suppose $\eta<\zeta$ and we shall confine ourselves to partitions $\pi=\{x_0=\eta,x_1,\ldots,x_\omega=\zeta\}$ which are given by $x_j=\eta+(j/\omega)(\zeta-\eta)$, $j=0,1,\ldots,\omega$, ω an infinite integer, while $\zeta_j=x_{j-1}$, $j=1,\ldots,\omega$. Then the right hand side of 3.5.1 becomes

$$\sum_{j=1}^{\omega} f(x_{j-1})\frac{\zeta-\eta}{\omega}=\frac{\zeta-\eta}{\omega}\sum_{j=0}^{\omega-1} f(x_j).$$

Notice that since $\zeta-\eta$ may now be infinite, $(\zeta-\eta)/\omega$ may be finite even for infinite ω. We shall include this possibility in our considerations to begin with.

3.5.9 THEOREM. With $f(x)$ defined as above, let ω be an arbitrary but fixed infinite natural number. Then there exists an *infinite* positive real number b such that for all η, ζ which satisfy $a \le \eta < \zeta \le b$,

3.5.10
$$\int_\eta^\zeta f(x)\,\mathrm{d}x \simeq \frac{\zeta-\eta}{\omega} \sum_{j=0}^{\omega-1} f(x_j)$$

where the x_j are defined as above.

PROOF. Let n be any finite natural number. Then, for the given infinite ω and for any c and d such that $a \le c \le d \le n$, the difference

$$\int_c^d f(x)\,\mathrm{d}x - \frac{d-c}{\omega} \sum_{j=0}^{\omega-1} f(x_j)$$

is infinitesimal and so by 3.5.2, it is true in $^*R$ that

3.5.11
$$n\left|\int_c^d f(x)\,\mathrm{d}x - \frac{d-c}{\omega} \sum_{j=0}^{\omega-1} f(x_j)\right| < 1, \qquad x_j = c + \frac{j}{\omega}(d-c).$$

If in 3.5.11, we replace ω by a finite natural number m then we obtain an inequality which may or may not be true for a particular n. However, if we express this inequality by a sentence $X(n,m)$ within our formal language then it is true in R that 'for every natural number m, if there exists any natural number y such that $\neg X(y,m)$ holds then there exists a smallest y of this kind'.

The usual argument now shows that the statement just given in quotation marks holds also in $^*R$. Accordingly, there must be a smallest number $y = n_0$ in $^*N$ which satisfies $\neg X(y,\omega)$ if any y satisfies $\neg X(y,\omega)$ at all. If such an n_0 exists then it must be infinite since $X(n,\omega)$ is true for all finite n. In that case, we put $b = n_0 - 1$. If $\neg X(y,\omega)$ is not true for any y,

i.e., if $X(y,\omega)$ holds for all natural numbers y then we set b equal to an arbitrary positive infinite real number. In either case,

$$\left|\int_c^d f(x)\,\mathrm{d}x - \frac{d-c}{\omega} \sum_{j=0}^{\omega-1} f(x_j)\right| < \frac{1}{n}$$

for all $a \le c < d \le n \le b$, which shows, for $n = b$, that

$$\int_c^d f(x)\,\mathrm{d}x \simeq \frac{d-c}{\omega} \sum_{j=0}^{\omega-1} f(x_j) \quad \text{for} \quad a \le c < d \le b\,.$$

This proves the theorem. If we want to make sure that the sum on the right hand side involves a *fine* partition, we only have to take b infinite but smaller than min $(n_0 - 1, \sqrt{\omega})$ if n_0 as above exists and as $\sqrt{\omega}$ otherwise. For, with this choice,

$$\frac{\zeta-\eta}{\omega} \le \frac{|\zeta|+|\eta|}{\omega} \le \frac{2}{\sqrt{\omega}},$$

so that the length of the intervals (x_j, x_{j+1}) is infinitesimal.

3.5.12 THEOREM. With $f(x)$ defined as above, the integral $\int_a^\infty f(x)\,\mathrm{d}x$ exists if and only if for some infinite natural number ω and for some infinite positive real number b

$$\frac{\zeta-\eta}{\omega} \sum_{j=0}^{\omega-1} f(x_j) \simeq 0$$

for all infinite η, ζ such that $\eta < \zeta \le b$.

PROOF. Suppose the integral exists. Then $\int_\eta^\zeta f(x)\,\mathrm{d}x \simeq 0$ for all positive infinite η and ζ, by 3.5.8. At the same time, for arbitrary infinite natural ω and for a number b which satisfies the conclusions of 3.5.11,

$$\frac{\zeta-\eta}{\omega} \sum_{j=0}^{\omega-1} f(x_j) \simeq \int_\eta^\zeta f(x)\,\mathrm{d}x \qquad \text{for} \qquad a \le \eta < \zeta \le b\,.$$

Hence, for infinite η and ζ which satisfy $\eta<\zeta\le b$,

$$\frac{\zeta-\eta}{\omega}\sum_{j=0}^{\omega-1} f(x_j)\simeq 0.$$

This shows that the condition of the theorem is necessary.

Now suppose that the condition of the theorem is satisfied. Thus, for some infinite natural ω and positive infinite b,

$$\frac{\zeta-\eta}{\omega}\sum_{j=0}^{\omega-1} f(x_j)$$

is infinitesimal for all positive infinite η and ζ such that $\eta<\zeta\le b$. Now we know that, for the given ω and for sufficiently small positive infinite b'

$$\frac{\zeta-\eta}{\omega}\sum_{j=0}^{\omega-1} f(x_j)\simeq\int_\eta^\zeta f(x)\,\mathrm{d}x \qquad \text{for} \qquad a\le\eta<\zeta\le b'.$$

Hence, for positive infinite η and ζ such that $\eta<\zeta\le\min(b,b')$, $\int_\eta^\zeta f(x)\ \mathrm{d}x\simeq 0$. Now let ε be any standard positive number, then $|\int_\eta^\zeta f(x)\,\mathrm{d}x|<\varepsilon$ for $\eta<\zeta\le\min(b,b')$. Let A be the set of natural numbers n, such that $|\int_\eta^\zeta f(x)\,\mathrm{d}x|<\varepsilon$ for all ζ and η such that $n\le\eta<\zeta\le\min(b,b')$. As we have just seen, A includes all infinite natural numbers smaller than $\min(b,b')$. (A includes all infinite natural numbers $\ge\min(b,b')$, trivially.) Also, A is an internal set, so it possesses a smallest element, which must therefore be finite. Accordingly, there exists a standard natural number n such that for all η and ζ for which $n\le\eta<\zeta\le\min(b,b')$, $|\int_\eta^\zeta f(x)\,\mathrm{d}x|<\varepsilon$. Hence, for any standard η and ζ such that $n\le\eta<\zeta$, we have $|\int_\eta^\zeta f(x)\,\mathrm{d}x|<\varepsilon$, in $^*R$ and hence also in R. But this shows that $\int_a^\infty f(x)\,\mathrm{d}x$ satisfies the necessary and sufficient condition for the convergence of an improper integral in the classical sense and completes the proof of 3.5.12.

The last part of the proof makes use of an interesting phenomenon which appears in various forms. Thus, let $\{s_n\}$ be any standard infinite sequence. We know that a necessary and sufficient condition for $\{s_n\}$ to converge to 0 is that $s_n\simeq 0$ for all infinite n. However, an apparently weaker condition is already sufficient for convergence.

3.5.13 THEOREM. Let $\{s_n\}$ be a standard infinite sequence. Suppose that there exists an infinite natural number ν such that s_n is infinitesimal for all infinite $n<\nu$. Then $\lim_{n\to\infty} s_n=0$ in R.

PROOF. Let ε be a standard positive number. Then $|s_n|<\varepsilon$ for all infinite n such that $n<\nu$ since s_n is actually infinitesimal for such n. Thus, the set A which consists of all natural numbers μ such that $|s_n|<\varepsilon$ for all n which satisfy $\mu\le n<\nu$ is not empty, and includes all infinite natural μ. Accordingly, the first μ which satisfies this condition is finite, $\mu=\mu_0$, say. Then $|s_n|<\varepsilon$ for all finite n such that $n\ge n_0$, both in $^*R$ and in R. This shows that $\lim_{n\to\infty} s_n=0$ and proves 3.5.13.

3.6 Differentials. Let $\{a_1,\dots,a_n\}$ be a set of real numbers in $^*R$, where n is a finite positive natural number. The set of all linear combinations $r_1a_1+\cdots+r_na_n$ where $r_1,\dots,r_n$ are finite (i.e., belong to M_0) will be denoted by $O(a_1,\dots,a_n)$. Similarly, the set of all $r_1a_1+\cdots+r_na_n$ such that $r_1,\dots,r_n$ are infinitesimal (belong to M_1) will be denoted by $o(a_1,\dots,a_n)$. In particular, $M_0=O(1)$ and $M_1=o(1)$.

For any given $a_1,\dots,a_n$, put $a=\max(|a_1|,\dots,|a_n|)$. Then it is easy to see that $O(a_1,\dots,a_n)=O(a)$, $o(a_1,\dots,a_n)=o(a)$. Both $O(a)$ and $o(a)$ are additive groups which admit M_0 as domain of (multiplicative) operators. M_0 is isomorphic to $O(a)$ as an additive group under the isomorphism under which $m\in M_0$ is mapped on $ma\in O(a)$. Under this isomorphism, M_1 is mapped on $o(a)$. It follows that the difference group $O(a)/o(a)$ is isomorphic to the additive group of R.

Let $y=f(x)$ be an internal function which is defined in an open interval $(a,b), a<b$, a and b standard. We choose an infinitesimal positive real number, calling it $\mathrm{d}x$. For x in (a,b), we then put $\mathrm{d}y=\mathrm{d}f=f(x+\mathrm{d}x)-f(x)$. Thus, d may be regarded as an operator which maps any function in $^*R$ on another function which is defined in a slightly smaller interval (i.e., $(a,b-\mathrm{d}x)$). Applying d to $\mathrm{d}y$, we obtain

$$\mathrm{d}^2y=\mathrm{d}(\mathrm{d}y)=f(x+2\mathrm{d}x)-2f(x+\mathrm{d}x)+f(x),$$

which is defined for $a<x<b-2\mathrm{d}x$. More generally, putting $\mathrm{d}^0y=y$,

$d^n y = d(d^{n-1}y)$, $n=1,2,\ldots$, we obtain functions which are defined for $a<x<x-n\,dx$, respectively.

For standard $f(x)$ and standard x, $a<x<b$, it follows immediately from 3.4.9 that if the function $f(x)$ is differentiable at x then

$$\frac{dy}{dx} \simeq f'(x).$$

More generally, we propose to show that, under certain conditions which will be discussed presently,

3.6.1 $$\frac{d^n y}{dx^n} \simeq f^{(n)}(x),\ n=1,2,\ldots,$$

where dx^n stands for $(dx)^n$. Since 3.6.1 indicates equivalence modulo $M_1 = o(1)$, 3.6.1 implies that $(d^n y/dx^n) - f^{(n)}(x) \in o(1)$. Hence,

3.6.2 $$d^n y - f^{(n)}(x)\,dx^n \in o(dx^n).$$

We shall write $\alpha \simeq \beta \bmod o(a_1,\ldots,a_n)$ or $\alpha \simeq \beta \bmod O(a_1,\ldots,a_n)$ in order to indicate equivalence modulo $o(a_1,\ldots,a_n)$ or $O(a_1,\ldots,a_n)$ respectively. With this notation, 3.6.2 becomes

3.6.3 $$d^n y \simeq f^{(n)}(x)\,dx^n \bmod o(dx^n).$$

3.6.4 THEOREM. Let $f(x)$ be a standard function which is defined in the open interval $a<x<b$ and possesses continuous derivatives up to the nth order in that interval, for standard a, b, and n, then 3.6.1 and 3.6.3 hold for any finite real x such that $a<{}^0x<b$.

PROOF. If $a<x<x+nh<b$ in R then, by a classical generalization of the mean value theorem, there exists a (standard) number θ, $0<\theta<1$ such that

3.6.5 $$\frac{\Delta^n f(x)}{h^n} = f^{(n)}(x+n\theta h),$$

where $\Delta^n f(x)$ is the nth formal difference for the interval length h. Thus $\Delta f(x) = f(x+h)-f(x)$, $\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$, etc. Since this result is true in R it must be true also in $^*R$. But, for $h=dx$, $\Delta^n f(x)$ becomes $d^n y$, and so

3.6.6 $$\frac{d^n y}{dx^n} = f^{(n)}(x+n\theta dx), \qquad 0<\theta<1.$$

By assumption, $a<{}^0x<b$ and $f^{(n)}(x)$ is continuous at ${}^0x$. Hence, $f^{(n)}({}^0x) \simeq f^{(n)}(x) \simeq f^{(n)}(x+n\theta dx)$ and so

$$\frac{d^n y}{dx^n} \simeq f^{(n)}(x).$$

This proves 3.6.4.

3.7 Total differentials. Let $f(x_1,\ldots,x_n)$ be a function of n variables in R and let $(\xi_1,\ldots,\xi_n)$ be a point within the domain of definition of $f(x_1,\ldots,x_n)$ such that the function exists in the interior of a sphere S which is given by $(x_1-\xi_1)^2+(x_2-\xi_2)^2+\ldots+(x_n-\xi_n)^2 = r^2$, $r>0$. The function extends to $^*R$ in the usual way. Various results established previously on continuity and differentiation of functions apply also to functions of several variables. Continuity will be considered more systematically in the next chapter, within the general framework of topological and metric spaces. For the present we shall require only the following result.

3.7.1 THEOREM. Suppose $f(x_1,\ldots,x_n)$ as introduced above is continuous at the point $\xi_1,\ldots,\xi_n$. Then $f(x_1',\ldots,x_n') \simeq f(\xi_1,\ldots,\xi_n)$ for all $(x_1',\ldots,x_n')$ such that the distance $\sqrt{\{(x_1'-\xi_1)^2+\cdots+(x_n'-\xi_n)^2\}}$ is infinitesimal.

PROOF. For every standard positive ε there exists a standard positive δ such that $|f(x_1,\ldots,x_n)-f(\xi_1,\ldots,\xi_n)|<\varepsilon$ provided $\sqrt{\{(x_1-\xi_1)^2+\cdots+(x_n-\xi_n)^2\}}<\delta$. But $\sqrt{\{(x_1'-\xi_1)^2+\cdots+(x_n'-\xi_n)^2\}}$ is smaller than δ since it is even infinitesimal. Hence, $|f(x_1',\ldots,x_n')-f(\xi_1,\ldots,\xi_n)|<\varepsilon$. Since this is true for arbitrary standard positive ε, $|f(x_1',\ldots,x_n')-f(\xi_1,\ldots,\xi_n)|$ is infinitesimal. This proves the theorem.

Now suppose that the standard function $f(x_1,\ldots,x_n)$ exists and possesses continuous first derivatives within a sphere S as above. Taking $(x_1',\ldots,x_n')$ and $(h_1,\ldots,h_n)$ as two n-tuples of standard real numbers such that $(x_1',\ldots,x_n')$ and $(x_1'+h_1,\ldots,x_n'+h_n)$ belong to the interior of S, we put $g(t) = f(x_1'+h_1t,\ldots,x_n'+h_nt)$.

By a classical result (which can be established also by means of Nonstandard Analysis),

$$g'(t)=\frac{\partial f}{\partial x_1}\frac{dx_1}{dt}+\cdots+\frac{\partial f}{\partial x_n}\frac{dx_n}{dt}=\frac{\partial f}{\partial x_1}h_1+\cdots+\frac{\partial f}{\partial x_n}h_n,$$

where the partial derivatives are taken for the appropriate $x_i=x_i'+h_i t$. Hence, by the mean value theorem

3.7.2
$$g(1)-g(0)=f(x_1'+h_1,\ldots,x_n'+h_n)-f(x_1',\ldots,x_n')$$
$$=\left(\frac{\partial f}{\partial x_1}\right)_\theta h_1+\cdots+\left(\frac{\partial f}{\partial x_n}\right)_\theta h_n,$$

where the partial derivatives are taken at some point $(x_1'+\theta h_1,\ldots,x_n'+\theta h_n)$, $0<\theta<1$.

Passing to $^*R$ we choose $(x_1',\ldots,x_n')$ such that $^0x_i'=\xi_i$, $i=1,\ldots,n$ and we make the $h_i=dx_i$ infinitesimal but not all equal to zero. We *define* the *total differential* df by

$$df=f(x_1'+dx_1,\ldots,x_n'+dx_n)-f(x_1',\ldots,x_n').$$

Then, from 3.7.2,

3.7.3
$$df=\left(\frac{\partial f}{\partial x_1}\right)_0 dx_1+\cdots+\left(\frac{\partial f}{\partial x_n}\right)_0 dx_n+r_1\,dx_1+\cdots+r_n\,dx_n,$$

where the subscript 0 indicates that the partial derivatives are taken at $(x_1',\ldots,x_n')$ and where

$$r_i=\left(\frac{\partial f}{\partial x_i}\right)_\theta-\left(\frac{\partial f}{\partial x_i}\right)_0,\qquad i=1,\ldots,n.$$

We claim that the r_i are infinitesimal. Indeed, denoting by $(\partial f/\partial x_i)_\xi$ the partial derivatives of f at $(\xi_1,\ldots,\xi_n)$ and applying 3.7.1 twice (with the partial derivatives of f in place of f in the formulation of that theorem) we find that

$$\left(\frac{\partial f}{\partial x_i}\right)_0\simeq\left(\frac{\partial f}{\partial x_i}\right)_\xi,\quad\left(\frac{\partial f}{\partial x_i}\right)_\theta\simeq\left(\frac{\partial f}{\partial x_i}\right)_\xi$$

and so

$$\left(\frac{\partial f}{\partial x_i}\right)_\theta\simeq\left(\frac{\partial f}{\partial x_i}\right)_0,$$

the r_i are infinitesimal, as asserted. It follows that $r_1\,dx_1+\cdots+r_n\,dx_n\in o(dx_1,\ldots,dx_n)$. Referring to 3.7.3 we see that we have proved the following theorem.

3.7.4 THEOREM. Let $f(x_1,\ldots,x_n)$ be a standard function which is defined and possesses continuous first derivatives in the interior of a sphere with center $(\xi_1,\ldots,\xi_n)$ and radius $r>0$ where $\xi_1,\ldots,\xi_n$, and r are standard. If $^0x_1'=\xi_1,\ldots,{}^0x_n'=\xi_n$ and $dx_1,\ldots,dx_n$ are infinitesimal, then

3.7.5
$$df\simeq\left(\frac{\partial f}{\partial x_1}\right)_0 dx_1+\cdots+\left(\frac{\partial f}{\partial x_n}\right)_0 dx_n \bmod o(dx_1,\ldots,dx_n),$$

where $df=f(x_1'+dx_1,\ldots,x_n'+dx_n)-f(x_1',\ldots,x_n')$ and where the subscripts 0 indicate that the partial derivatives are taken at $(x_1',\ldots,x_n')$.

3.8 Elementary Differential Geometry. Quite early in the history of the Differential Calculus, infinitesimals were employed in the development of the theory of curves, and long after the classical ε, δ-method had displaced the naive use of infinitesimals in Analysis they survived in Differential Geometry and in several branches of Theoretical Physics. Even now there are many classical results in Differential Geometry which have never been established in any other way, the assumption being that somehow the rigorous but less intuitive, ε, δ-method would lead to the same result. So far as one can see without a complete check this assumption is usually correct. However, although no striking results will be obtained in this domain, it seems appropriate to investigate some simple examples from elementary Differential Geometry by means of the infinitesimals of Nonstandard Analysis.

Let the curve C in three-dimensional (x,y,z)-space over the real number R be given by

3.8.1
$$x=x(t),\quad y=y(t),\quad z=z(t),\qquad 0\le t\le 1,$$

where $x(t)$, $y(t)$, $z(t)$ have continuous derivatives in their interval of definition.

Passing to $^*R$, we choose a positive infinitesimal dt and we put, correspondingly,