Chapter 20 Suslin's Problem

In this chapter we investigate Suslin's Problem, which is the question whether every linearly ordered set which is dense, unbounded, complete, and satisfies the countable chain condition is also separable. First it will be shown that the answer to Suslin's Problem is "no" if and only if there exists a Suslin line, and then it will be proved that the existence of a Suslin line is independent of ZFC.

A Topological Characterisation of the Real Line

Let (P, <) be a linearly ordered set, in particular, for any two distinct elements $a, b \in P$ we have either a < b or b < a. We say that

- P is dense if for all $a, b \in P$ with a < b there is a $c \in P$ such that a < c < b;
- *P* is **unbounded** if for all $a \in P$ there are $b_0, b_1 \in P$ such that $b_0 < a < b_1$;
- *P* is **complete** if every non-empty, bounded subset of *P* has a supremum and an infimum;
- P is separable if there exists a countable set C ⊆ P such that for all a, b ∈ P with a < b there is a c ∈ C such that a < c < b;
- P satisfies ccc if every family of pairwise disjoint, non-empty open intervals

 (a, b) = {c ∈ P : a < c < b} is countable.

It is well-known that the real line \mathbb{R} together with the natural ordering "<" is—up to isomorphism—the unique linearly ordered set that is dense, unbounded, complete, and separable. Now, **Suslin's Problem** is the question whether *separable* can be replaced with *satisfying ccc*, that is, whether every linearly ordered set which is dense, unbounded, complete, and satisfies *ccc* is isomorphic to the real line.

First we show "separable" implies "ccc".

FACT 20.1. Every separable, linearly ordered set satisfies ccc.

Proof. Let (P, <) be a separable, linearly ordered set and let

$$\mathscr{I} := \left\{ (a_{\xi}, b_{\xi}) : \xi \in \Lambda \right\}$$

be a family of pairwise disjoint, non-empty open intervals. Since P is separable, there exists a countable set $C \subseteq P$ such that for all $a, b \in P$ with a < b there is a $c \in C$ such that a < c < b. In particular, for each interval $(a_{\xi}, b_{\xi}) \in \mathscr{I}$ there is a $c_{\xi} \in C$ such that $a_{\xi} < c_{\xi} < b_{\xi}$. Since the intervals in \mathscr{I} are pairwise disjoint, for any distinct $\xi, \xi' \in \Lambda$ we get $c_{\xi} \neq c_{\xi'}$. Now, since C is countable, also \mathscr{I} is countable, and since \mathscr{I} was arbitrary, we get that P satisfies ccc.

With FACT 20.1, Suslin's Problem is equivalent to the question whether "*ccc*" implies "separable", which leads to the following definition: A linearly ordered set which is dense, unbounded, complete, and satisfies *ccc*, but which is not separable, is called a **Suslin line**. So, Suslin's Problem is equivalent to the question whether there exists a Suslin line.

The following result shows that every dense, unbounded, linearly ordered set which satisfies *ccc* can be completed.

LEMMA 20.2. Let (Q, <) be a dense, unbounded, linearly ordered set. Then there is a complete, dense, unbounded, linearly ordered set (P, \prec) and a set $Q' \subseteq P$, such that:

- (Q', \prec) is isomorphic to (Q, <);
- Q' is dense in P, i.e., if x, y ∈ P with x ≺ y, then there is a z ∈ Q' such that x ≺ z ≺ y.

Furthermore, if (Q, <) satisfies *ccc* but is not separable, then also (P, \prec) satisfies *ccc* but is not separable.

Proof. A so-called *Dedekind cut* in Q is a pair (A, B) of disjoint, non-empty subsets of Q such that :

- $A \cup B = Q;$
- for all $a \in A$ and $b \in B$ we have a < b;
- if the infimum of B exists (which is the same as the supremum of A), then it belongs to B.

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Let P be the set of all Dedekind cuts in Q and let

$$(A_1, B_1) \prec (A_2, B_2) \iff A_1 \subsetneq A_2$$

Then, by the properties of Dedekind cuts, " \prec " is a linear ordering on P, and moreover, P is complete. To see that every non-empty bounded subset of P has a supremum, let $\{(A_{\iota}, B_{\iota}) : \iota \in \Lambda\}$ be a non-empty bounded subset of P. Then the Dedekind cut

$$\left(\bigcup_{\iota\in\Lambda}A_{\iota},\ \bigcap_{\iota\in\Lambda}B_{\iota}\right)$$

is its supremum. Similarly, every non-empty bounded subset of P has an infimum. For each $q \in Q$ let

$$A_q := \{ x \in Q : x < q \} \text{ and } B_q := \{ x \in Q : q \le x \} .$$

Then, for $Q' := \{(A_q, B_q) : q \in Q\}, (Q', \prec)$ is obviously isomorphic to (Q, <). We leave it as an exercise to the reader to show that Q' is dense in P, and that if Q satisfies *ccc* but is not separable, then also P satisfies *ccc* but is not separable. \dashv

So, in order to construct a Suslin line, it is enough to construct a linearly ordered set which is not separable, but which is dense, unbounded, and satisfies *ccc*.

Suslin Lines and Suslin Trees

In order to solve Suslin's Problem we first state it in terms of trees. A **tree** is a partially ordered set (T, <) with the property that for each $x \in T$, the set $\{y \in T : y < x\}$ of all *predecessors* of x is well-ordered by "<". For $x \in T$ let o.t.(x) be the order type of the well-ordered set $\{y \in T : y < x\}$. For $\alpha \in \Omega$, the α^{th} level of T, denoted $T|_{\alpha}$, consists of all $x \in T$ such that $o.t.(x) = \alpha$, *i.e.*, $T|_{\alpha} = \{x \in T : o.t.(x) = \alpha\}$. We require that $T|_0$ contains a single element, called the **root** of T. The **height** of T, denoted h(T), is the least α such that $T|_{\alpha} = \emptyset$, or in other words,

$$h(T) := \bigcup \{ \text{o.t.}(x) + 1 : x \in T \}.$$

For $x \in T$ let $T_x := \{y \in T : x < y\}$ and

$$succ_T(x) := \{ y \in T_x : \neg \exists z \in T(x < z < y) \},\$$

i.e., $succ_T(x)$ is the set of immediate successors of x. A **branch** of T is a maximal linearly ordered subset of T. Notice that if $\eta \subseteq T$ is a branch of T and $x \in \eta$, then $\{y \in T : y \leq x\} \subseteq \eta$. The **length** of a branch is the order type of the branch, and a branch is called **countable** if its order type is countable. An **anti-chain** in T is a set

 $A \subseteq T$ such that any two distinct elements $x, y \in A$ are incomparable, *i.e.*, neither x < y nor y < x.

Now, a tree (T, <) is called a **Suslin tree** if it has the following properties:

- (a) The height of T is ω_1 .
- (b) Every branch of T is countable.
- (c) Every anti-chain in T is countable.

The goal is now to show that there exists a Suslin line if and only if there exists a Suslin tree. The construction of a Suslin tree from a Suslin line will be straightforward, whereas the other construction will require some more work.

LEMMA 20.3. There exists a Suslin line if and only if there exists a Suslin tree.

Proof. (\Rightarrow) Let S be a Suslin line. The tree T we construct from S will consist of non-degenerate closed intervals $[a,b] \subseteq S$, where $a,b \in S$, a < b, and $[a,b] := \{c \in S : a \leq c \leq b\}$. The partial ordering "<" of T is defined by stipulating

$$I < J \iff J \subsetneq I.$$

The construction of T is by induction on ω_1 . Let $I_0 := [a_0, b_0]$ be an arbitrary nondegenerate closed interval (*i.e.*, $a_0 < b_0$). Assume that for some $\alpha \in \omega_1$, we have already constructed non-degenerated closed intervals $I_\beta = [a_\beta, b_\beta]$ for all $\beta \in \alpha$. Then, since α is countable, the set $C := \{a_\beta : \beta \in \alpha\} \cup \{b_\beta : \beta \in \alpha\}$ is a countable subset of S. Now, since S is a Suslin line, S is not separable, or in other words, no countable subset of S is dense. In particular, C is not dense, and therefore, there exists a non-degenerate interval $I_\alpha = [a_\alpha, b_\alpha]$ such that $C \cap I_\alpha = \emptyset$. The set $T = \{I_\alpha : \alpha \in \omega_1\}$ is an uncountable set. Moreover, for any $\alpha, \beta \in \omega_1$ with $\beta \in \alpha$, we have either $I_\beta < I_\alpha$ (*i.e.*, $I_\alpha \subsetneq I_\beta$), or I_β and I_α are incomparable (*i.e.*, $I_\alpha \cap I_\beta = \emptyset$). Thus, for each $\alpha \in \omega_1$, $\{I \in T : I < I_\alpha\}$ is well-ordered. Hence, by adjoining the entire set S as a root to T, (T, <) is a tree. It remains to show that Thas no uncountable anti-chains, no uncountable branches, and that $h(T) = \omega_1$.

no uncountable anti-chains: By construction of T, two elements $I, J \in T$ are incomparable if and only if I and J are disjoint intervals of S, and since S satisfies ccc, every anti-chain in T is countable.

no uncountable branches: Assume towards a contradiction that $\eta = \{[a_{\alpha}, b_{\alpha}] : \alpha \in \omega_1\} \subseteq T$ is a branch of T of length ω_1 such that for all $\alpha, \alpha' \in \omega_1, \alpha \in \alpha'$ implies $a_{\alpha} < a_{\alpha'}$, *i.e.*, for the corresponding intervals we have $I_{\alpha'} \subsetneq I_{\alpha}$. Then the intervals $(a_{\alpha}, a_{\alpha+1})$ (for $\alpha \in \omega_1$) form an uncountable collection of non-empty, pairwise disjoint open intervals in S, contradicting the fact that S satisfies *ccc*.

 $h(T) = \omega_1$: Since all branches of T are countable, the height of T is at most ω_1 . On the other hand, since every level of T is an anti-chain and therefore countable, and $|T| = \omega_1$, the height of T is at least ω_1 . Hence, $h(T) = \omega_1$.

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 (\Leftarrow) We first prove the following

CLAIM. If there is a Suslin tree, then there exists a Suslin tree \tilde{T} with the following additional properties:

- (d) For all x ∈ T̃ and all α ∈ ω₁ with o.t.(x) ∈ α, T̃_x|_α ≠ Ø, i.e., there is a y ∈ T̃|_α with x < y.
- (e) For each limit ordinal λ ∈ ω₁ and for all x, y ∈ T|_λ, if x and y have the same predecessors, then x = y, i.e.,

$$x, y \in T|_{\lambda} \land \{ z \in T : z < x \} = \{ z \in T : z < y \} \to x = y$$

(f) For all $x \in \tilde{T}$, $|succ_{\tilde{T}}(x)| = \omega$.

Proof of Claim. Let T be a Suslin tree. First, let $T' := \{x \in T : |T_x| = \omega_1\}$. Notice that since $h(T) = \omega_1$ and every anti-chain in T is countable, for every $x \in T'$ and every $\alpha \in \omega_1$ with $o.t.(x) \in \alpha$, there is a $y \in T'|_{\alpha}$ such that x < y. Next, let $T'' := \{x \in T' : |succ_{T'}(x)| \ge 2\}$, *i.e.*, $T'' \subseteq T'$ is the set of all branching points of T'. Then every limit level of T'' is infinite, and since every anti-chain in T'' is countable, every limit level of T'' is countably infinite. Moreover, since $|T'_x| = \omega_1$ (for all $x \in T'$) and every branch in T' is countable, T'' is still a Suslin tree, and in addition, it satisfies property (d).

Now, for each $x \in T''$ with o.t. $(x) = \lambda$ for some limit ordinal $\lambda \in \omega_1$ we add an extra node w_x to T'' and stipulate

$$z < w_x \iff z < x \text{ and } w_x < z \iff x \leq z$$
.

Roughly speaking, w_x is a node between $\{z \in T'' : z < x\}$ and x. Let

$$T''' := T'' \cup \{w_x : x \in T'' \land \text{o.t.}(x) = \lambda\}$$

where $\lambda \in \omega_1$ is a limit ordinal. Notice that the root of T''' is w_{x_0} , where x_0 is the root of T''. Since every level of T'' is countable and there are just ω_1 limit ordinals in ω_1 , T''' is still a Suslin tree, and in addition, it satisfies properties (d) and (e).

Finally, let \tilde{T} consist of all $x \in T'''$ at limit levels, *i.e.*, of all $x \in T'''$ such that o.t. $(x) = \lambda$ for some limit ordinal $\lambda \in \omega_1$. In fact, $\tilde{T} = T''' \setminus T''$. Notice that by the construction of T'' we get that for all $x \in \tilde{T}$, $succ_{\tilde{T}}(x)$ is infinite, and since every anti-chain in \tilde{T} is countable, we obtain $|succ_{\tilde{T}}(x)| = \omega$. Notice also that \tilde{T} still satisfies property (d). Hence, \tilde{T} is a Suslin tree which satisfies properties (d), (e), and (f).

Let now T be a Suslin tree which satisfies properties (d), (e), and (f) of the CLAIM. The Suslin line \overline{S} will be the completion of the set S consisting of all branches $\eta \subseteq T$. Since T satisfies property (f), every $x \in T$ has countably many successors. For each $x \in T$ fix a bijection $\sigma_x : succ_T(x) \to \mathbb{Q}$. With respect to σ_x we define an ordering " \prec " on the elements of S (*i.e.*, on the countable branches of T) as follows: Let $\eta = \{x_{\beta} : \beta \in \lambda\}$ and $\eta' = \{x'_{\beta} : \beta \in \lambda'\}$ be two branches of T. Then, by (e) and (f), both λ and λ' are limit ordinals. If α is the least level where η and η' differ, then, by property (e), α is a successor ordinal. So, $\alpha = \beta + 1$, $x_{\beta} = x'_{\beta}$ and $x_{\beta+1} \neq x'_{\beta+1}$; in particular we have $\sigma_{x_{\beta}} = \sigma_{x'_{\beta}}$ and $\sigma_{x_{\beta}}(x_{\beta+1}) \neq \sigma_{x_{\beta}}(x'_{\beta+1})$. Now, let

$$\eta \prec \eta' \iff \sigma_{x_{\beta}}(x_{\beta+1}) <_{\mathbb{Q}} \sigma_{x_{\beta}}(x'_{\beta+1})$$

where " $<_{\mathbb{Q}}$ " denotes the natural linear ordering on \mathbb{Q} . It remains to show that the completion \overline{S} of the linearly ordered S is a Suslin line, *i.e.*, \overline{S} is linearly ordered, unbounded, dense, complete, satisfies *ccc*, but is not separable. Recall that by LEMMA 20.2, it is enough to show that S is linearly ordered, unbounded, dense, satisfies *ccc*, but is not separable.

S is linearly ordered and unbounded: Let $\eta = \{x_{\beta} : \beta \in \lambda\}$ and $\eta' = \{x'_{\beta} : \beta \in \lambda'\}$ be two different elements of S and let $\beta + 1$ be the least level where η and η' differ. Then we have $\sigma_{x_{\beta}}(x_{\beta+1}) \neq \sigma_{x_{\beta}}(x'_{\beta+1})$ which shows that either $\eta < \eta'$ or $\eta' < \eta$. To show that S is unbounded is left as an exercise for the reader.

S is dense: Let $\eta = \{x_{\beta} : \beta \in \lambda\}$ and $\eta' = \{x'_{\beta} : \beta \in \lambda'\}$ be two elements of *S* such that $\eta < \eta'$, and let $\beta + 1$ be the least level where η and η' differ. Then there is a branch $\eta'' = \{x''_{\beta} : \beta \in \lambda''\}$ of *T* such that for all $\gamma \in \beta + 1$ we have $x''_{\gamma} = x_{\gamma}$, and

$$\sigma_{x_{\beta}}(x_{\beta+1}) <_{\mathbb{Q}} \sigma_{x_{\beta}}(x_{\beta+1}'') <_{\mathbb{Q}} \sigma_{x_{\beta}}(x_{\beta+1}'),$$

which implies that $\eta < \eta'' < \eta'$.

S satisfies ccc: For every $x \in T$ let $I_x := \{\eta \in S : x \in \eta\}$. Now, for every non-empty open interval $(\eta, \eta') \subseteq S$ we find an $x \in T$ such that $I_x \subseteq (\eta, \eta')$. To see this, let $\beta + 1$ be the least level where η and η' differ and choose $x \in T$ such that $\sigma_{x_\beta}(x_{\beta+1}) <_{\mathbb{Q}} \sigma_{x_\beta}(x) <_{\mathbb{Q}} \sigma_{x_\beta}(x'_{\beta+1})$. Notice that if I_x and I_y are disjoint, then x and y are incomparable in T. So, to any collection \mathscr{I} of pairwise disjoint, non-empty open intervals of S we obtain an anti-chain in T of the same size as \mathscr{I} , and since every anti-chain in T is countable, S satisfies ccc.

S is not separable: Let \mathscr{C} be an arbitrary countable set of branches of *T* and let $\alpha \in \omega_1$ be such that α is bigger than the order type of any $\eta \in \mathscr{C}$. Notice that since all branches of *T* are countable and ω_1 is regular, such an ordinal α exists. Now, choose an element $x \in T|_{\alpha}$ and two distinct branches $\eta, \eta' \in S$ which both contain x. Then $(\eta, \eta') \cap \mathscr{C} = \emptyset$, which shows that \mathscr{C} is not dense in *S*. Hence, since \mathscr{C} was arbitrary, *S* is not separable.

There May Be No Suslin Line

In this section we show that $MA(\omega_1)$ implies that there exists no Suslin line. In particular, if we assume $MA(\omega_1)$, then every linearly ordered set which is dense, unbounded, complete, and satisfies *ccc*, is isomorphic to the real line.

PROPOSITION 20.4. $MA(\omega_1)$ implies that there exists no Suslin line.

Proof. Assume towards a contradiction that there exists a Suslin line. Then, by the proof of LEMMA 20.3, there exists a Suslin tree (T, <) with the additional properties (d), (e), and (f). With respect to (T, <) we define the forcing notion $\mathbb{P}_T := (T, \leq)$. Since T is a Suslin tree, every anti-chain in T is countable, hence, \mathbb{P}_T satisfies *ccc*. For each $\beta \in \omega_1$ let

$$D_{\beta} := \bigcup \left\{ T|_{\alpha} : \beta \in \alpha \in \omega_1 \right\}.$$

Then, by property (d), for each $\beta \in \omega_1$, D_α is an open dense subset of \mathbb{P}_T . Finally, let $\mathscr{D} := \{D_\beta : \beta \in \omega_1\}$ and let $G \subseteq T$ be a \mathscr{D} -generic filter on T. Then G is a branch of T of length ω_1 , which contradicts the fact that all branches of the Suslin tree T are countable.

As an immediate consequence we get

COROLLARY 20.5. It is consistent with ZFC that there exists no Suslin line.

There May Be a Suslin Line

In this section we show that it is consistent with ZFC that there exists a Suslin line. In particular, it is consistent with ZFC that there exists a linearly ordered set which is dense, unbounded, complete, and satisfies *ccc*, but which is not isomorphic to the real line.

We first construct a particular tree T_{ω_1} and show that with a Cohen real we can transform the tree T_{ω_1} to a Suslin tree. We start with the following preliminary result.

LEMMA 20.6. There exists a sequence $\langle e_{\alpha} : \alpha \in \omega_1 \rangle$ such that:

- (a) for each $\alpha \in \omega_1, e_\alpha : \alpha \hookrightarrow \omega$ is an injective function from α into ω ;
- (b) for all $\alpha \in \beta \in \omega_1$, the set $\{\xi \in \alpha : e_\alpha(\xi) \neq e_\beta(\xi)\}$ is finite;
- (c) for each $\alpha \in \omega$, the set $\omega \setminus e_{\alpha}[\alpha]$ is infinite.

Proof. The proof is by induction on ω_1 . For $\alpha = 0$, let e_0 be the empty function. If $\alpha = \gamma + 1$, then let $e_\alpha := e_\gamma \cup \{\langle \gamma, n \rangle\}$, where $n \in \omega \setminus e_\alpha[\alpha]$. Notice that by property (c), such an *n* exists and that e_α has the required properties.

Suppose now that $\alpha \in \omega_1$ is a limit ordinal and that the sequence $\langle e_{\gamma} : \gamma \in \alpha \rangle$ is already constructed. Let $\langle \gamma_k : k \in \omega \rangle$ be an increasing sequence such that $\bigcup \{\gamma_k : k \in \omega\} = \alpha$. Notice that since $\alpha \in \omega_1$, α is countable, in particular, $cf(\alpha) = \omega$. By (c), for each $k \in \omega$ we have $|\omega \setminus e_{\gamma_k}[\gamma_k]| = \omega$, and by (b), for $k, l \in \omega$ with k < lwe have $e_{\gamma_k}[\gamma_k] \subseteq^* e_{\gamma_l}[\gamma_l]$. So, there exists an infinite set $A \in [\omega]^{\omega}$ which is almost contained in $\omega \setminus e_{\gamma_k}[\gamma_k]$ for each $k \in \omega$. In other words, for all $k \in \omega$, $A \cap e_{\gamma_k}[\gamma_k]$ is finite. Split A into two disjoint infinite sets A_1 and A_2 , *i.e.*, $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, and $A_1, A_2 \in [\omega]^{\omega}$. Without loss of generality assume $\gamma_0 = 0$ and for each $k \in \omega$ let $[\gamma_k, \gamma_{k+1}) := \{\xi : \gamma_k = \xi \lor \gamma_k \in \xi \gamma_{k+1}\}$, and for the sake of simplicity let $I_k := [\gamma_k, \gamma_{k+1})$.

CLAIM. For each $k \in \omega$ there is an injective function $f_k : I_k \hookrightarrow \omega$ such that:

- $f_k[I_k] \cap A_1 = \emptyset;$
- $\{\xi \in I_k : f_k(\xi) \neq e_{\gamma_{k+1}}(\xi)\}$ is finite;
- $\bigcup_{k' < k} f_{k'} : \gamma_{k+1} \hookrightarrow \omega$ is injective.

Proof of Claim. The construction is by induction on ω . First we define $f_0 := e_{\gamma_1}$. Assume now that $f_{k'}$ is already constructed for all k' < k. For $\xi \in I_k$ define $f_k(\xi) := e_{\gamma_{k+1}}(\xi)$ if $e_{\gamma_{k+1}}(\xi) \notin (A_1 \cup \bigcup_{k' < k} f_{k'}[I_{k'}])$, otherwise, let $f_k(\xi) := n$ where $n \in A_2 \setminus (\bigcup_{k' < k} f_{k'}[I_{k'}] \cup \bigcup_{\xi' \in I_k \cap \xi} f_k(\xi'))$. Then for all $k \in \omega$, f_k has the required properties (the details are left as an exercise for the reader).

Finally, let

$$e_{\alpha} := \bigcup_{k \in \omega} f_k \,.$$

Then, by the properties of f_k (for $k \in \omega$), $e_\alpha : \alpha \hookrightarrow \omega$ is an injection, for all $\gamma \in \alpha$, $\{\xi \in \alpha : e_\gamma(\xi) \neq e_\alpha(\xi)\}$ is finite, and since $e_\alpha[\alpha] \cap A_1 = \emptyset, \omega \setminus e_\alpha[\alpha]$ is infinite. \dashv

Let $\langle e_{\alpha} : \alpha \in \omega_1 \rangle$ be the sequence of LEMMA 20.6 and define

$$T_{\omega_1} := \Big\{ \eta : \exists \alpha \in \omega_1 \big(\eta : \alpha \hookrightarrow \omega \land \big| \big\{ \xi \in \alpha : \eta(\xi) \neq e_\alpha(\xi) \big\} \big| < \omega \big) \Big\}.$$

Furthermore, for distinct $\eta, \eta' \in T_{\omega_1}$ let

$$\eta < \eta' \iff \operatorname{dom}(\eta) \subsetneq \operatorname{dom}(\eta') \land \eta'|_{\operatorname{dom}(\eta)} = \eta$$

Then $(T_{\omega_1}, <)$ is a tree of height ω_1 , where all branches and all levels of T_{ω_1} are countable. We leave it as an exercise for the reader to show that T_{ω_1} is a tree. To see that all branches of T_{ω_1} are countable, notice that a branch of length ω_1 would yield

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an injection from ω_1 into ω , which is obviously impossible. Furthermore, to see that every level of T_{ω_1} is countable, notice that for each $\alpha \in \omega_1$, the α^{th} level

$$T_{\omega_1}|_{\alpha} = \left\{ \eta : \operatorname{dom}(\eta) = \alpha \land \left| \left\{ \xi \in \alpha : \eta(\xi) \neq e_{\alpha}(\xi) \right\} \right| < \omega \right) \right\}$$

is countable. A tree with these properties is called **Aronszajn tree**. In particular, the existence of Aronszajn trees is provable in ZFC, whereas the existence of Suslin trees is not provable in ZFC, as we have seen above. However, there are models of ZFC in which Suslin trees exist, as we will see below; but first we introduce the following notion:

For a real $x \in {}^{\omega}\omega$ we define

$$x \circ T_{\omega_1} := \{x \circ \eta : \eta \in T_{\omega_1}\}$$

where $x \circ \eta(\xi) := x(\eta(\xi))$. Notice that for $\eta \in T_{\omega_1}|_{\alpha}$, $x \circ \eta : \alpha \to \omega$ is still a function from α to ω , but not necessarily an injection. Furthermore, notice that for different $\eta, \eta' \in T_{\omega_1}|_{\alpha}$, we may have $x \circ \eta = x \circ \eta'$. However, $(x \circ T_{\omega_1}, <)$ is still an Aronszajn tree.

Now we are ready to prove the following result.

PROPOSITION 20.7. Let V be a model of ZFC and let $c \in {}^{\omega}\omega$ be a Cohen real over V. Then there exists a Suslin tree in V[c], in fact

$$\mathbf{V}[c] \vDash c \circ T_{\omega_1}$$
 is a Suslin tree.

Proof. We consider Cohen forcing $\mathbb{C} = (C, \leq)$ for $C = \bigcup_{n \in \omega} {}^n \omega$. Since the height of $c \circ T_{\omega_1}$ is ω_1 , in order to prove that $c \circ T_{\omega_1}$ is a Suslin tree, we just have to show that $c \circ T_{\omega_1}$ has neither uncountable anti-chains nor uncountable branches.

First we prove a preliminary result: Let $\{\eta_{\beta} : \beta \in \omega_1\} = T_{\omega_1}$ be an enumeration of T_{ω_1} such that for all $\beta, \beta' \in \omega_1$ with $\beta \in \beta'$ we have $\operatorname{dom}(\eta_{\beta}) \subseteq \operatorname{dom}(\eta_{\beta'})$. Furthermore, let

$$\mathscr{A} \subseteq \left\{ \langle \eta_{\beta}, p \rangle : \beta \in \omega_1 \land p \in C \right\}$$

be a \mathbb{C} -name for an uncountable subset of $T_{\omega_1}[c]$, *i.e.*,

 $\mathbf{V}[c] \models \mathscr{A}[c]$ is an uncountable subset of $T_{\omega_1}[c]$.

Since the set C of \mathbb{C} -conditions is countable, there is a \mathbb{C} -condition p_0 such that

$$U := \left\{ \beta \in \omega_1 : \langle \eta_\beta, p_0 \rangle \in \mathscr{A} \right\}$$

is uncountable. Let $n_0 := \operatorname{dom}(p_0)$ and for each $\beta \in U$ let

$$E_{\beta} := \{ \gamma \in \operatorname{dom}(\eta_{\beta}) : \eta_{\beta}(\gamma) \in n_0 \}$$

CLAIM. There is an uncountable set $W \subseteq U$ and a finite set $\Delta \subseteq \omega_1$, such that for all $\beta, \beta' \in W$ with $\beta \in \beta'$ we have

$$E_{\beta} \cap E_{\beta'} = \Delta, \ \eta_{\beta}|_{\Delta} = \eta_{\beta'}|_{\Delta}, \ \text{and} \ \forall \xi \in \operatorname{dom}(\eta_{\beta}) \left(\eta_{\beta'}(\xi) \in n_0 \to \xi \in \Delta\right).$$

Notice that the last condition is equivalent to $(E_{\beta'} \setminus \Delta) \cap \operatorname{dom}(\eta_{\beta}) = \emptyset$.

Proof of Claim. By applying the Δ -SYSTEM LEMMA 14.3 to the family $\{E_{\beta} : \beta \in U\}$ we find an uncountable family $U_{\Delta} \subseteq U$ and a finite set $\Delta \subseteq \omega_1$, such that for any distinct $\beta, \beta' \in U_{\Delta}$ we have $E_{\beta} \cap E_{\beta'} = \Delta$. Now, since n_0 is finite, uncountably many functions η_{β} , where $\beta \in U_{\Delta}$, must agree on Δ . Hence, there is an uncountable set $U' \subseteq U_{\Delta}$ such that for any $\beta, \beta' \in U'$ we have $\eta_{\beta}|_{\Delta} = \eta_{\beta'}|_{\Delta}$.

Furthermore, we can construct an increasing sequence $\langle \alpha_{\delta} : \delta \in \omega_1 \rangle$ of elements of U' such that for all $\gamma, \delta \in \omega_1$ with $\gamma \in \delta$ (*i.e.*, $\alpha_{\gamma} \in \alpha_{\delta}$) we have

$$\forall \xi \in \operatorname{dom}(\eta_{\alpha_{\gamma}}) \left(\eta_{\alpha_{\delta}}(\xi) \in n_0 \to \xi \in \Delta \right).$$

The construction is by induction on ω_1 . Let $\alpha_0 = \bigcap U'$ and let $U_0 = U'$. Now, assume that for all $\gamma \in \delta$, where $\delta \in \omega_1$, we have already constructed ordinals $\alpha_\gamma \in \omega_1$ and uncountable sets $U_\gamma \subseteq U'$. If $\delta = \nu + 1$, then let

$$U_{\delta} = \left\{ \beta \in \left(U_{\nu} \setminus \{ \alpha_{\nu} \} \right) : \forall \xi \in \operatorname{dom}(\alpha_{\nu}) \left(\eta_{\beta}(\xi) \in n_{0} \to \xi \in \Delta \right) \right\}$$

and let

$$\alpha_{\delta} = \bigcap U_{\delta} \, .$$

To see that U_{δ} is uncountable, consider the set $(U_{\nu} \setminus \{\alpha_{\nu}\}) \setminus U_{\delta}$, which consists of all $\beta \in (U_{\nu} \setminus \{\alpha_{\nu}\})$ such that for some $\xi \in \text{dom}(\alpha_{\nu})$, $\eta_{\beta}(\xi) \in n_0 \setminus \Delta$. Now, since $(E_{\beta} \cap E_{\beta'}) \setminus \Delta = \emptyset$ (for any distinct $\beta, \beta' \in U'$), and since $\text{dom}(\alpha_{\nu})$ is countable, the set $(U_{\nu} \setminus \{\alpha_{\nu}\}) \setminus U_{\delta}$ is countable. Hence, since U_{ν} is uncountable, this shows that U_{δ} is an uncountable subset of U'.

In the case when δ is a limit ordinal, let $U_{\delta} = \bigcap_{\gamma \in \delta} U_{\eta}$ and let $\alpha_{\delta} = \bigcap U_{\delta}$. Since each set $U_{\nu} \setminus U_{\nu+1}$ (for $\nu \in \delta$) is countable and since δ is a countable ordinal, the set U_{δ} is an uncountable subset of U'.

Finally, let $W = \{\alpha_{\delta} : \delta \in \omega_1\}$. Then W is an uncountable subset of U' with the required properties.

Now, we show that $c \circ T_{\omega_1}$ has no uncountable anti-chains. For this, assume towards a contradiction that

 $\mathbf{V}[c] \vDash "c \circ T_{\omega_1}$ contains an uncountable anti-chain"

and let

$$\mathscr{A} \subseteq \{ \langle \eta_{\beta}, p \rangle : \beta \in \omega_1 \land p \in C \}$$

There May Be a Suslin Line

be a \mathbb{C} -name for an uncountable subset of $T_{\varphi_1}[c]$ such that in $\mathbf{V}[c]$, the set

$$\left\{c \circ \eta_{\beta} : \exists p \in C\left(\langle \eta_{\beta}, p \rangle \in \mathscr{A}\right)\right\}$$

is an uncountable anti-chain in $c \circ T_{\omega_1}$. With respect to \mathscr{A} , let p_0, n_0, W , and Δ be as in the claim above, *i.e.*, p_0 is a \mathbb{C} -condition, $n_0 = \operatorname{dom}(p_0)$, and $W, \Delta \subseteq \omega_1$ where W is uncountable and Δ is finite. Notice that for

$$\mathscr{A}' := \left\{ \langle \eta_{\beta}, p_0 \rangle : \beta \in W \right\}$$

we have $p_0 \Vdash_{\mathbb{C}} \mathscr{A}' \subseteq \mathscr{A}$, moreover,

$$p_0 \Vdash_{\mathbb{C}} ``\{c \circ \eta_\beta : \langle \eta_\beta, p_0 \rangle \in \mathscr{A}'\}$$
 is an uncountable anti-chain in $c \circ T_{\omega_1}$.

The goal is now to construct a \mathbb{C} -condition $q \ge p_0$ such that for distinct $\beta_0, \beta_1 \in W$ with $\beta_0 \in \beta_1$ we have

$$q \vdash_{\mathbb{C}} c \circ \eta_{\beta_0} \leq c \circ \eta_{\beta_1}.$$

For this, fix two ordinals $\beta_0, \beta_1 \in W$ with $\beta_0 \in \beta_1$. By the properties of T_{ω_1} and since $\beta_0, \beta_1 \in W$, we have:

- (a) $\left|\left\{\xi \in \operatorname{dom}(\eta_{\beta_0}) : \eta_{\beta_1}(\xi) \neq \eta_{\beta_0}(\xi)\right\}\right| < \omega$
- (b) $\eta_{\beta_0}|_{\Delta} = \eta_{\beta_1}|_{\Delta}$
- (c) $\forall \xi \in \operatorname{dom}(\eta_{\beta_0}) \left(\eta_{\beta_1}(\xi) \in n_0 \to \xi \in \Delta \right)$

Now, let

$$\Xi := \left\{ \xi \in \operatorname{dom}(\eta_{\beta_0}) : \eta_{\beta_0}(\xi) \neq \eta_{\beta_1}(\xi) \right\}$$

and define

$$F := \left\{ \left\langle \eta_{\beta_0}(\xi), \eta_{\beta_1}(\xi) \right\rangle : \xi \in \Xi \right\}.$$

Notice that by (a), Ξ is a finite subset of ω_1 , and consequently, F is a finite subset of $\omega \times \omega$. In particular, $F \subseteq m \times m$ for some $m \in \omega$. Since both functions, η_{β_0} and η_{β_1} , are injective, we can identify F with a bijection $f : A \to B$ where

$$A := \left\{ \eta_{\beta_0}(\xi) : \xi \in \Xi \right\} \quad \text{and} \quad B := \left\{ \eta_{\beta_1}(\xi) : \xi \in \Xi \right\}.$$

Furthermore, $B \cap n_0 = \emptyset$. To see this, notice that by (c),

$$D := \left\{ \xi \in \operatorname{dom}(\eta_{\beta_0}) : \eta_{\beta_1}(\xi) \in n_0 \right\} \subseteq \Delta,$$

and by (b) we have $\eta_{\beta_0}|_{\Delta} = \eta_{\beta_1}|_{\Delta}$, hence, $\eta_{\beta_0}(\xi) = \eta_{\beta_1}(\xi)$ for every $\xi \in D$. As a consequence we get that for every $k \in A$ we have $f(k) \ge n_0$. Now, for every $k \in A \cap n_0$ let $O_k := \{k_i : i \in \omega\}$ such that $k_0 := k$ and

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$$f(k_{i+1}) := \begin{cases} f(k_i) & \text{if } k_i \in A, \\ k_i & \text{otherwise.} \end{cases}$$

Since f is a bijection and $f(k) \ge n_0$ for all $k \in A \cap n_0$, for any distinct $k, k' \in A \cap n_0$ we have $O_k \cap O_{k'} = \emptyset$. Finally, for every $k \in A \cap n_0$ let $j_k := p_0(k)$, and define $q \in {}^m \omega$ by stipulating $q|_{n_0} := p_0$, and in general, for $l \in m$, we define

$$q(l) := \begin{cases} j_k & \text{if } l \in O_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $q \ge p_0$ and by construction we have

$$q \vdash_{\mathbb{C}} c \circ \eta_{\beta_0} \leq c \circ \eta_{\beta_1}.$$

It remains to show that $c \circ T_{\omega_1}$ has no uncountable branches. For this, assume towards a contradiction that

$$\mathbf{V}[c] \vDash "c \circ T_{\omega_1}$$
 has an uncountable branch"

and let

$$\mathscr{B} \subseteq \{ \langle \eta_{\beta}, p \rangle : \beta \in \omega_1 \land p \in C \}$$

be a \mathbb{C} -name for an uncountable subset of $T_{\varphi_1}[c]$ such that in $\mathbf{V}[c]$, the set

$$\left\{c\circ\eta_{\beta}:\exists p\in C\big(\langle \eta_{\beta},p\rangle\in\mathscr{B}\big)\right\}$$

is an uncountable branch of $c \circ T_{\omega_1}$. With respect to the \mathbb{C} -name \mathscr{B} , let $p_0 \in {}^{n_0}\omega$, W, B, Δ, Ξ , be as in the claim above. Notice that for

$$\mathscr{B}' := \left\{ \langle \eta_{\beta}, p_0 \rangle : \beta \in W \right\}$$

we have $p_0 \Vdash_{\mathbb{C}} \mathscr{B}' \subseteq \mathscr{B}$, in particular,

$$p_0 \Vdash_{\mathbb{C}}$$
 "the elements of $\{c \circ \eta_\beta : \langle \eta_\beta, p_0 \rangle \in \mathscr{B}'\}$ are pairwise compatible".

The goal is now to construct a $\mathbb{C}\text{-condition}\ q\geq p_0$ such that for distinct $\beta_0,\beta_1\in W$ we obtain

 $q \vdash_{\mathbb{C}} "c \circ \eta_{\beta_0}$ and $c \circ \eta_{\beta_1}$ are incomparable".

Then, by similar arguments as above, one can show that there are $\beta_0, \beta_1 \in W$ with $\beta_0 \in \beta_1$ such that for some $\xi_0 \in \operatorname{dom}(\eta_{\beta_0}), \eta_{\beta_0}(\xi_0) \neq \eta_{\beta_1}(\xi_0)$ and $\eta_{\beta_1}(\xi_0) \notin n_0$. To see this, assume first that for all $\beta, \beta' \in W$ with $\beta \in \beta'$ we have $\eta_\beta < \eta_{\beta'}$. Then, since W is uncountable, this yields an injection from an uncountable subset of ω_1 into ω , which is obviously a contradiction. Now, take any two $\beta_0, \beta_1 \in W$ with $\beta_0 \in \beta_1$ such that $\eta_{\beta_0} \neq \eta_{\beta_1}$. Then there is a $\xi_0 \in \operatorname{dom}(\eta_{\beta_0})$ such that Notes

 $\eta_{\beta_0}(\xi_0) \neq \eta_{\beta_1}(\xi_0)$, in particular, $\xi_0 \in \Xi$. Moreover, since $B \cap n_0 = \emptyset$, we have $\eta_{\beta_1}(\xi_0) \notin n_0$.

Finally, let $q \ge p_0$ be a \mathbb{C} -condition such that

dom $(q) = dom(p_0) \cup \{\eta_{\beta_0}(\xi_0), \eta_{\beta_1}(\xi_0)\}$ and $q(\eta_{\beta_0}(\xi_0)) \neq q(\eta_{\beta_1}(\xi_0))$.

Notice that since $\eta_{\beta_1}(\xi_0) \notin n_0$, there is no restriction on the value of $q(\eta_{\beta_1}(\xi_0))$. Clearly, $q \Vdash_{\mathbb{C}} "c \circ \eta_{\beta_0}$ and $c \circ \eta_{\beta_1}$ are incomparable", which implies that $c \circ \eta_{\beta_0}$ and $c \circ \eta_{\beta_1}$ cannot belong to the same branch.

By combining LEMMA 20.3 and PROPOSITION 20.7 we get the following

COROLLARY 20.8. It is consistent with ZFC that there exists a Suslin line.

As a further consequence of the preceding results we get

COROLLARY 20.9. If $\mathfrak{c} = \mathfrak{p} > \omega_1$, then MA(ω_1) fails after adding a Cohen real.

Proof. Let V be a model of $\mathfrak{c} = \mathfrak{p} > \omega_1$ in which MA(ω_1) holds in V. So, by PROPOSITION 20.4, there is no Suslin line in V. Now, let c be a Cohen real over V. Then, by PROPOSITION 20.7, there exists a Suslin line in $\mathbf{V}[c]$. So, MA(ω_1) fails in $\mathbf{V}[c]$.

NOTES

Suslin's Problem. Suslin's Problem was posed by Suslin and published posthumously in [10]. Translated into English it states: *Is a (linearly) ordered set without jumps or gaps, such that every set of its intervals (containing more than one element) not overlapping each other is at most denumerable, necessarily an (ordinary) linear continuum?*

The Existence of Suslin Lines and Suslin Trees. LEMMA 20.3, which gives the relation between Suslin lines and Suslin trees, is taken from Jech [3, Section 22]. The first models in which a Suslin line exists were discovered by Jech [2] and Tennenbaum [11]. The construction for adding a Suslin tree by Cohen forcing is due to Shelah [9, §1]. However, the approach taken here is due to Todorčević [12, p. 292 f.] (see also Bartoszyński and Judah [1, Section 3.3.A]). Finally, we would like to mention that one of the main reasons to formulate Martin's Axiom was that $MA(\omega_1)$ implies that there is no Suslin tree (see Martin and Solovay [8]).

RELATED RESULTS

- 110. There is a Suslin Line in L. Jensen showed that a certain combinatorial principle, denoted ◊, implies the existence of a Suslin line (see Jensen [5, 6]). Now, since ◊ holds in Gödel's constructible universe L, there is a Suslin line in L (see also Jech [3, Section 22] and [4, Chapter 27]).
- 111. $S \times S$ does not satisfy ccc. Let S be a Suslin line. Then S satisfies ccc but is not separable. By the latter property one can construct an uncountable family of pairwise disjoint non-empty open subsets of $S \times S$. So, $S \times S$ does not satisfy ccc (see Kunen [7, Chapter II, §4]).

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