4

Semantics for GL and other modal logics

The semantical treatment of modal logic that we now present is due to Kripke and was inspired by a well-known fantasy often ascribed to Leibniz, according to which we inhabit a place called *the actual world*, which is one of a number of *possible worlds*. (It is a further part of the fantasy, which we can ignore, that because of certain of its excellences God selected the possible world that we inhabit to be the one that he would make actual. Lucky us.) Each of our statements is true or false in – we shall say *at* – various possible worlds. A statement is true at a world if it correctly describes that world and false if it does not. We sometimes call a particular statement true or false, *tout court*, but when we do, we are to be understood as speaking about the actual world and saying that the statement is true or false *at it*. Some of the statements we make are true at all possible worlds, including of course the actual world; these are the so-called *necessary* statements. A statement to the effect that another is necessary will thus be true if the other statement is true at all possible worlds. It follows that if a statement is necessary, then it is true. Some statements are true at at least one possible world; these are the *possible* statements. Since what is true at the actual world is true at at least one possible world, whatever is true is possible. A statement is necessary if and only if its negation is not possible, for the negation of a statement will be true at precisely those worlds at which the statement is false. And if a conditional and its antecedent are both necessary, then the consequent of the conditional is necessary too.

There is a question, raised by Kripke, to which this description of Leibniz's system of possible worlds does not supply the answer. We are said to inhabit the actual world. Are the other possible worlds of whose existence we have been apprised absolutely all of the other worlds that there really are, or are they only those that are possible *relative to* the actual world? The description leaves it open whether or not, if we had inhabited some other world than the actual world, there might have been worlds other than those we

now acknowledge that were possible *relative to* that other possible world; in brief, our description does not answer the question whether or not exactly the same worlds are possible relative to each possible world as are possible relative to the actual world.

A possible world is called *accessible from* another if it is possible relative to that other. If we do not assume that the worlds accessible from the actual world are precisely the worlds accessible from each world – even though it may appear self-evident that they are – then questions arise about the nature of the accessibility relation. For example, is the relation transitive? If so, then all worlds accessible from worlds that are accessible from the actual world will themselves be worlds that are accessible from the actual world. It follows that if a statement A is necessary, then A will be true at all worlds x accessible from the actual world; and therefore A will be true at every world y that is accessible from some world x accessible from the actual world (for all such worlds y are accessible from the actual world if accessibility is transitive); and therefore the statement that A is necessary will be true at every world x accessible from the actual world; and therefore the statement that A is necessary will itself be necessary. Thus, on the assumption that the accessibility relation is transitive, if a statement A is necessary, then the statement that A is necessary will also be necessary. In like manner other determinations of the character of the accessibility relation can guarantee the correctness of other modal principles. (The system of semantics for GL that we shall give in this chapter will differ from Leibniz's system in that no world will ever be accessible from itself!)

Set-theoretical analogues of these metaphysical notions were defined by Kripke in providing what has become the standard sort of model-theoretical semantics for the most common systems of propositional modal logic.[1]

Definitions, most of them familiar:

R is a relation *on* W if for all w, x, if wRx, then $w, x \in W$.
A relation R on W is *reflexive on* W if for all w in W, wRw.
R is *irreflexive* if for no w, wRw.
R is *antisymmetric* if for all w, x, if wRx and xRw, then $w = x$.
R is *transitive* if for all w, x, y, if wRx and xRy, then wRy.
R is *symmetric* if for all w, x, if wRx, then xRw.
R is *euclidean* if for all w, x, y, if wRx and wRy, then xRy. (Thus also, if wRx and wRy, then yRx.)

R is an *equivalence relation on* W if R is reflexive on W, symmetric, and transitive.

A symmetric relation is transitive if and only if it is euclidean, and a reflexive relation on W that is euclidean is symmetric. Thus a relation is an equivalence relation on W if and only if it is euclidean and reflexive on W.

A *frame* is an ordered pair $\langle W, R\rangle$ consisting of a nonempty set W and a binary relation R on W. $\langle W, R\rangle$ is finite iff W is. The elements of W are called "possible worlds" or sometimes just "worlds". W is called the *domain* of $\langle W, R\rangle$ and R the *accessibility relation.* (It is occasionally useful to read "R" as "sees". Thus a world *sees* those worlds accessible from it.)

A frame $\langle W, R\rangle$ is said to have some property of binary relations, e.g., transitivity, iff R has that property. ($\langle W, R\rangle$ is called reflexive if R is reflexive on W.)

A *valuation*[2] V on a set W is a relation between members of W and sentence letters, i.e., a set of ordered pairs of members of W and sentence letters. (It is sometimes convenient to read "V" as "verifies".)

A *model* is a triple $\langle W, R, V\rangle$, where $\langle W, R\rangle$ is a frame and V is a valuation on W. A model $\langle W, R, V\rangle$ is said to be *based on* the frame $\langle W, R\rangle$.

A model is finite, reflexive, transitive, etc., iff the frame on which it is based is finite, reflexive, transitive, etc.

For each modal sentence A, each model $M, = \langle W, R, V\rangle$, and each world w in W, we define the relation

$$M, w \vDash A$$

as follows:

if $A = p$ (a sentence letter), then $M, w \vDash A$ iff wVp;
if $A = \bot$, then not: $M, w \vDash A$;
if $A = (B \to C)$, then $M, w \vDash A$ iff either $\neg M, w \vDash B$ or $M, w \vDash C$; and
if $A = \Box B$, then $M, w \vDash A$ iff for all x such that wRx, $M, x \vDash B$.

Some evident consequences of this definition: if $A = \neg B$, then $M, w \vDash A$ iff it is not the case that $M, w \vDash B$; if $A = (B \wedge C)$, then $M, w \vDash$ A iff $M, w \vDash B$ and $M, w \vDash C$; if $A = (B \vee C)$, then $M, w \vDash A$ iff $M, w \vDash B$ or $M, w \vDash C$, etc. Moreover, if $A = \Diamond B$, then $M, w \vDash A$ iff for some x such that wRx, $M, x \vDash B$.

It is worth mentioning that $M, w \vDash \boxdot A$ iff for all x such that either wRx or $w = x$, $M, x \vDash A$.

A sentence A is said to be *true* at a world w *in* a model M iff $M, w \vDash A$. A sentence A is said to be *valid in a model* $M, = \langle W, R, V\rangle$, iff for all w in W, A is true at w in M. And A is said to be *valid in a frame* $\langle W, R\rangle$ iff A is valid in all models based on $\langle W, R\rangle$.

Similarly, a sentence is *satisfiable in a model* $M, = \langle W, R, V\rangle$, iff for some w in W, A is true at w in M. And A is said to be *satisfiable in a frame* $\langle W, R\rangle$ iff A is satisfiable in some model based on $\langle W, R\rangle$.

Important notational conventions. Unless there is some clear indication to the contrary, when 'M' is used to denote a model, it will denote the model also denoted: $\langle W, R, V\rangle$. Moreover, where context makes it clear which model is in question, we shall feel free to write, e.g., '$w \vDash A$', instead of '$M, w \vDash A$' or '$\langle W, R, P\rangle, w \vDash A$'. When we do so, '$w$' is of course understood to denote a member of the set W of worlds of the model M in question.

Suppose that M is a model and $w \in W$. Then every tautology is true at w. And if A and $(A \to B)$ are true at w, so is B. Moreover, every distribution axiom $\Box(A \to B) \to (\Box A \to \Box B)$ is true at w as well: for suppose that $w \vDash \Box(A \to B)$ and $w \vDash \Box A$. Then if wRx, both $x \vDash (A \to B)$ and $x \vDash A$, whence $x \vDash B$. Thus if wRx, $x \vDash B$; $w \vDash \Box B$. So if $w \vDash \Box(A \to B)$ and $w \vDash \Box A$, then $w \vDash \Box B$; it follows that $w \vDash \Box(A \to B) \to (\Box A \to \Box B)$.

Thus all tautologies and all distribution axioms are true at every world in every model and the set of sentences true at a world in a model is closed under modus ponens.

Furthermore, if A is valid in M, so is $\Box A$: for assume A valid in M, i.e., true at every world in M. Let w be an arbitrary member of W. Then for all x such that wRx, $x \vDash A$; therefore, $w \vDash \Box A$. Since w was arbitrary, $\Box A$ is valid in M.

Thus all tautologies and all distribution axioms are valid in every model and the set of sentences valid in a model is closed under both modus ponens and necessitation.

Thus all theorems of K are valid in all models and hence in all frames.

It is not in general true that if a sentence is valid in a model, then every substitution instance is valid in that model: let $\langle W, R, V\rangle$ be a model in which wVp and not: wVq for every w in W. Then p is valid in $\langle W, R, V\rangle$, but q, which is a substitution instance of p, is not. What *is* true is that if a sentence is valid in a frame, then every substitution instance of it is also true in that frame.

Theorem 1. *Suppose F is valid in the frame $\langle W, R\rangle$. Then every substitution instance $F_p(A)$ of F is also valid in $\langle W, R\rangle$.*

Proof. Let V be an arbitrary valuation on W. Let $M = \langle W, R, V\rangle$. Define the valuation V^* on W by: wV^*p iff $M, w \models A$, and wV^*q iff wVq for every sentence letter q other than p. Let $M^* = \langle W, R, V^*\rangle$. It follows by an easy induction on the complexity of subsentences G of F that $M^*, w \models G$ iff $M, w \models G_p(A)$. So $M^*, w \models F$ iff $M, w \models F_p(A)$. Since F is valid in $\langle W, R\rangle$, $M^*, w \models F$. Thus $M, w \models F_p(A)$. Since w and V were arbitrary, $F_p(A)$ is valid in $\langle W, R\rangle$. ⊣

Let R be a binary relation on a set W. For each natural number i, define R^i as follows: R^0 is the identity relation on W; $R^{i+1} = \{\langle w, y\rangle: \exists x(wR^ix \wedge xRy)\}$. Thus $R^1 = R$ and wR^ny iff $\exists x_0 \cdots \exists x_n(w = x_0R \ldots Rx_n = y)$.

Let A be a modal sentence. Define $\Box^iA$ as follows: $\Box^0A = A$; $\Box^{i+1}A = \Box\Box^iA$. Define $\Diamond^iA$ similarly.

Theorem 2. $w \models \Box^iA$ *iff for all y, if wR^iy, $y \models A$;* $w \models \Diamond^iA$ *iff for some y, wR^iy and $y \models A$.*

Proof. Induction on i. The basis step is trivial. As for the induction step, $w \models \Diamond^{i+1}A$ iff $w \models \Diamond\Diamond^iA$; iff for some x, wRx and $x \models \Diamond^iA$; iff by the induction hypothesis, for some x, wRx and for some y, xR^iy and $y \models A$; iff for some y, $wR^{i+1}y$ and $y \models A$. The result for $\Box$ holds by de Morgan. ⊣

Here is a theorem about what the truth-value of a sentence at a world depends upon. Let A be a modal sentence, M a model, and $w \in W$.

Define $d(A)$ as follows: $d(p) = d(\bot) = 0$; $d(A \to B) = \max(d(A), d(B))$; and $d(\Box A) = d(A) + 1$. Thus $d(A)$ is the maximum number of nested occurrences of $\Box$ in A. $d(A)$ is called the *(modal) degree* of A.

Theorem 3 (the "continuity" theorem). *Let M and $N = \langle X, S, U\rangle$ be models, $w \in W$. Let P be a set of sentence letters. Suppose that $d(A) = n$, all sentence letters that occur in A are in P, $X \supseteq \{x: \exists i \leqslant n\ wR^ix\}$, $S = \{\langle x, y\rangle: x, y \in X \wedge xRy\}$,*

and xUp iff xVp for all $x \in X$ and all sentence letters in P. Then $M, w \models A$ iff $N, w \models A$.

Proof. We show that for all subsentences B of A, if for some i, wR^ix and $d(B) + i \leqslant n$ (so that $i \leqslant n$ and $x \in X$), then $M, x \models B$ iff $N, x \models B$. Since wR^0w and $d(A) = n$, the theorem follows.

The cases in which $B = \bot$ and B is a sentence letter are trivial. If $B = (C \to D)$, then $d(C), d(D) \leqslant d(B)$, and the result holds for B if it holds for C and D.

Suppose $B = \Box C$, wR^ix, and $d(B) + i \leqslant n$. Then $x \in X$ and $d(B) = d(C) + 1$. If xRy, then $wR^{i+1}y$, $d(C) + i + 1 \leqslant n$, $y \in X$, and so xSy, and by the induction hypothesis, $M, y \models C$ iff $N, y \models C$; since $S \subseteq R$, xRy iff xSy. But then $M, x \models B$ iff for all y such that xRy, $M, y \models C$; iff for all y such that xSy, $M, y \models C$; iff, by the i.h., for all y such that xSy, $N, y \models C$; iff $N, x \models B$. ⊣

Theorem 4 (the generated submodel theorem). *Let M be a model, $w \in W$, $X = \{x: \exists i\, wR^ix\}$, $S = \{\langle x, y\rangle: x, y \in X \wedge xRy\}$, and xUp iff xVp for all $x \in X$ and all sentence letters p. Let $N = \langle X, S, U\rangle$. Then $M, w \models A$ if $N, w \models A$. (N is called the submodel of M generated from w.)*

Proof. Let P be the set of all sentence letters, and $n = d(A)$. Then $X \supseteq \{x: \exists i \leqslant n\, wR^ix\}$, and the generated submodel theorem follows from the continuity theorem. ⊣

The following corollary is a useful immediate consequence of the continuity theorem.

Corollary. *Let A be a sentence. Let M and N, $= \langle W, R, U\rangle$ be models, and wVp iff wUp for all w in W and all p contained in A. Then $M, w \models A$ iff $N, w \models A$.*

We now want to investigate the conditions under which each of the modal sentences $\Box p \to p$, $\Box p \to \Box\Box p$, $p \to \Box\Diamond p$, $\Diamond p \to \Box\Diamond p$, and $\Box(\Box p \to p) \to \Box p$ is valid in a frame $\langle W, R\rangle$.

Theorem 5. *$\Box p \to p$ is valid in $\langle W, R\rangle$ iff R is reflexive on W.*

Proof. Suppose $\Box p \to p$ is valid in $\langle W, R\rangle$. Let w be an arbitrary member of W. We want to show that wRw.

Let V be a valuation on W such that for all x in W, xVp iff wRx.

If wRx, then xVp and $M,x\vDash p$; thus $M,w\vDash\Box p$. Since $M,w\vDash\Box p\to p$, $M,w\vDash p$, wVp, and wRw.

Conversely, suppose R is reflexive on W. Let V be a valuation on W, and suppose $w\in W$. Then if $M,w\vDash\Box p$, for all x such that wRx, $M,x\vDash p$; since wRw by reflexivity, $M,w\vDash p$. Thus if $M,w\vDash\Box p$, then $M,w\vDash p$; so $M,w\vDash\Box p\to p$. ⊣

Theorem 6. $\Box p\to\Box\Box p$ *is valid in* $\langle W,R\rangle$ *iff* R *is transitive.*

Proof. Suppose $\Box p\to\Box\Box p$ is valid in $\langle W,R\rangle$, wRx and xRy. Let V be a valuation on W such that for all z in W, zVp iff wRz. Then $w\vDash\Box p$, for if wRz, zVp. So $w\vDash\Box\Box p$, whence $x\vDash\Box p$, $y\vDash p$, and wRy. Conversely, suppose R is transitive. Let V be an arbitrary valuation. Suppose $w\vDash\Box p$ and wRx. If xRy, then by transitivity, wRy and $y\vDash p$. Thus $x\vDash\Box p$. So $w\vDash\Box\Box p$. ⊣

Theorem 7. $p\to\Box\Diamond p$ *is valid in* $\langle W,R\rangle$ *iff* R *is symmetric.*

Hint for proof. Suppose wRx. Let V be such that zVp iff $z=w$. ⊣

Theorem 8. $\Diamond p\to\Box\Diamond p$ *is valid in* $\langle W,R\rangle$ *iff* R *is euclidean.*

Hint for proof. Suppose wRy, wRx. Let V be such that zVp iff $z=y$. ⊣

Theorem 9 (six soundness theorems)

(a) *if* K$\vdash A$, *then* A *is valid in all frames.*
(b) *if* K4$\vdash A$, *then* A *is valid in all transitive frames.*
(c) *if* T$\vdash A$, *then* A *is valid in all reflexive frames.*
(d) *if* S4$\vdash A$, *then* A *is valid in all reflexive and transitive frames.*
(e) *if* B$\vdash A$, *then* A *is valid in all reflexive and symmetric frames.*
(f) *if* S5$\vdash A$, *then* A *is valid in all reflexive and euclidean frames.*

Proof of (d). Suppose that S4$\vdash A$ and $\langle W,R\rangle$ is reflexive and transitive. We must show A valid in $\langle W,R\rangle$. But $\Box p\to p$ and $\Box p\to\Box\Box p$ are valid in $\langle W,R\rangle$ by Theorems 5 and 6, and therefore every sentence $\Box A\to A$ and $\Box A\to\Box\Box A$ is valid in $\langle W,R\rangle$, for $\Box A\to A$ is a substitution instance of $\Box p\to p$, as is $\Box A\to\Box\Box A$ of $\Box p\to\Box\Box p$. Since all tautologies and all distribution axioms are valid in all models, all axioms of S4 are valid in $\langle W,R\rangle$. And since the sentences valid in $\langle W,R\rangle$ are closed under modus ponens and necessitation, A is also valid in $\langle W,R\rangle$.

The proofs of (a), (b), (c), (e), and (f) are similar. ⊣

What about GL?

A relation R is called *wellfounded* if for every nonempty set X, there is an R-least element of X, that is to say, an element w of X such that xRw for no x in X.

And a relation R is called *converse wellfounded* if for every nonempty set X, there is an R-greatest element of X, an element w of X such that wRx for no x in X.

If R is converse wellfounded, then R is irreflexive, for if wRw, then $\{w\}$ is a nonempty set with no R-greatest element.

And if R is a converse wellfounded relation on W, then to prove that every member of W has a certain property ψ, it suffices to deduce that an arbitrary object w has ψ from the assumption that all x such that wRx have ψ. (This technique of proof is called *induction on the converse of* R.) To see that the technique works, assume that for all w, w has ψ if all x such that wRx have ψ, and let $X=\{w\in W: w$ does not have $\psi\}$. We show that X has no R-greatest element: suppose $w\in X$. Then w does not have ψ, and by our assumption, for some x, wRx and x does not have ψ. $x\in W$ (since R is a relation on W), and so $x\in X$. Thus X indeed has no R-greatest element. Since R is converse wellfounded, X must be empty, and every w in W has ψ.

Theorem 10. $\Box(\Box p\to p)\to\Box p$ *is valid in* $\langle W,R\rangle$ *iff* R *is transitive and converse wellfounded.*

Proof. Suppose that $\Box(\Box p\to p)\to\Box p$ is valid in $\langle W,R\rangle$. Then all sentences $\Box(\Box A\to A)\to\Box A$ are also valid in $\langle W,R\rangle$, and as above, all theorems of GL are valid in $\langle W,R\rangle$. By Theorem 18 of Chapter 1, $\Box p\to\Box\Box p$ is valid in $\langle W,R\rangle$, and so by Theorem 6, $\langle W,R\rangle$ is transitive.

And R is converse wellfounded: for suppose that there is a nonempty set X with no R-greatest element. Let $w\in X$, and let V be a valuation on W such that for every $a\in W$, aVp iff $a\notin X$. We shall show that $w\vDash\Box(\Box p\to p)$ and $w\nvDash\Box p$, contradicting the validity in $\langle W,R\rangle$ of $\Box(\Box p\to p)\to\Box p$.

Suppose wRx, whence $x\in W$. Assume $x\nvDash p$. Then not: xVp, $x\in X$, and therefore for some $y \in X$, xRy, $y \in W$, not: yVp, $y \nvDash p$, and therefore $x \nvDash \Box p$. Thus $x \vDash \Box p \to p$ and $w \vDash \Box(\Box p \to p)$.

And since $w \in X$, for some $x \in X$, wRx, and $x \in W$. Thus not: xVp, $x \nvDash p$, and so $w \nvDash \Box p$.

Conversely, suppose that $\langle W, R \rangle$ is transitive and converse wellfounded and that $\langle W, R, V \rangle$, $w \nvDash \Box p$. Let $X = \{x \in W: wRx \wedge x \nvDash p\}$. Since $w \nvDash \Box p$, for some z, wRz and $z \nvDash p$. Thus $z \in X$, X is nonempty, and by converse wellfoundedness, for some $x \in X$, xRy for no y in X. Since $x \in X$, wRx, and $x \nvDash p$. Suppose xRy. Then $y \notin X$ and since wRy by transitivity, $y \vDash p$. Thus $x \vDash \Box p$, $x \nvDash \Box p \to p$, and $w \nvDash \Box(\Box p \to p)$. So $\Box(\Box p \to p) \to \Box p$ is valid in $\langle W, R \rangle$. ⊣

We will need an alternative characterization of the finite transitive and converse wellfounded relations.

Theorem 11. *Suppose that $F, = \langle W, R \rangle$ is finite and transitive. Then F is irreflexive if and only if F is converse wellfounded.*

Proof. We have already observed that if F is converse wellfounded, F is irreflexive. Suppose that F is irreflexive. If $x_1, \ldots, x_n$ is a sequence of elements of W such that $x_i R x_{i+1}$ for all $i < n$, then $x_i \neq x_j$ if $i < j$: otherwise $x_i = x_j$, and by transitivity $x_i R x_j$, contra irreflexivity. Now assume that F is not converse wellfounded. Let X be a nonempty subset of W such that $\forall w \in X \exists x \in X\ wRx$. Then it is clear by induction that for each positive n, there is a sequence $x_1, \ldots, x_n$ of elements of X such that $x_i R x_{i+1}$ for all $i < n$. Therefore for each n, there are at least n elements of $X \subseteq W$. Thus W is infinite, contradiction. ⊣

Thus a frame is finite transitive and converse wellfounded if and only if it is finite transitive and irreflexive.

We thus have established the following soundness theorem for GL.

Theorem 12. *If* GL$\vdash A$, *then A is valid in all transitive and converse wellfounded frames, and A is also valid in all finite transitive and irreflexive frames.*

We conclude with two remarks on the non-characterizability of converse wellfounded frames.

Frames $\langle W, R \rangle$ are naturally thought of as models interpreting formal languages that contain a single two-place predicate letter

ρ. A frame is reflexive, transitive, symmetric, or euclidean if and only if the first-order sentence $\forall w w\rho w$, $\forall w \forall x \forall y(w\rho x \wedge x\rho y \to w\rho z)$, $\forall w \forall x(w\rho x \to x\rho w)$, or $\forall w \forall x \forall y(w\rho x \wedge w\rho y \to x\rho y)$, respectively, is true in the frame. For "converse wellfounded" it is otherwise: there is no first-order sentence that is true in $\langle W, R \rangle$ iff $\langle W, R \rangle$ is converse wellfounded.

Proof. Suppose that σ is a counterexample. Let $\alpha_0, \alpha_1, \ldots$ be an infinite sequence of distinct new constants. Then every finite subset of $\{\sigma\} \cup \{\alpha_i \rho \alpha_j : i < j\}$ has a model, and by the compactness theorem, the entire set has a model $\langle W, R, a_0, a_1, \ldots \rangle$. But the binary relation R that interprets ρ is not converse wellfounded (because $a_0 R a_1 R \ldots$), and thus $\langle W, R \rangle$ is not converse wellfounded either, even though σ is true in $\langle W, R, a_0, a_1, \ldots \rangle$ and hence in $\langle W, R \rangle$. ⊣

The same argument also shows that there is no first-order sentence that is true in just those frames that are transitive and converse wellfounded.

We know that $\Box(\Box p \to p) \to \Box p$ is a modal sentence that is valid in just the transitive converse wellfounded frames[3]; however, no modal sentence is valid in exactly those frames that are converse wellfounded.

Proof. Suppose that A is a counterexample. Let W be the set of natural numbers and R the successor relation on W, i.e., $\{\langle w, x \rangle : w, x \in W \wedge w + 1 = x\}$. Then $\langle W, R \rangle$ is not converse wellfounded, and so for some valuation V on W, some w in W, $\langle W, R, V \rangle$, $w \nvDash A$. Let $n = d(A)$, and let $X = \{w, w+1, \ldots, w+n\}$, $S = \{\langle x, y \rangle : x, y \in X$ and $xRy\}$, and xUp iff xVp for every p contained in A. By the continuity theorem, $\langle X, S, U \rangle$, $w \nvDash A$, But $\langle X, S \rangle$ is converse wellfounded, contradiction. ⊣

Exercise. True or false: if A is satisfiable in some finite transitive and irreflexive model and contains at most one sentence letter, then A is satisfiable in some finite transitive and irreflexive model in which for all $w_0, w_1, \ldots, w_{d(A)}$ in W, not: $w_0 R w_1 R \ldots R w_{d(A)}$.