Chapter 5
Models of Countable Theories

As in the previous chapter, we require that all formulae are written in Polish notation and that the variables are among $v_0, v_1, v_2, \ldots$ Furthermore, let $\mathscr{L}$ be a countable signature, let T be a consistent $\mathscr{L}$-theory, and let σ_0 be an $\mathscr{L}$-sentence which is not provable from T. Finally, let $\overline{\mathsf{T}}$ be the maximally consistent extension of $\mathsf{T} + \neg\sigma_0$ as above.

We shall now construct a model of $\overline{\mathsf{T}}$. For this, we first extend the signature $\mathscr{L}$ by adding some new constant symbols, then we extend the theory $\overline{\mathsf{T}}$, and finally we construct the model.

Extending the Language

A string of symbols is a **term-constant**, if it results from applying F I N I T E L Y many times the following rules:

(C0) Each closed (*i.e.*, variable-free) $\mathscr{L}$-term is a term-constant.
(C1) If $\tau_0, \ldots, \tau_{n-1}$ are any term-constants which we have already built and F is an n-ary function symbol, then $F\tau_0 \cdots \tau_{n-1}$ is a term-constant.
(C2) For any natural numbers i, n, if $\tau_0, \ldots, \tau_{n-1}$ are any term-constants which we have already built, then $(i, \tau_0, \ldots, \tau_{n-1}, n)$ is a term-constant.

The strings $(i, \tau_0, \ldots, \tau_{n-1}, n)$ which are built with rule (C2) are called **special constants**. Notice that for $n = 0$, $(i, \tau_0, \ldots, \tau_{n-1}, n)$ becomes $(i, 0)$.

Let $\mathscr{L}_c$ be the signature $\mathscr{L}$ extended with the countably many special constants. In order to write the special constants in a list, we first encode them and then define an ordering on the codes.

First we encode closed $\mathscr{L}$-terms as above with strings of `0`'s and `2`'s. Now, let $c^{\bar{\tau}}_{i,n} \equiv (i, \tau_0, \ldots, \tau_{n-1}, n)$ be a special constant, where the codes of $\tau_0, \ldots, \tau_{n-1}$ are already defined. Then we encode $c^{\bar{\tau}}_{i,n}$ as follows:

$$
\begin{array}{rcccccccccccc}
c^{\bar{\tau}}_{i,n} & \equiv & (& \boldsymbol{i} & , & \boldsymbol{\tau_0} & , & \ldots & , & \boldsymbol{\tau_{n-1}} & , & \boldsymbol{n} &) \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\#c^{\bar{\tau}}_{i,n} & \equiv & 6 & \underbrace{1\ldots1}_{i\text{-times }1} & 8 & \#\tau_0 & 8 & \ldots & 8 & \#\tau_{n-1} & 8 & \underbrace{1\ldots1}_{n\text{-times }1} & 9
\end{array}
$$

The codes of special constants are ordered by their length and lexicographically, where $0 < 1 < \ldots < 8 < 9$.

Finally, let $\Lambda_c = [c_0, c_1, \ldots]$ be the potentially infinite list of all special constants, ordered with respect to the ordering of their codes.

Extending the Theory

In this section we shall add witnesses for certain existential $\mathscr{L}_c$-sentences σ_i in the list $\overline{\mathsf{T}} = [\sigma_0, \sigma_1, \ldots, \sigma_i, \ldots]$, where an $\mathscr{L}_c$-sentence is existential if it is of the form $\exists\nu\varphi$. The witnesses we choose from the list Λ_c of special constants. In order to make sure that we have a witness for each existential $\mathscr{L}_c$-sentence (and not just for $\mathscr{L}$-sentences), and also to make sure that the choice of witnesses do not lead to a contradiction, we have to choose the witnesses carefully. For this we introducte the following notion: An $\mathscr{L}$-sentence $\sigma_i \in \overline{\mathsf{T}}$ is in **special prenex normal form**, denoted sPNF, if σ_i is in PNF and

$$\sigma_i \equiv \exists\!\!\!\forall_0 v_0 \exists\!\!\!\forall_1 v_1 \ldots \exists\!\!\!\forall_n v_n \sigma_{i,n}$$

where each $\exists\!\!\!\forall_m$ (for $0 \leqslant m \leqslant n$) stands for either "$\exists$" or "$\forall$", $\sigma_{i,n}$ is quantifier free, and each variable $v_0, \ldots, v_n$ appears free in $\sigma_{i,n}$. Notice that by the PRENEX NORMAL FORM THEOREM 1.12 and the VARIABLE SUBSTITUTION THEOREM 1.13, for every $\mathscr{L}$-sentence σ there is an equivalent $\mathscr{L}$-sentence σ' which is in sPNF.

Let $\sigma_i \in \overline{\mathsf{T}}$ and let $c^{\bar{t}}_{i,n} \equiv (i, t_0, \ldots, t_{n-1}, n)$ be a special constant. Then we say that $\boldsymbol{c^{\bar{t}}_{i,n}}$ **witnesses** $\boldsymbol{\sigma_i}$ if:

- σ_i is in sPNF,
- "$\exists v_n$" appears in σ_i, and
- for all $m < n$: if "$\exists v_m$" appears in σ_i, then $t_m \equiv (i, t_0, \ldots, t_{m-1}, m)$.

If an $\mathscr{L}$-sentence $\sigma_i \in \overline{\mathsf{T}}$ is in sPNF and "$\exists v_n$" or "$\forall v_n$" appear in σ_i, then

$$\sigma_i \equiv \exists\!\!\!\forall_0 v_0 \exists\!\!\!\forall_1 v_1 \cdots \exists\!\!\!\forall_n v_n \sigma_{i,n}(v_0, \ldots, v_n)$$

where $\sigma_{i,n}(v_0, \ldots, v_n)$ is an $\mathscr{L}$-formula in which each variable $v_0, \ldots, v_n$ appears free.

Now, we go through the list $\Lambda_c = [c_0, c_1, \ldots]$ of special constants and extend step by step the list $\overline{\mathsf{T}} = [\sigma_0, \sigma_1, \ldots]$: For this, we first stipulate $T_0 := \overline{\mathsf{T}}$. If T_j is already defined and that $c_j \equiv (i, t_0, \ldots, t_{n-1}, n)$. We have the following two cases:

Case 1. The special constant c_j does not witness the $\mathscr{L}$-sentence $\sigma_i \in \overline{\mathsf{T}}$. In this case we set $T_{j+1} := T_j$.

Case 2. The special constant c_j witnesses $\sigma_i \in \overline{\mathsf{T}}$. In this case we insert the $\mathscr{L}_c$-sentence

$$\sigma_{i,n}[c_j] \equiv \sigma_{i,n}(v_0/t_0, \ldots, v_{n-1}/t_{n-1}, v_n/c_j)$$

into the list T_j on the place which corresponds to the code $\#\sigma_{i,n}[c_j]$. The extended list is then T_{j+1}.

Finally, let $\overline{\mathsf{T}}_c$ be the resulting list, *i.e.*, $\overline{\mathsf{T}}_c$ is the union of all the T_j's.

LEMMA 5.0. *$\overline{\mathsf{T}}_c$ is consistent.*

Proof. By construction of $\overline{\mathsf{T}}$ we have $\mathrm{Con}(\overline{\mathsf{T}})$. Now, assume towards a contradiction that $\mathrm{Con}(\overline{\mathsf{T}}_c)$ is inconsistent. Then, by the COMPACTNESS THEOREM 1.15, we find finitely many $\mathscr{L}_c$-sentences $\sigma_{i,n}[c_j]$ in $\overline{\mathsf{T}}_c$ such that

$$\neg\,\mathrm{Con}\left(\overline{\mathsf{T}} + \left\{\sigma_{i_1,n_1}[c_{j_1}], \ldots, \sigma_{i_k,n_k}[c_{j_k}]\right\}\right).$$

Without loss of generality we may assume that $\sigma_{i_1,n_1}[c_{j_1}], \ldots, \sigma_{i_k,n_k}[c_{j_k}]$ are such that the sum $n_1 + \ldots + n_k + k$ is minimal.

Now, for term-constants τ we define the height $h(\tau)$ as follows: If τ is a closed $\mathscr{L}$-term, then $h(\tau) := 0$. If $\tau_0, \ldots, \tau_{n-1}$ are term-constants und $F \in \mathscr{L}$ is an n-ary function symbol, then

$$h(F\tau_0 \cdots \tau_{n-1}) := \max\left\{h(\tau_0), \ldots, h(\tau_{n-1})\right\}.$$

Finally, if $\tau \equiv (i, \tau_0, \ldots, \tau_{n-1}, n)$ is a special constant, then

$$h(\tau) := 1 + \max\left\{h(\tau_0), \ldots, h(\tau_{n-1})\right\}.$$

Without loss of generality we may assume that $h(c_{j_k}) = \max\left\{h(c_{j_1}, \ldots, h(c_{j_k})\right\}$. To simplify the notation, let $\Sigma := \left\{\sigma_{i_1,n_1}[c_{j_1}], \ldots, \sigma_{i_k,n_{k-1}}[c_{j_{k-1}}]\right\}$; furthermore we write i, n, j instead of i_k, n_k, j_k respectively.

Now, we consider the $\mathscr{L}_c$-sentence $\sigma_{i,n}[c_j]$. For this, let $c_j \equiv (i, t_0, \ldots, t_{n-1}, n)$, *i.e.*,

$$\sigma_{i,n}[c_j] \equiv \sigma_{i,n}(v_0/t_0, \ldots, v_{n-1}/t_{n-1}, v_n/c_j).$$

Since c_j witnesses σ_i, "$\exists v_n$" appears in σ_i, *i.e.*,

$$\sigma_{i,n-1}(v_0, \ldots, v_{n-1}) \equiv \exists v_n \sigma_{i,n}(v_0, \ldots, v_{n-1}, v_n).$$

To simplify the notation again we set

$$\tilde{\sigma}(v_n) :\equiv \sigma_{i,n}(v_0/t_0, \ldots, v_{n-1}/t_{n-1}, v_n).$$

CLAIM. $\neg\,\mathrm{Con}\left(\overline{\mathsf{T}} + \Sigma + \sigma_{i,n}[c_j]\right) \implies \neg\,\mathrm{Con}\left(\overline{\mathsf{T}} + \Sigma + \exists v_n \tilde{\sigma}(v_n)\right)$

Proof of Claim. If $\overline{\mathsf{T}} + \Sigma + \sigma_{i,n}[c_j]$ is inconsistent, then $\overline{\mathsf{T}} + \Sigma + \sigma_{i,n}[c_j] \vdash \boxtimes$ and with the DEDUCTION THEOREM we get

$$\overline{\mathsf{T}} + \Sigma \vdash \sigma_{i,n}[c_j] \rightarrow \boxtimes .$$

In the latter proof we replace the special constant c_j throughout the proof with a variable ν which does not occur, neither in the proof nor in $\sigma_{i,n}$. Notice that every logical axiom becomes a logical axiom of the same type and that $\mathscr{L}$-sentences of $\overline{\mathsf{T}}$ are not affected (which do not contain any of the special constants). Furthermore, also $\mathscr{L}_c$-sentences of Σ are not affected since they do not contain the special constant c_j (otherwise, the height $h(c_j)$ would not be maximal). Finally, each application of MODUS PONENS or GENERALISATION becomes a new application of the same inference rule (notice that we do not apply GENERALISATION to ν, since otherwise, we would have applied GENERALISATION to c_j, but c_j is a constant). It follows that we obtain a proof of $\tilde{\sigma}(\nu) \rightarrow \boxtimes$ from $\overline{\mathsf{T}} + \Sigma$:

$\overline{\mathsf{T}} + \Sigma \vdash \tilde{\sigma}(\nu) \rightarrow \boxtimes$	by construction
$\overline{\mathsf{T}} + \Sigma \vdash \forall \nu \big(\tilde{\sigma}(\nu) \rightarrow \boxtimes\big)$	by GENERALISATION
$\overline{\mathsf{T}} + \Sigma \vdash \forall \nu \big(\tilde{\sigma}(\nu) \rightarrow \boxtimes\big) \rightarrow \big(\exists \nu \tilde{\sigma}(\nu) \rightarrow \boxtimes\big)$	L_{14}
$\overline{\mathsf{T}} + \Sigma \vdash \exists \nu \tilde{\sigma}(\nu) \rightarrow \boxtimes$	by MODUS PONENS
$\overline{\mathsf{T}} + \Sigma \vdash \exists v_n \tilde{\sigma}(v_n) \rightarrow \boxtimes$	TAUTOLOGY (Q.2)

Therefore, we finally have $\neg \operatorname{Con}\big(\overline{\mathsf{T}} + \Sigma + \exists v_n \tilde{\sigma}(v_n)\big)$. $\dashv_{\text{Claim}}$

We now write again i_k, n_k, j_k instead of i, n, j respectively and consider the following three cases:

Case 1. If $n_k = 0$, then $\sigma_{i_k} \equiv \exists v_0 \tilde{\sigma}$, *i.e.*, $\neg \operatorname{Con}\big(\overline{\mathsf{T}} + \Sigma\big)$. So,

$$\neg \operatorname{Con}\left(\overline{\mathsf{T}} + \big\{\sigma_{i_1,n_1}[c_{j_1}], \dots, \sigma_{i_{k-1},n_{k-1}}[c_{j_{k-1}}]\big\}\right)$$

which is a contradiction to the minimality of $n_1 + \ldots + n_k + k$ (*i.e.*, the choice of $\sigma_{i_1,n_1}[c_{j_1}], \dots, \sigma_{i_k,n_k}[c_{j_k}]$), since

$$n_1 + \ldots + n_{k-1} + (k-1) < n_1 + \ldots + n_k + k \,.$$

Case 2. If $n_k > 0$ and "$\exists v_m$" appears in σ_{i_k} for some $m < n_k$, then

$$\operatorname{Con}\left(\overline{\mathsf{T}} + \Sigma + \sigma_{i_k,m}(v_0/t_0, \ldots v_m/t_m)\right) .$$

Otherwise, we would have

$$n_1 + \ldots + n_{k-1} + m + k < n_1 + \ldots + n_k + k$$

which is again a contradiction to the minimality of $n_1 + \ldots + n_k + k$.

Case 3. If, for some $m + 1 < n_k$, we have

$$\mathrm{Con}\left(\overline{\mathsf{T}} + \Sigma + \sigma_{i_k,m}(v_0/t_0, \dots v_m/t_m)\right)$$

and "$\forall v_{m+1}$" appears in σ_{i_k}, then, by L_{11}, we get

$$\mathrm{Con}\left(\overline{\mathsf{T}} + \Sigma + \sigma_{i_k,m}(v_0/t_0, \dots v_m/t_m, v_{m+1}/t_{m+1})\right).$$

Combining the Cases 1–3 we get that $\overline{\mathsf{T}} + \Sigma + \sigma_{i_k}[c_{j_k}]$ is consistent, which contradicts our primary assumption. Hence, the $\mathscr{L}_c$-theory $\overline{\mathsf{T}}_c$ is consistent. ⊣

Completeness Theorem for Countable Signatures

In this section we shall construct a model of the $\mathscr{L}_c$-theory $\overline{\mathsf{T}}_c$, which is of course also a model of the $\mathscr{L}$-theory $\mathsf{T} + \neg\sigma_0$. However, since we extended the signature $\mathscr{L}$, we first have to extend the binary relation "$=$" as well as relation symbols in $\mathscr{L}$ to the new closed $\mathscr{L}_c$-terms.

LEMMA 5.1. *The list $\overline{\mathsf{T}}_c$ can be extended to a consistent list $\widetilde{\mathsf{T}}$ of $\mathscr{L}_c$-sentence, such that the new $\mathscr{L}_c$-sentences are variable-free and for each variable-free $\mathscr{L}_c$-senctence σ we have*

$$\text{either} \quad \sigma \in \widetilde{\mathsf{T}} \quad \text{or} \quad \neg\sigma \in \widetilde{\mathsf{T}}.$$

Proof. Like in the proof of LINDENBAUM'S LEMMA 4.5, we go through the list of all variable-free $\mathscr{L}_c$-sentences and successively extend the list $\overline{\mathsf{T}}_c$ to a maximally consistent list $\widetilde{\mathsf{T}}$. ⊣

Now we are ready to construct the domain of a model of $\widetilde{\mathsf{T}}$, which shall be a list of lists: For this, let

$$\Lambda_\tau = [t_0, t_1, \dots, t_n, \dots]$$

be the list of all term-constants (ordered with respect to the encoding above). We go through the list Λ_τ and construct step by step a list of lists: First, we set $A_0 := \big[[\ ]\big]$. Now, assume that A_n is already defined. Then consider the $\mathscr{L}_c$-sentences

$$t_n = t_0, t_n = t_1, \dots, t_n = t_{n-1}.$$

If $t_n = t_m$ is one of these sentences and $t_n = t_m$ belongs to $\widetilde{\mathsf{T}}$, then we append t_n to the list in A_n which contains t_m; the resulting list is A_{n+1}. If none of the sentences $t_n = t_m$ belongs to $\widetilde{\mathsf{T}}$, then $A_{n+1} := A_n + \big[[t_n]\big]$.

Let $A = \big[[t_{n_0}, \dots], [t_{n_1}, \dots] \dots\big]$ be the resulting list, *i.e.*, A is the union of all the A_n's.

The lists in the list A is the domain of our model $\mathbf{M}$ of $\widetilde{\mathsf{T}}$. In order to simplify the notation, for term-constants τ let $\tilde{\tau}$ be the unique list of A which contains τ.

In order to get an $\mathscr{L}_c$-structure $\mathbf{M}$ with domain A, we have to define a mapping which assigns to each constant symbol $c \in \mathscr{L}_c$ an element $c^{\mathbf{M}} \in A$, to each n-ary function symbol $F \in \mathscr{L}$ a function $F^{\mathbf{M}} : A^n \to A$, and to each n-ary relation symbol $R \in \mathscr{L}$ a set $R^{\mathbf{M}} \subseteq A^n$:

- If $c \in \mathscr{L}_c$ is a constant symbol of $\mathscr{L}$ or a special constant, then let
$$c^{\mathbf{M}} := \widetilde{c}\,.$$
- If $F \in \mathscr{L}$ is an n-ary function symbol and $\widetilde{t}_1, \ldots, \widetilde{t}_n$ are elements of A, then let
$$F^{\mathbf{M}}\widetilde{t}_1 \cdots \widetilde{t}_n := \widetilde{Ft_1 \cdots t_n}\,.$$
- If $R \in \mathscr{L}$ is an n-ary relation symbol and $\widetilde{t}_1, \ldots, \widetilde{t}_n$ are elements of A, then we define
$$\langle \widetilde{t}_1, \ldots, \widetilde{t}_n \rangle \in R^{\mathbf{M}} \quad :\Longleftrightarrow \quad Rt_1 \cdots t_n \in \widetilde{\mathsf{T}}\,.$$

FACT 5.2. *The definitions above, which rely on representatives of the lists in A, are well-defined.*

Proof. This follows easily by L_{15}–L_{17} and the construction of $\widetilde{\mathsf{T}}$; the details are left as an exercise to the reader. ⊣

THEOREM 5.3. *The $\mathscr{L}_c$-structure $\mathbf{M}$ is a model of $\widetilde{\mathsf{T}}$, and consequently also of $\mathsf{T} + \neg\sigma_0$.*

Proof. We have to show that for each $\mathscr{L}_c$-sentence $\sigma \in \widetilde{\mathsf{T}}$, $\mathbf{M} \vDash \sigma$, *i.e.*,
$$\forall \sigma \big(\sigma \in \widetilde{\mathsf{T}} \implies \mathbf{M} \vDash \sigma\big)\,.$$

First notice that for $\mathscr{L}_c$-sentences $\sigma \,\&\, \sigma'$ with $\sigma \Leftrightarrow \sigma'$ (*i.e.*, $\vdash \sigma \leftrightarrow \sigma'$), by the SOUNDNESS THEOREM **??** we get
$$\mathbf{M} \vDash \sigma \quad \Longleftrightarrow \quad \mathbf{M} \vDash \sigma'\,.$$

So, by the 3-SYMBOLS THEOREM 1.11 it is enough to prove the theorem only for $\mathscr{L}_c$-sentences σ which are either atomic or of the form $\neg\sigma'$, $\wedge\sigma_1\sigma_2$, or $\exists\nu\sigma'$.

We first consider the case when σ is variable-free. By LEMMA 5.1 we know that for each variable-free $\mathscr{L}_c$-sentences σ we have either $\sigma \in \widetilde{\mathsf{T}}$ or $\neg\sigma \in \widetilde{\mathsf{T}}$. Thus, we must show that for these sentences we have
$$\sigma \in \widetilde{\mathsf{T}} \quad \Longleftrightarrow \quad \mathbf{M} \vDash \sigma\,.$$

If σ is atomic, then either $\sigma \equiv t_1 = t_2$ (for some term-constants $t_1 \,\&\, t_1$) or $\sigma \equiv Rt_1 \cdots t_n$ (for term-constants $t_1, \ldots, t_n$ and an n-ary relation symbol $R \in \mathscr{L}$), and by construction of $\mathbf{M}$ we get $\sigma \in \widetilde{\mathsf{T}} \Longleftrightarrow \mathbf{M} \vDash \sigma$.

Now, assume towards a contradiction that there exists a variable-free $\mathscr{L}_c$-sentence σ_0 such that either $\sigma_0 \in \widetilde{\mathsf{T}}$ *and* $\mathbf{M} \nvDash \sigma_0$, or $\sigma_0 \notin \widetilde{\mathsf{T}}$ *and* $\mathbf{M} \vDash \sigma_0$. Without loss of generality we may assume that σ_0 has as few logical symbols as possible. Notice that we already know that σ_0 is not atomic. We consider the following cases:

$\sigma_0 \equiv \neg\sigma$: Since σ has less logical symbols than σ_0, we have $\sigma \in \widetilde{\mathsf{T}}$ if and only if $\mathbf{M} \vDash \sigma$. This shows that

$$\neg\sigma \notin \widetilde{\mathsf{T}} \iff \mathbf{M} \nvDash \neg\sigma$$

which is a contradiction to the choice of σ_0.

$\sigma_0 \equiv \wedge\sigma_1\sigma_2$: Since σ_1 as well as σ_2 has less logical symbols than σ_0, we have $\sigma_1 \in \widetilde{\mathsf{T}}$ if and only if $\mathbf{M} \vDash \sigma_1$, as well as $\sigma_2 \in \widetilde{\mathsf{T}}$ if and only if $\mathbf{M} \vDash \sigma_2$. This shows that

$$\wedge\sigma_1\sigma_2 \in \widetilde{\mathsf{T}} \iff \mathbf{M} \vDash \wedge\sigma_1\sigma_2$$

which is a contradiction to the choice of σ_0.

Now, we consider the case when σ contains variables and show that for every $\sigma \in \widetilde{\mathsf{T}}$ we have $\mathbf{M} \vDash \sigma$; If σ is an $\mathscr{L}_c$-sentence which belongs to $\widetilde{\mathsf{T}}$, then there exists a $\sigma' \in \overline{\mathsf{T}}_c$ in sPNF such that $\sigma \Leftrightarrow \sigma'$. In particular we get $\mathbf{M} \vDash \sigma$ if and only if $\mathbf{M} \vDash \sigma'$.

Assume towards a contradiction that there is an $\mathscr{L}_c$-sentence $\sigma' \in \overline{\mathsf{T}}_c$ in sPNF for which we have $\mathbf{M} \nvDash \sigma'$. Notice that since $\sigma' \in \overline{\mathsf{T}}_c$, we have $\sigma' \in \widetilde{\mathsf{T}}$, in particular we get $\widetilde{\mathsf{T}} \vdash \sigma'$. For σ' there are natural numbers i, m, n with $m < n$ and term-constants $t_0, \ldots, t_{m-1}$, such that

$$\sigma' \equiv \exists\!\!\!\forall_m v_m \cdots \exists\!\!\!\forall_n v_n \sigma_{i,m}(v_0/t_0, \ldots, v_{m-1}/t_{m-1}, v_m, \ldots, v_n)\,,$$

where each $\exists\!\!\!\forall_k$ (for $m \leqslant k \leqslant n$) stands for either "$\exists$" or "$\forall$" and $\sigma_{i,n}$ is quantifier free.

Because $\mathbf{M} \nvDash \sigma'$, we get $\mathbf{M} \vDash \neg\sigma'$, and for $\neg\sigma'$ we have:

$$\neg\sigma' \equiv \overline{\exists\!\!\!\forall}_m v_m \cdots \overline{\exists\!\!\!\forall}_n v_n \neg\sigma_{i,n}(v_0/t_0, \ldots, v_{m-1}/t_{m-1}, v_m, \ldots, v_n)$$

where for $m \leqslant k \leqslant n$, the quantifier $\overline{\exists\!\!\!\forall}_k$ is "$\exists$" if $\exists\!\!\!\forall_k$ is "$\forall$", and vice versa.

For each k with $m \leqslant k \leqslant n$, we replace in $\sigma_{i,n}$ step by step the variable v_k with a term-constant t_k as follows:

- If $\exists\!\!\!\forall_k$ is the quantifier "$\forall$", then

$$\mathbf{M} \vDash \exists v_k \cdots \neg\sigma_{i,n}(v_0/t_0, \ldots, v_k, \ldots)\,.$$

  Hence, there exists a $\tilde{t}_k \in A$ such that

$$\mathbf{M} \vDash \overline{\exists\!\!\!\forall}_{k+1} v_{k+1} \cdots \neg\sigma_{i,n}(v_0/t_0, \ldots, v_k/t_k, \ldots)\,.$$

  On the other hand, if $\exists\!\!\!\forall_k$ is the quantifier "$\forall$", then

$$\widetilde{\mathsf{T}} \vdash \forall v_k \cdots \sigma_{i,n}(v_0/t_0, \ldots, v_k, \ldots)\,,$$

which implies, by L_{11},

$$\widetilde{\mathsf{T}} \vdash \forall\!\!\!\exists_{k+1} v_{k+1} \cdots \sigma_{i,n}(v_0/t_0, \ldots, v_k/t_k, \ldots) \,.$$

- If $\forall\!\!\!\exists_k$ is the quantifier "$\exists$", then, for $t_k \equiv (i, t_0, \ldots, t_{k-1}, k)$,

$$\forall\!\!\!\exists_{k+1} v_{k+1} \cdots \sigma_{i,n}(v_0/t_0, \ldots, v_k/t_k, \ldots) \in \overline{\mathsf{T}}_c \,,$$

which implies

$$\widetilde{\mathsf{T}} \vdash \forall\!\!\!\exists_{k+1} v_{k+1} \cdots \sigma_{i,n}(v_0/t_0, \ldots, v_k/t_k, \ldots) \,.$$

On the other hand, if $\forall\!\!\!\exists_k$ is the quantifier "$\exists$", then

$$\mathbf{M} \vDash \forall v_k \cdots \neg\sigma_{i,n}(v_0/t_0, \ldots, v_k, \ldots) \,,$$

which implies, by L_{11},

$$\mathbf{M} \vDash \overline{\forall\!\!\!\exists}_{k+1} v_{k+1} \cdots \neg\sigma_{i,n}(v_0/t_0, \ldots, v_k/t_k, \ldots) \,.$$

Proceeding this way, we finally get

$$\mathbf{M} \vDash \neg\sigma_{i,n}(v_0/t_0, \ldots, v_n/t_n) \quad \text{and} \quad \widetilde{\mathsf{T}} \vdash \sigma_{i,n}(v_0/t_0, \ldots, v_n/t_n) \,.$$

Since the latter implies $\neg\sigma_{i,n}(v_0/t_0, \ldots, v_n/t_n) \notin \widetilde{\mathsf{T}}$ and since $\sigma_{i,n}$ is variable-free, this is a contradiction to what we have proved above. $\dashv$

The following theorem just summarises what we have achieved so far:

COUNTABLE GÖDEL-HENKIN COMPLETENESS THEOREM 5.4. *If $\mathscr{L}$ is a countable signature and T is a consistent set of $\mathscr{L}$-sentences, then T has a model. Moreover, if $\mathsf{T} \nvdash \sigma_0$ (for some $\mathscr{L}$-sentence σ_0), then $\mathsf{T} + \neg\sigma_0$ has a model.*

In the next chapter, we shall prove the COMPLETENESS THEOREM for arbitrarily large signatures, but before, we would like to present a few consequences which follow directly from the COUNTABLE GÖDEL-HENKIN COMPLETENESS THEOREM (or its proof), or in combination with the COMPACTNESS THEOREM.

Some Consequences

Let $\mathscr{L}$ be a countable signature, T a set of $\mathscr{L}$-sentences, and σ_0 an $\mathscr{L}$-sentence.

- If $\mathsf{T} \nvdash \sigma_0$, then there is an $\mathscr{L}$-structure $\mathbf{M}$ such that $\mathbf{M} \vDash \mathsf{T} + \neg\sigma_0$:

$$\mathsf{T} \nvdash \sigma_0 \quad \Longrightarrow \quad \exists \mathbf{M} \left(\mathbf{M} \vDash \mathsf{T} + \neg\sigma_0\right)$$

- If T is consistent, then T has a model:

$$\mathrm{Con}(\mathsf{T}) \implies \exists \mathbf{M}\,(\mathbf{M} \vDash \mathsf{T})$$

- If each model of T is also a model of σ_0, then $\mathsf{T} \vdash \sigma_0$:

$$\forall \mathbf{M}\,(\mathbf{M} \vDash \mathsf{T} \implies \mathbf{M} \vDash \sigma_0) \implies \mathsf{T} \vdash \sigma_0$$

- In combination with the COMPACTNESS THEOREM 1.15 we get

$$\mathrm{Con}(\mathsf{T}) \iff \exists \mathbf{M}\,(\mathbf{M} \vDash \mathsf{T})$$

  and finally:

$$\underbrace{\forall \mathbf{M}\,(\mathbf{M} \vDash \mathsf{T} \implies \mathbf{M} \vDash \sigma_0)}_{\mathsf{T} \vDash \sigma_0} \iff \mathsf{T} \vdash \sigma_0$$

The last consequence allows us to replace *formal proofs* with *mathematical proofs*: For example, instead of proving formally the uniqueness of the neutral element in groups from the axioms of Group Theory GT, we just show that in every model of GT (*i.e.*, in every group), the neutral element is unique. So, instead of $\mathsf{GT} \vdash \sigma_0$, we just show $\mathsf{GT} \vDash \sigma_0$.

As a last consequence we would like to mention the DOWNWARD LÖWENHEIM–SKOLEM THEOREM, which is also known as SKOLEM'S PARADOX.

DOWNWARD LÖWENHEIM-SKOLEM THEOREM 5.5. *If $\mathscr{L}$ is a countable signature and T is a consistent set of $\mathscr{L}$-sentences, then T has a countable model.*

Proof. In the previous chapter, we began with a countable signature $\mathscr{L}$ and a consistent set of $\mathscr{L}$-sentences T; and at the end, the domain A of the model $\mathbf{M}$ of T was a finite or potentially infinite list of lists. So, the model $\mathbf{M}$ we constructed is countable. $\dashv$