The Universal Computer

The Road From Leibniz to Turing

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It is tempting to indulge in a bit of: "What if?" What if Leibniz had not been shackled to his patrons' family history, and been free to devote more time to his *calculus rationcinator*? Might he not have accomplished what Boole was only to do so much later? But of course, such speculation is useless. What Leibniz has left us is his dream, but even this dream can fill us with admiration for the power of human speculative thought and can serve as a yardstick for judging later developments.

CHAPTER 2

Boole Turns Logic into Algebra

George Boole's Hard Life

The beautiful and intelligent Princess Caroline von Ensbach, one day to be Queen of England as the wife of George II, met Leibniz in Berlin in 1704 when she was 18. After she went to England with the court, their friendship continued by correspondence. She tried to persuade her father-in-law, now George I of England, to bring Leibniz to England, but as we have seen, the king insisted that Leibniz remain in Germany to complete the Hanovarian family history.

Caroline found herself entangled in the foolish continuing dispute between Leibniz and Newton and his followers, each side accusing the other of plagiarism over the invention of the calculus. She tried to convince Leibniz that the issue was of no great importance, but he was having none of it. Indeed, Leibniz sought her support before the king for his desire to be appointed "Historiographer of England" so as to match Newton's position as "Master of the Mint," asserting that only in this way could the honor of Germany vis a vis England be maintained. Leibniz wrote Caroline that when Newton held that a grain of sand exerted a gravitational force on the distant sun without any evident means by which such a force could be transmitted, he was in effect calling on miraculous means to explain a natural phenomenon, something he assured her was inadmissable. For her part, Caroline tried to get some of Leibniz's writings translated into English. This effort brought her into contact with Samuel Clarke who had been recommended to her as a possible translator.

Clarke was a philosopher and theologian and also a convinced disciple of Newton. In his *Being and Attributes of God,* dated 1704, Clarke had developed a proof of the existence of God. Caroline showed him a letter

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from Leibniz attacking certain of Newton's ideas and asked him to reply. This initiated a correspondence between the two men that continued until just a few days before Leibniz's death. Not surprisingly, there was no meeting of minds. From the point of view of our story, the most interesting fact about Samuel Clarke is that almost a century and a half after Leibniz's death, George Boole would demonstrate the efficacy of his own methods by using Clarke's proof of the existence of God as an example. In effect, with these methods, Boole had so far succeeded in bringing to life part of Leibniz's dream, that Clarke's complicated deduction could be reduced to a simple set of equations.[1]

In proceeding from the world of Leibniz and the seventeenth century European nobility to that of George Boole, we move forward not only two centuries in time, but also down several layers of social class. George, the first of four children, was born on November 2, 1815, in the town of Lincoln in the eastern part of England, to John and Mary Boole who had been childless for the first nine years of their marriage. John Boole, a cobbler who eked out only a meager living from his trade, had a passion for learning, and especially for scientific instruments. He proudly displayed a telescope he had made in his shop window. Unfortunately, he was not an effective business man and his talented, conscientious son soon found himself carrying the burden of supporting the whole family.[2]

In June 1830, the citizens of Lincoln were treated to a silly controversy in a local newspaper over the originality of an English translation of one of the poems of the ancient Greek writer Meleager. The translation had appeared in the *Lincoln Herald* as the work of "G. B. of Lincoln, aged 14 years," and one P. W. B. took the trouble to write accusing G. B. of plagiarism. P. W. B. admitted that he was unable to provide a reference to the source from which he was accusing G. B. of copying, but regarded it as simply beyond belief that the work could have been produced by a 14-year-old. The battle led to an exchange of several letters between G .B. and P. W. B. all duly published in the *Herald*.

George's family, who early recognized his ability, were far too poor to furnish him with a proper formal education, and so, with the important help of his father, George was mainly self-taught. George studied not only Latin and Greek but also taught himself French and German and was able (much later, of course) to write mathematical research papers in these languages. George Boole never belonged to any particular religious denomination, and found it impossible to believe in the divinity of Christ, but throughout his life he held strong religious convictions. He soon abandoned his original ambition to join the clergy of the Church of England, in part because of his beliefs, but crucially because of his family's need for immediate financial help when his father's business collapsed. George was not yet 16 when he began his career as a teacher.

After two years at a small Methodist school some 40 miles from home he was fired, mainly it seems, owing to complaints about his irreligious behavior: he worked on mathematics on Sundays, and even in chapel! Indeed, it was at this time that Boole's efforts turned more and more to mathematics. In later years, reminiscing about this period in his life, he explained that having a very limited budget for buying books, he found that mathematics books provided the best value because it took longer to work through them than books on other subjects. He also liked to speak of the inspiration that suddenly came to him during his stay at the Methodist school. While walking across a field, the thought flashed across his mind that it should be possible to express logical relationships in algebraic form. This experience, which a biographer compares to that of Paul on the road to Damascus, was to bear fruit only many years later.[3]

After the Methodist school, Boole took a position in Liverpool. But after six months of living and teaching there, he felt compelled to leave because of (in the words of his sister), "the spectacle of gross appetites and passions unrestrainedly indulged ..." presumably by the school headmaster.[4] His next job, in a village only four miles from home, was also of brief duration. This time, the reason was that, at the age of 19, concerned to put his family's finances on a sound basis, George Boole had decided to start his own school in his home town, Lincoln. For fifteen years, until accepting a professorship at a newly founded university at Cork, Ireland, Boole managed a successful career as a schoolmaster. His schools (there were three in succession) were the sole support of his parents and his siblings, although eventually his sister Mary Ann and brother William did participate in the work.

Although running a day and boarding school, and teaching numerous classes might be thought to be a full-time job, Boole managed during this period to make the transition from student of mathematics to creative mathematician. In addition, he somehow found time for activities of social improvement. He was a founder and trustee of a "Female Penitent's Home"

in Lincoln whose purpose was "to provide a temporary home in which, by moral and religious instruction and the formation of industrious habits, females, who have deviated from the paths of virtue, may be restored to a reputable place in society." Boole's biographer speaks of prostitutes (who were evidently numerous in Victorian Lincoln) as the "penitent" women who were to be helped by this institution.[5] More likely, the typical client was a young woman of the servant class who found herself pregnant and abandoned after having been promised marriage by a lover of her own social class.* Some insight into George Boole's personal attitudes towards sexual matters may perhaps be gleaned from what he said in two of his lectures on non-mathematical subjects. In one, a lecture on education, he warned:

> A very large proportion of the extant literature of Greece and Rome ... is deeply stained with allusions and all too often with more than allusions to the vices of Heathenism. ... But that the innocence of youth can be exposed to the contamination of evil without danger I do not believe.[6]

And a lecture on the proper uses of leisure (given after a successful campaign by the "Lincoln Early Closing Association" to obtain a ten-hour working day) included Boole's stern words:

> If you seek gratification in those pursuits from which virtue turns aside, you do so without excuse.[7]

Boole, following in his father's footsteps, was also deeply involved with the Lincoln Mechanics' Institute. These mechanics' institutes, mainly devoted to after-hours education for artisans and other workers, had sprung up all over Victorian Britain. Boole did committee work for the one in Lincoln, made recommendations for improving the library, gave lectures, and provided teaching on a variety of subjects without remuneration.

Yet somehow, amidst all of this, he found time to study some of the most important English and continental mathematical treatises, and to begin making his own contributions. Much of Boole's early work bears witness to Leibniz's belief in the power of appropriate mathematical symbolism, of the manner in which the symbols seem to magically produce correct answers to problems almost unaided. Leibniz had pointed to the example of algebra. In England, as Boole began his own work, it was coming to be

*The study (Barret-Ducrocq, 1989) of a similar institution in London recounts many such tales of woe.

realized that the power of algebra comes from the fact that the symbols representing quantities and operations obeyed a small number of basic rules or laws. This implied that this same power could be applied to objects and operations of the most varied kind so long as they obeyed some of these same laws.[8]

In Boole's early work, he applied algebraic methods to the objects that mathematicians call *operators*. These "operate" on expressions of ordinary algebra to form new expressions. Boole was particularly interested in *differential operators*, so called because they involve the differentiation operation of the calculus mentioned in the previous chapter.[9] These operators were seen to be of particular importance because many of the fundamental laws of the physical universe take the form of differential equations, that is equations involving differential operators. Boole showed how certain differential equations could be solved by using methods of ordinary algebra applied to differential operators. Engineering and science students typically learn some of these methods nowadays in their sophomore or junior year in a course in differential equations.

During his years as a schoolmaster, Boole published a dozen research papers in the *Cambridge Mathematical Journal*. In addition, he submitted a very long paper to the *Philosophical Transactions of the Royal Society*. At first the Royal Society was loathe to consider a submission from such an outsider. But finally they decided to accept it, and later awarded it their Gold Medal.[10] Boole's method was to introduce a technique and then to apply it to a number of examples. He generally asked for no more in the way of *proof* that his methods were correct than that his examples worked out.[11]

At this time, Boole developed professional correspondences and friendships with a number of England's leading young mathematicians. A quarrel with the Scottish philosopher Sir William Hamilton that his friend Augustus De Morgan had fallen into, brought Boole's thoughts back to his long ago flash of insight—that logical relationships might be expressible as a kind of algebra. Although Hamilton was an erudite scholar in aspects of metaphysics, he seems to have been something of a quarrelsome fool. Out of what can only have been his colossal ignorance of the subject, he published diatribes against mathematics as a subject. What had set him off was De Morgan's publication on logic that Hamilton claimed plagiarized what he thought of as his great discovery in logic, what he called the "quantification

of the predicate." We need waste no time trying to understand this idea or the fierce controversy it generated—it is of importance only because of the stimulus it provided to George Boole.[12]

The classical logic of Aristotle that had so fascinated the young Leibniz involved sentences like:

1. All plants are alive.

2. No hippopotamus is intelligent.

3. Some people speak English.

Boole came to realize that what is significant in logical reasoning about such words as "alive," "hippopotamus," or "people" is the *class* or *collection* of all individuals described by the word in question: the *class* of living things, the *class* of hippopotamuses, the *class* of people. Moreover, he came to see how this kind of reasoning can be expressed in terms of an algebra of such classes. Boole used letters to represent classes just as letters had previously been used to represent numbers or operators. If the letters x and y stand for two particular classes, then Boole wrote xy for the class of things that are both in x and in y. As Boole himself put it:

> ... if an adjective, as "good," is employed as a term of description, let us represent by a letter, as y, all things to which the description "good" is applicable, i.e. "all good things," or the class "good things." Let it further be agreed, that by the combination xy shall be represented that class of things to which the names or descriptions represented by x and y are simultaneously applicable. Thus, if x alone stands for "white things," and y for "sheep," let xy stand for "white sheep;" and in like manner, if z stand for "horned things," ... let zxy represent "horned white sheep," ...[13]

Boole thought of this operation applied to classes as being in some ways like the operation of multiplication applied to numbers. However, he noticed a crucial difference: If once again y is the class of sheep, what is yy? It must be the class of things that are sheep and are also ... sheep. But this is the very same thing as the class of sheep; so $yy = y$. It is only a small exaggeration to say that Boole based his entire system of logic on the fact that when x stands for a class, the equation $xx = x$ is always true. We will return to this point later.*

*Boole's equation $xx = x$ can be compared to Leibniz's $A \oplus A = A$. In both cases, an operation that is intended to be applied to pairs of items, when applied to an item and *itself*, yields that very same item as a result.

George Boole was 32 when his first revolutionary monograph on logic as a form of mathematics was published. His more polished exposition, *The Laws of Thought* appeared seven years later. These were eventful years in Boole's life. Boole's social class and unconventional education had apparently ruled out his chances for an appointment at an English university. Strangely, it was the Irish "problem" that gave Boole an opening. Among the many bitter complaints in Ireland concerning English rule was the Protestant character of their only university, Trinity College in Dublin. In response it was proposed by the British government to found three new universities to be called "Queen's Colleges" in Cork, Belfast, and Galway. Remarkably for the time, they would be non-denominational. Despite denunciations by Irish political and religious figures, who demanded institutions of a definitely Catholic character, the plans moved forward. Boole decided to apply for an appointment at one of these universities, and finally three years later, in 1849, he was appointed Professor of Mathematics at Queen's College in Cork.

By 1849, Ireland had come through the worst of the disaster of famine and disease brought by the potato blight, a devastating fungus that destroyed most of the potato crops on which the Irish poor depended. Many of those who did not starve to death were killed by the epidemics of typhus, dysentery, cholera, and relapsing fever to which their weakened immune systems had laid them open. The English rulers, slow to recognize the fungus as the underlying cause of the catastrophe, instead blamed the supposed indolence of the Irish. This social analysis was used to justify the continuing export of food from Ireland while millions went hungry and starved. Between 1845 and 1852, out of eight million Irish, at least a million died and another one and a half million emigrated.[14]

Boole had little if anything to say about this: his strong expressions of indignation centered on cruelty to animals. Indeed, his attitude to the Irish people was rather equivocal as emerges from these lines from a sonnet to Ireland that Boole wrote just as the college in Cork was being inaugurated:

> Yet thou in *wisdom* still art young, though old
> In misery and tears. Oh that thy store
> Of bitter thoughts, which brood upon the past,
> Were from thy bosom quite erased and worn.[15]

Although Cork was certainly no major intellectual or cultural center, the position provided Boole with the possibility of a life far more appropriate to his stature as one of the great mathematicians of the century than that of a schoolmaster. His father had recently died and, after making suitable provision for his mother, he was finally freed from the burden of being the family provider, and could think of having a personal life. The mathematics taught at Cork was at a rather low level for a university. The syllabus began with "Fractional and Decimal Arithmetic" and continued with topics taught today in secondary school. Boole's annual salary was £250 in addition to a direct tuition fee of about £2 per term from each student. Since he had no assistant, he did all the grading of the weekly homework assignments himself.

Controversy over the Queen's Colleges continued. Although Cork's president was the distinguished Catholic scientist Sir Robert Kane, Catholics were certainly under-represented: of the academic staff of 21, only one other was Catholic. The Catholic Church hierarchy had actually gone so far as to forbid members of the clergy from participating in the work of the Colleges. Some felt that Irish candidates for positions were deliberately passed over for relatively mediocre Englishmen or Scots. Nor did President Kane endear himself to his faculty. His wife had no wish to live in Cork, and so the President tried to run the college from Dublin. This, combined with his arbitrary pugnacious manner, led to one fight after another between the President and the faculty, sterile battles in which Boole usually found himself involved.[16]

Mary Everest, Boole's wife-to-be, later recounted some of her first impressions of the attitudes of some of the residents of Cork towards the man she would marry. One lady's answer to the question "What is the Professor of Mathematics like?" was "Oh he's like—the sort of man to trust your daughter with." Another lady explained the absence of her young children by informing Miss Everest that George Boole had taken them for a walk and that she was always happy when he had them. To the reply that Boole seemed to be a general favorite, the lady demurred:

> He is no favorite of mine, ... at least, I don't enjoy his society. I don't care to be with such very good people. ... he never shows you that he thinks you wicked, but when you are near anyone so pure and holy, you can't help feeling how shocked he must be at you. He makes me feel very wicked; but I am always at ease when the children are with him; I know they are getting some good.[17]

Mary Everest was the daughter of an eccentric clergyman and a niece of Lieutenant-Colonel Sir George Everest, whose name was given to the world's tallest mountain. She was also a niece of Boole's friend and colleague, John Ryall, Vice-President and Professor of Greek at Cork; this family connection brought George and Mary together. As a child Mary had displayed an aptitude for mathematics and after George began to tutor her, they grew to be good friends and frequent letter writers. It seems that Boole believed that their 17-year age difference precluded anything more, but five years after their first meeting when Boole was 40, matters came to a head with the death of Mary's father. As Mary was apparently left financially impoverished, George proposed at once, and they were married before the year was out.

Their marriage would last a mere nine years, for Boole died at the age of only 49, after walking three miles to class in a cold October rainstorm. The ensuing bronchitis soon became pneumonia, and he died two weeks later. Tragically, his death may have been hastened by his wife's crank medical views—apparently she treated his pneumonia by placing him between cold soaking bed sheets.[18]

The marriage had evidently been a very happy one.[19] Mary Boole recalled it as being "like the remembrance of a sunny dream." They had five children, all girls. Boole's widow lived well into the twentieth century, dying at the age of 84 while the First World War raged across the channel. She became attached to various systems of mystical belief and wrote a great deal of nonsense. Boole's daughters all had interesting lives. The third daughter, Alicia, possessed a very remarkable geometric ability: she was able to visualize clearly geometric objects in four dimensions. This enabled her to make a number of important mathematical discoveries. However, the youngest daughter Ethel Lilian was the most astonishing. She was only six months old when her father died and she remembered her childhood as one of terrible poverty. Lily, as she was called, became involved with the circle of Russian revolutionary emigres that had made London their home during the late years of the nineteenth century. While on a trip to the Russian empire (which at that time included much of Poland) to help her revolutionary friends, she was seen by her future husband, Wilfred Voynich, from his prison cell, as she stared up at the Warsaw Citadel. Voynich recognized her years later after he had made his escape to London. This romantic beginning led to their marriage.

Lily became famous later as the author of *The Gadfly*, a novel inspired by her short but passionate love affair with the man who became known as Sidney Riley and whose incredible life formed the basis for a television mini-series called *Riley: Ace of Spies*. With irony piled upon irony, Riley, a fervent anti-communist, was executed in Russia by the Bolsheviks, while his lover's novel, its true inspiration unknown, became required reading for Russian schoolchildren. In 1955 *Pravda* reported to its Moscow readers that the author of *The Gadfly* was alive and well in New York, and she received from Russia a royalty check for $15,000. She died five years later at the age of 96.[20]

George Boole's Algebra of Logic

Returning to Boole's new algebra applied to logic, we recall that if x and y represent two classes, then Boole would write xy to stand for the class of those things that belong to both x and y, and that he intended the notation to suggest an analogy with multiplication in ordinary algebra. In contemporary terminology, xy is called the *intersection* of x and y.[21] We also saw that the equation $xx = x$ is always true when x represents a class. This led Boole to ask the question: *in ordinary algebra, where x stands for a number, when is the equation $xx = x$ true?* The answer is straightforward: the equation is true when x is 0 or 1 and for no other numbers. This led Boole to the principle that the algebra of logic was precisely what ordinary algebra would become if it were restricted to the two values 0 and 1. However, to make sense of this, it became necessary to reinterpret the symbols 0 and 1 as classes. A clue is provided by the behavior of 0 and 1, respectively, with respect to ordinary multiplication: 0 *times any number is* 0; 1 *times any number is that very number.* In symbols,

$$0x = 0, \quad 1x = x.$$

Now for classes, $0x$ will be identical to 0 for every x, if *we interpret* 0 *to be a class to which nothing belongs*, in modern terminology, 0 is the *empty set*. Likewise, $1x$ will be identical to x for every x, if 1 *contains every object under consideration*, or, as we may say, 1 is the "universe of discourse."

Ordinary algebra deals with addition and subtraction as well as multiplication. Thus, if Boole was to present the algebra of logic as just ordinary algebra with the special rule $xx = x$, he had to provide an interpretation for + and −. So, if x and y represent two classes, Boole took $x + y$ to represent the class of all things to be found either in x or in y, nowadays called the *union* of x and y. Thus, to use Boole's own example, if x is the class of men and y is the class of women, then $x + y$ is the class consisting of all men and women. Also, Boole wrote $x - y$ for the class of things in x that are not in y.[22] If x represents the class of all people and y represents the class of all children, then $x - y$ would represent the class of adults. In particular, $1 - x$ would be the class of things not in x, so that

$$x + (1 - x) = 1.$$

Let us see how Boole's algebra works. Using ordinary algebraic notation, let us write x^2 for xx. So Boole's basic rule can be written as $x^2 = x$ or $x - x^2 = 0$. Factoring this equation, following the usual rules of algebra,

$$x(1 - x) = 0.$$

In words: *nothing can both belong and fail to belong to a given class x.* For Boole, this was an exciting result, helping to convince him that he was on the right track. For as he said, quoting Aristotle's *Metaphysics*, this equation expresses precisely:

> ... that "principle of contradiction" which Aristotle has described as the fundamental axiom of all philosophy. "It is impossible that the same quality should both belong and not belong to the same thing ... This is the most certain of all principles ... Wherefore they who demonstrate refer to this as an ultimate opinion. For it is by nature the source of all the other axioms ... "[23]

Boole must have been delighted to obtain confirmation such as every scientist seeks when introducing new and general ideas: to see an important earlier landmark turn out to be a mere particular application of the new ideas, in this case Aristotle's principle of contradiction. In fact in Boole's time, it was common for writers on logic to equate the entire subject with what Aristotle had done so many centuries earlier. As Boole put it, this was to maintain that "the science of Logic enjoys an immunity from those conditions of imperfection and of progress to which all other sciences are subject ..." The part of logic that Aristotle studied deals with inferences, called *syllogisms*, of a very special and restricted kind. They are inferences from a pair of propositions called *premises* to another proposition, called

the *conclusion*. The premises and conclusions must be representable by sentences of one of the following four types:*

Sentence type	Example
All X are Y	All horses are animals.
No X are Y	No trees are animals.
Some X are Y	Some horses are pure-bred.
Some X are not Y	Some horses are not pure-bred.

The following is an example of a valid syllogism:

All X are Y
<u>All Y are Z</u>
All X are Z

That this syllogism is *valid* means that whatever properties are substituted for X, Y, and Z, so long as the given two premises are true, the conclusion will be as well. Here are two instances of this syllogism:

All horses are mammals.	All boojums are snarks.
<u>All mammals are vertebrates.</u>	<u>All snarks are purple.</u>
All horses are vertebrates.	All boojums are purple.

Boole's algebraic methods can easily be used to demonstrate that this syllogism is valid. To say that everything in X also belongs to Y is the same as to say that there is nothing that belongs to X but not to Y, i.e., $X(1 - Y) = 0$ or equivalently $X = XY$. Likewise, the second premise can be written $Y = YZ$. Using these equations we get

$$X = XY = X(YZ) = (XY)Z = XZ,$$

the desired conclusion.[24]

Of course, not every proposed syllogism is valid. An example of an *invalid syllogism* can be obtained by interchanging the second premise with the conclusion in the previous example:

*In Carroll (1988, pp. 258–259), Lewis Carroll tells us that in a "sillygism" one proceeds from two "prim Misses" to a "delusion."

All X are Y
<u>All X are Z</u>
All Y are Z

This time there is no way to use the premises $X = XY$ and $X = YZ$ to obtain the supposed conclusion $Y = YZ$.

In retrospect, it is difficult to understand the widespread belief that syllogistic reasoning constituted the whole of logic, and Boole was quite scathing in his denunciation of this idea. He pointed out that much ordinary reasoning involves what he calls *secondary propositions*, that is, propositions that express relations between other propositions. Such reasoning is not syllogistic.

For a simple example of such reasoning, let us listen in on a conversation between Joe and Susan. Joe can't find his checkbook and Susan is helping him.

SUSAN: Did you leave it in the supermarket when you were shopping?

JOE: No, I telephoned them, and they didn't find it. If I had left it there, they surely would have found it.

SUSAN: Wait a minute! You wrote a check at the restaurant last night and I saw you put your checkbook in your jacket pocket. If you haven't used it since, it must still be there.

JOE: You're right. I haven't used it. It's in my jacket pocket.

Joe looks and (if it's a good day for logic), the missing checkbook is there. Let us see how Boole's algebra could be used to analyze Joe and Susan's reasoning.

In their reasoning, Joe and Susan were dealing with the following propositions (each labeled with a letter):

L Joe left his checkbook at the supermarket

F Joe's checkbook was found at the supermarket

W Joe wrote a check at the restaurant last night

P After writing the check last night, Joe put his check book in his jacket pocket

H Joe hasn't used his check book since last night

S Joe's checkbook is still in his jacket pocket

They used the following pattern:

PREMISES. If L, then F
Not F
W & P
If W & P & H, then S
H

CONCLUSIONS. Not L
S

Like Aristotle's syllogisms, this pattern forms a valid inference. As with any valid inference, the truth of sentences called *conclusions* is inferred from the truth of other sentences called *premises*.

Boole saw that the same algebra that worked for classes would also work for inferences of this kind.[25] Boole used an equation like $X = 1$ to mean that the proposition X is true; likewise he used the equation $X = 0$ to mean that X is false. Thus, for "Not X," he could write the equation $X = 0$. Also, for "X&Y" he wrote the equation $XY = 1$. This works because X&Y is true precisely when X and Y are both true, while algebraically, $XY = 1$ if $X = Y = 1$, but $XY = 0$ if either $X = 0$ or $Y = 0$ (or both).

Finally, the statement "If X then Y" can be represented by the equation

$$X(1-Y) = 0.$$

To see this, think of this statement as asserting

$$\textit{if } X = 1 \textit{ then } Y = 1$$

But indeed, substituting $X = 1$ in the proposed equation leads to $1-Y = 0$, that is, to $Y = 1$.

Using these ideas, Joe and Susan's premises can be expressed by the equations

$$\begin{aligned} L(1-F) &= 0, \\ F &= 0, \\ WP &= 1, \\ WPH(1-S) &= 0, \\ H &= 1. \end{aligned}$$

Substituting the second equation in the first, we get $L = 0$, the first desired conclusion. Substituting the third and fifth equations in the fourth, we get $1 - S = 0$ that is $S = 1$, the other desired conclusion.

Now of course, Joe and Susan had no need for this algebra. But the fact that the kind of reasoning that ordinarily takes place informally and implicitly in ordinary human interactions could be captured by Boole's algebra encouraged the hope that more complicated reasoning could be captured as well. Mathematics may be thought of as systematically encapsulating highly complex logical inferences. This is part of the reason that mathematics is so useful in science. So an ultimate test of a theory of logic that aims at completeness is whether it encompasses all mathematical reasoning. We will return to this matter in the next chapter.

As a final example of Boole's methods, we turn to Samuel Clarke's proof of the existence of God mentioned at the beginning of this chapter. Without trying to follow Clarke's long complex deduction, it is at least amusing to see how Boole proceeds. We quote a small fragment:[26]

> The premises are:—
> 1st. Something is.
> 2nd. If something is, either something always was, or the things that now are have risen out of nothing.
> 3rd. If something is, either it exists in the necessity of its own nature, or it exists by the will of another being.
> 4th. If it exists in the necessity of its own nature, something always was.
> 5th. If it exists by the will of another being, then the hypothesis that the things which now are have risen out of nothing, is false.
> We must now express symbolically the above propositions.

Let

x = Something is.
y = Something always was.
z = The things that now are have risen out of nothing.
p = It exists in the necessity of its own nature (i.e., the *something* spoken of above).
q = It exists by the will of another Being.

Boole then obtains from the premises the equations

$$\begin{aligned} 1-x &= 0, \\ x\{yz+(1-y)(1-z)\} &= 0, \\ x\{pq+(1-p)(1-q)\} &= 0, \\ p(1-y) &= 0, \\ qz &= 0. \end{aligned}$$

One wonders what Clarke would have made of this reduction of his intricate metaphysical reasoning to manipulations of simple equations. Likely, as a disciple of Newton, he would have been pleased. On the other hand, the pugnacious metaphysician Sir William Hamilton who hated mathematics so very much must have been horrified.

Boole and Leibniz's Dream

Boole's system of logic included Aristotle's and went far beyond it. But it still fell far short of what was needed to fulfill Leibniz's dream. Consider the following sentence:

All failing students are either stupid or lazy.

One might think of this sentence as being of the type:

All X are Y

However, this would require that the class of students being stupid or lazy be treated as a unit and would not permit any reasoning that sought to distinguish those who were failing because of stupidity from those who were failing because of laziness. In the next chapter we'll see how Gottlob Frege's system of logic does include reasoning of this subtler kind.

It is quite straightforward to use Boole's algebra as a system of rules for calculating, and so we may say, that within its limits, it provided the *calculus ratiocinator* Leibniz had sought. Leibniz's writings on these matters were in the form of letters and other unpublished documents, and it was only late in the nineteenth century that a serious effort to gather and publish these was undertaken. So, there is no reasonable way that Boole could have been aware of his predecessor's efforts. Nevertheless it is interesting to compare Boole's full-blown system with Leibniz's fragmentary attempts. Leibniz's fragment quoted in our first chapter included as its second axiom, $A \oplus A = A$. Thus the operation Leibniz considered was to obey Boole's fundamental rule: $xx = x$. Moreover, Leibniz proposed to present his logic as a full-fledged deductive system in which all of the rules are deduced from a small set of axioms. This is in accord with modern practice and shows Leibniz, in this respect, to have been ahead of Boole.

George Boole's great achievement was to demonstrate once and for all that logical deduction could be developed as a branch of mathematics. Although there had been some developments in logic after Aristotle's pioneering work (notably by the stoics in Hellenistic times and by the twelfth century scholastics in Europe), Boole had found the subject essentially as Aristotle left it two millennia earlier. After Boole, mathematical logic has been under uninterrupted development to the present day.*

*An international organization, the *Association for Symbolic Logic* publishes two quarterly journals and holds regular meetings for the dissemination of new research. European logicians have their own annual meetings. New work on the relationships between logic and computers is presented at the annual international *Logic in Computer Science* and *Computer Science Logic* conferences.

CHAPTER 3

Frege: From Breakthrough to Despair

In June 1902 a letter arrived in Jena, a medieval German town, addressed to the 53-year-old Gottlob Frege from the young British philosopher Bertrand Russell. Although Frege believed that he had made important and fundamental discoveries, his work had been almost totally ignored. It must then have been with some pleasure that he read, "I find myself in agreement with you in all essentials ... I find in your work discussions, distinctions, and definitions that one seeks in vain in the work of other logicians." But, the letter continued, "There is just one point where I have encountered a difficulty." Frege soon realized that this one "difficulty" seemed to lead to the collapse of his life's work. It cannot have helped too much that Russell went on to write, "The exact treatment of logic in fundamental questions has remained very much behind; in your works I find the best I know of our time, and therefore I have permitted myself to express my deep respect to you."

Frege replied at once to Russell acknowledging the problem. The second volume of his treatise in which he had applied his logical methods to the foundations of arithmetic was already at the printer, and he hastily added an Appendix beginning with the words, "There is nothing worse that can happen to a scientist than to have the foundation collapse just as the work is finished. I have been placed in this position by a letter from Mr. Bertrand Russell ..."

Many years later, more than four decades after Frege's death, Bertrand Russell had occasion to write:

> As I think about acts of integrity and grace, I realize that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less

"infinitesimal" numbers. Infinitesimals were supposed to be positive numbers so very tiny that no matter how many times such a number is added to itself, the number 1 (or even the number .0000001) will never be reached. The legitimacy of such quantities was challenged from the outset; the philosopher Bishop Berkeley scoffed at infinitesimals as "ghosts of departed quantities." By the end of the nineteenth century, mathematicians were in agreement that the use of infinitesimals could not be justified (although physicists and engineers continued to employ them). Discussion of infinitesimal methods as used by Leibniz as well as their eventual rehabilitation in the twentieth century by the logician Abraham Robinson will be found in the book (Edwards, 1979) already cited. The *Scientific American* article (Davis and Hersh, 1972) gives another account of Robinson's achievement.

[8]Aiton (1985, p. 53).

[9]Mates (1986, p. 27). See also pp. 26–27 of this source for more about about these remarkable women, for information about Leibniz's beliefs about the intellectual capabilities of women, and for further references.

[10]The letter to L'Hospital quoted was dated April 28, 1693. (Couturat, 1961, p. 83). The quote from Couterat is from the same page of the same source. For the "thread of Ariadne" see Bourbaki (1969, p. 16).

[11]The letter from Leibniz to Jean Galloys Leibniz (1849/1962) on his Universal Characteristic was dated December 1678. The translation from French is mine.

[12]Gerhardt (1978, vol. 7, p. 200).

[13]Parkinson (1966, p. 105).

[14]For Leibniz's logical calculus of which a small "sample" is exhibited here, see Lewis (1918/1960, pp. 297–305). Leibniz did not use the = symbol, but instead used ∞. The interesting article (Swoyer, 1994) gives a thorough reconstruction of this system from a twentieth-century perspective.

[15]For some discussion of Leibniz's attempts to go beyond Aristotle's analysis, see Mates (1986, pp. 178–183).

[16]Huber (1951, pp. 267–269).

[17]Aiton (1985, p. 212).

Chapter 2

[1]Information about Leibniz's friendship with Princess Caroline and his correspondence with Samuel Clarke is from Aiton (1985, pp. 232,341–346) and the articles "Caroline (1683–1737)" and "Clarke, Samuel (1675–1729)" in Britanica (1910/11).

[2]Biographical information about George Boole is principally from MacHale (1985).

[3]MacHale (1985, pp. 17–19).

[4]"Gross appetites and passions" (MacHale, 1985, p. 19).

[5]MacHale (1985, pp. 30–31).

[6]MacHale (1985, pp. 24–25).

[7]MacHale (1985, p. 41).

[8]Among the most important of the laws of algebra are the *commutative* laws for addition and multiplication:

$$x + y = y + x, \qquad xy = yx,$$

and the *distributive* law

$$x(y + z) = xy + xz.$$

We are using the usual algebraic convention of writing, for example, xy instead of $x \times y$.

[9]Multiplication of two differential operators (which is taken to mean applying first one and then the other) doesn't always obey the commutative law.

[10]Boole's gold medal (MacHale, 1985, pp. 59–62, 64–66). In addition to Boole's work employing the methods of the calculus, he published a paper in two parts in the *Cambridge Mathematical Journal* for 1842 that can be thought of as founding a new and important branch of algebra, the theory of invariants. However, after this first contribution, Boole never again worked on invariants. We will be considering invariants again in the chapter on David Hilbert.

[11]Boole's casual attitude to proof in connection with limit processes may be contrasted with contemporary efforts on the continent to develop an appropriate rigorous foundation for such matters. Interested readers are referred to Edwards (1979), especially Chapter 11.

[12]The Scottish philosopher Sir William Hamilton is not to be confused with his contemporary, the Scottish mathematician, Sir William Rowan Hamilton.

[13]Boole (1854, pp. 28–29).

[14]Daly (1996); Kinealy (1996)

[15]MacHale (1985, p. 173).

[16]MacHale (1985, p. 92).

[17]MacHale (1985, p. 107).

[18]MacHale (1985, pp. 240–243).

[19]MacHale (1985, p. 111).

[20]MacHale (1985, pp. 252–276).

[21]The modern notation for the intersection of x and y is $x \cap y$ rather than xy. Also the empty set is usually represented by the Danish letter $\emptyset$ rather than by 0. Of course the notation he used was important for Boole because it made it easy to connect with ordinary algebra.

[22]Boole restricted the operation + to classes having no elements in common. Here we follow contemporary usage and do not enforce this restriction. So $x + y$ is the class of things belonging to x or y or both. Nowadays one speaks of the *union* of x and y, written $x \cup y$. Also, Boole restricted the notation $x - y$ to the

case that the class that y represents is part of the class that x represents. But there is no need for this restriction either.

[23]Boole (1854, p. 49).

[24]As Boole emphasizes, what is involved algebraically in demonstrating the validity of a syllogism is the *elimination* of one variable from two simultaneous equations in three variables.

Although Boole realized perfectly well that propositions of the form "All X are Y" could be represented in his algebra as $X(1-Y)=0$, he preferred to use $X = vY$ where v is what he called an *indefinite symbol*. This was apparently suggested by the mathematician Charles Graves (MacHale, 1985, p. 70). It was really a terrible idea and a quite unnecessary complication of Boole's system.

[25]Boole's method of relating secondary propositions to his algebra of classes was to bring *time* into the picture. With each proposition Boole would in effect associate the *class* of instants of time for which that proposition was true. To say that proposition X is true, Boole would write $X = 1$ meaning that the class of instants in which the proposition is true encompasses the entire time span under consideration. Likewise, $X = 0$ would express that X is false, because there are no instants of time in which X is true. Given a proposition $X\&Y$ which expresses the truth of both X and Y, the set of instants in which it is true is just the set intersection XY. Finally, for a proposition *if* X *then* Y to be true, what is required is that any time that X is true, Y is also true, that is that there is no time when X is true and Y is false. As an equation: $X(1-Y)=0$. (Boole, 1854, pp. 162–164).

[26]Boole (1854, pp. 188–211).

Chapter 3

[1]For Russell's letter, Frege's reply, and Russell's later comment, see van Heijenoort (1967) pp. 124–128.

[2]For Frege's notorious diary as well as Michael Dummett's comment, see Frege (1996).

[3]I am very much indebted to Professor Lothar Kreiser of the University of Leipzig who graciously replied to my request for information about Frege. See his masterful biography (Kreiser, 2001). Terrall Bynum's brief biography in Bynum (1972) was also helpful.

[4]I found Craig (1978) an excellent source on German History. For the origins of the First World War, see also Geiss (1967); Kagan (1995). A number of postcards from Frege to the philosopher Ludwig Wittgenstein, who was an artillery observer in the Austrian army during the war, have survived. Not surprisingly, they show Frege to have been a patriotic German (Frege, 1976).

[5]Frege (1996).

[6]Sluga (1993); Frege (1976, pp. 8–9).

[7]For the quoted comment, see van Heijenoort (1967, p. 1). The same source includes an excellent translation of Frege's *Begriffsschrift* with commentary, pp. 1–82. Another translation is in Bynum (1972, pp. 101–166).

[8]The symbols we are using are those in common use today not those used by Frege. Of course the fundamental insight was recognizing what needed to be symbolized rather than what specific symbols were used. Frege's were not widely adopted in part because they presented difficulties for the typesetter but mainly because the notation used by the Italian logician Giuseppe Peano as adapted by Bertrand Russell became much better known.

[9]Frege wrote "... what I wanted to create was not a mere *calculus ratiocinator* but a *lingua characteta* in Leibniz's sense." Quoted in van Heijenoort (1967, p. 2). See also Kluge (1977).

[10]This rule is known as *modus ponens*. The terminology derives from the scholastic logicians of the twelfth century.

[11]What we are calling Frege's logic is usually called *first-order logic*. This is to distinguish it from systems of logic in which the quantifiers $\forall$ and $\exists$ are applied to properties as well as to individuals. Here is an example of a sentence in what is known as *second-order logic*:

$$(\forall F)(\forall G)[(\forall x)(F(x) \supset G(x)) \supset (\exists x)(F(x) \supset (\exists x)G(x))]$$

Actually, Frege went beyond first-order logic in that he did consider quantification of properties; so our speaking of first-order logic as "Frege's logic" is not quite accurate.

[12]Strictly speaking, this explanation of "number," is closer to what Bertrand Russell proposed than to Frege's own exposition. But it is close enough to show why it was vulnerable to Russell's paradox.

[13]Interesting work done while this book was being written showed that a considerable part of Frege's program for the logical development of arithmetic can be saved (Boolos, 1995).

[14]Frege (1960).

[15]Dummett (1981); Baker and Hacker (1984).

[16]For clarity it is important to be able to state precisely the meaning of the locutions that occur in computer programming languages, or as one says, to provide the *semantics* of such a language. One approach to this question that has been much studied, known as *denotational semantics*, is ultimately based on Frege's ideas. See Davis et al. (1994, pp. 465–556).

Chapter 4

[1]Rucker (1982, p. 3).

[2]Quoted from Dauben (1979, p.124). This is a translation from Leibniz's original in French (Cantor, 1932, p 179).