ARISTOTLE'S SYLLOGISTIC

FROM THE STANDPOINT OF MODERN FORMAL LOGIC

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are rejected, then the expression 'If α and β, then γ' must be rejected too.[1] This rule, together with the rules of rejection (*c*) and (*d*) and the axiomatically rejected expression 'If all *C* is *B* and all *A* is *B*, then some *A* is *C*', enables us to reject any false expression of the system. Besides, we suppose as given the four asserted axioms of the syllogistic, the definitions of the *E*- and the *O*-premiss, the rules of inference for asserted expressions, and the theory of deduction as an auxiliary system. In this way the problem of decision finds its solution: for any given significant expression of the system we can decide whether it is true and may be asserted or whether it is false and must be rejected.

By the solution of this problem the main investigations on Aristotle's syllogistic are brought to an end. There remains only one problem, or rather one mysterious point waiting for an explanation: in order to reject all the false expressions of the system it is necessary and sufficient to reject axiomatically only one false expression, viz. the syllogistic form of the second figure with universal affirmative premisses and a particular affirmative conclusion. There exists no other expression suitable for this purpose. The explanation of this curious logical fact may perhaps lead to new discoveries in the field of logic.

[1] J. Słupecki, 'Z badań nad sylogistyką Arystotelesa' (Investigation on Aristotle's Syllogistic), *Travaux de la Société des Sciences et des Lettres de Wrocław*, Sér. B, No. 9, Wrocław (1948). See chapter v, devoted to the problem of decision.

CHAPTER IV

ARISTOTLE'S SYSTEM IN SYMBOLIC FORM

§ 22. *Explanation of the symbolism*

THIS chapter does not belong to the history of logic. Its purpose is to set out the system of non-modal syllogisms according to the requirements of modern formal logic, but in close connexion with the ideas set forth by Aristotle himself.

Modern formal logic is strictly formalistic. In order to get an exactly formalized theory it is more convenient to employ a symbolism invented for this purpose than to make use of ordinary language which has its own grammatical laws. I have therefore to start from the explanation of such a symbolism. As the Aristotelian syllogistic involves the most elementary part of the propositional logic called theory of deduction, I shall explain the symbolic notation of both these theories.

In both theories there occur variables and constants. Variables are denoted by small Latin letters, constants by Latin capitals. By the initial letters of the alphabet *a*, *b*, *c*, *d*, ..., I denote term-variables of the Aristotelian logic. These term-variables have as values universal terms, as 'man' or 'animal'. For the constants of this logic I employ the capital letters *A*, *E*, *I*, and *O*, used already in this sense by the medieval logicians. By means of these two kinds of letters I form the four functions of the Aristotelian logic, writing the constants before the variables:

Aab	means	All *a* is *b*	or *b* belongs to all *a*,
Eab	,,	No *a* is *b*	,, *b* belongs to no *a*,
Iab	,,	Some *a* is *b*	,, *b* belongs to some *a*,
Oab	,,	Some *a* is not *b*	,, *b* does not belong to some *a*.

The constants *A*, *E*, *I*, and *O* are called functors, *a* and *b* their arguments. All Aristotelian syllogisms are composed of these four types of function connected with each other by means of the words 'if' and 'and'. These words also denote functors, but of a different kind from the Aristotelian constants: their arguments are not term-expressions, i.e. concrete terms or term-variables, but propositional expressions, i.e. propositions like

'All men are animals', propositional functions like '*Aab*', or propositional variables. I denote propositional variables by *p*, *q*, *r*, *s*, ..., the functor 'if' by *C*, the functor 'and' by *K*. The expression *Cpq* means 'if *p*, then *q*' ('then' may be omitted) and is called 'implication' with *p* as the antecedent and *q* as the consequent. *C* does not belong to the antecedent, it only combines the antecedent with the consequent. The expression *Kpq* means '*p* and *q*' and is called 'conjunction'. We shall meet in some proofs a third functor of propositional logic, propositional negation. This is a functor of one argument and is denoted by *N*. It is difficult to render the function *Np* either in English or in any other modern language, as there exists no single word for the propositional negation.[1] We have to say by circumlocution 'it-is-not-true-that *p*' or 'it-is-not-the-case-that *p*'. For the sake of brevity I shall use the expression 'not-*p*'.

The principle of my notation is to write the functors before the arguments. In this way I can avoid brackets. This symbolism without brackets, which I invented and have employed in my logical papers since 1929,[2] can be applied to mathematics as well as to logic. The associative law of addition runs in the ordinary notation thus:

$$(a+b)+c = a+(b+c),$$

and cannot be stated without brackets. If you write, however, the functor + before its arguments, you get:

$$(a+b)+c = ++abc \quad \text{and} \quad a+(b+c) = +a+bc.$$

The law of association can be now written without brackets:

$$++abc = +a+bc.$$

Now I shall explain some expressions written down in this symbolic notation. The symbolic expression of a syllogism is easy to understand. Take, for instance, the mood Barbara:

If all *b* is *c* and all *a* is *b*, then all *a* is *c*.

It reads in symbols:

CKAbcAabAac.

[1] The Stoics used for propositional negation the single word οὐχί.

[2] See, for instance, Łukasiewicz and Tarski, 'Untersuchungen über den Aussagenkalkül', *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie*, xxiii (1930), Cl. III, pp. 31–2.

The conjunction of the premisses *Abc* and *Aab*, viz. *KAbcAab*, is the antecedent of the formula, the conclusion *Aac* is its consequent.

Some expressions of the theory of deduction are more complicated. Take the symbolic expression of the hypothetical syllogism:

If (if *p*, then *q*), then [if (if *q*, then *r*), then (if *p*, then *r*)].

It reads:

CCpqCCqrCpr.

In order to understand the construction of this formula you must remember that *C* is a functor of two propositional arguments which follow immediately after *C*, forming together with *C* a new compound propositional expression. Of this kind are the expressions *Cpq*, *Cqr*, and *Cpr* contained in the formula. Draw brackets around each of them; you will get the expression:

C(*Cpq*)*C*(*Cqr*)(*Cpr*).

Now you can easily see that (*Cpq*) is the antecedent of the whole formula, and the rest, i.e. *C*(*Cqr*)(*Cpr*), is the consequent, having (*Cqr*) as its antecedent and (*Cpr*) as its consequent.

In the same way we may analyse all the other expressions, for instance the following, which contains *N* and *K* besides *C*:

CCKpqrCKNrqNp.

Remember that *K*, like *C*, is a functor of two arguments, and that *N* is a functor of one argument. By using different kinds of brackets we get the expression:

C[*C*(*Kpq*)*r*]{*C*[*K*(*Nr*)*q*](*Np*)}.

[*C*(*Kpq*)*r*] is here the antecedent of the whole formula while {*C*[*K*(*Nr*)*q*](*Np*)} is its consequent, having the conjunction [*K*(*Nr*)*q*] as its antecedent and the negation (*Np*) as its consequent.

§ 23. *Theory of deduction*

The most fundamental logical system on which all the other logical systems are built up is the theory of deduction. As every logician is bound to know this system, I shall here describe it in brief.

The theory of deduction can be axiomatized in several different ways, according to which functors are chosen as primitive terms. The simplest way is to follow Frege, who takes as primitive terms the functors of implication and negation, in our symbolism *C* and *N*. There exist many sets of axioms of the *C–N*-system; the simplest of them and the one almost universally accepted was discovered by myself before 1929.[1] It consists of three axioms:

T1. *CCpqCCqrCpr*
T2. *CCNppp*
T3. *CpCNpq*.

The first axiom is the law of the hypothetical syllogism already explained in the foregoing section. The second axiom, which reads in words 'If (if not-*p*, then *p*), then *p*', was applied by Euclid to the proof of a mathematical theorem.[2] I call it the law of Clavius, as Clavius (a learned Jesuit living in the second half of the sixteenth century, one of the constructors of the Gregorian calendar) first drew attention to this law in his commentary on Euclid. The third axiom, in words 'If *p*, then if not-*p*, then *q*', occurs for the first time, as far as I know, in a commentary on Aristotle ascribed to Duns Scotus; I call it the law of Duns Scotus.[3] This law contains the venom usually imputed to contradiction: if two contradictory sentences, like α and $N\alpha$, were true together, we could derive from them by means of this law the arbitrary proposition *q*, i.e. any proposition whatever.

There belong to the system two rules of inference: the rule of substitution and the rule of detachment.

The rule of substitution allows us to deduce new theses from a thesis asserted in the system by writing instead of a variable a significant expression, everywhere the same for the same variable. Significant expressions are defined inductively in the following way: (*a*) any propositional variable is a significant expression; (*b*) $N\alpha$ is a significant expression provided α is a

[1] First published in Polish: 'O znaczeniu i potrzebach logiki matematycznej' (On the Importance and Requirements of Mathematical Logic), *Nauka Polska*, vol. x, Warsaw (1929), pp. 610–12. Cf. also the German contribution quoted in p. 78, n. 2: Satz 6, p. 35.

[2] See above, section 16.

[3] Cf. my paper quoted in p. 48, n.

significant expression; (*c*) $C\alpha\beta$ is a significant expression provided α and β are significant expressions.

The rule of detachment is the *modus ponens* of the Stoics referred to above: if a proposition of the type $C\alpha\beta$ is asserted and its antecedent α is asserted too, it is permissible to assert its consequent β, and detach it from the implication as a new thesis.

By means of these two rules we can deduce from our set of axioms all the true theses of the *C–N*-system. If we want to have in the system other functors besides *C* and *N*, e.g. *K*, we must introduce them by definitions. This can be done in two different ways, as I shall show on the example of *K*. The conjunction '*p* and *q*' means the same as 'it-is-not-true-that (if *p*, then not-*q*)'. This connexion between *Kpq* and *NCpNq* may be expressed by the formula:

$$Kpq = NCpNq,$$

where the sign = corresponds to the words 'means the same as'. This kind of definition requires a special rule of inference allowing us to replace the *definiens* by the *definiendum* and vice versa. Or we may express the connexion between *Kpq* and *NCpNq* by an equivalence, and as equivalence is not a primitive term of our system, by two implications converse to each other:

$$CKpqNCpNq \quad \text{and} \quad CNCpNqKpq.$$

In this case a special definition-rule is not needed. I shall use definitions of the first kind.

Let us now see by an example how new theses can be derived from the axioms by the help of rules of inference. I shall deduce from T1–T3 the law of identity *Cpp*. The deduction requires two applications of the rule of substitution and two applications of the rule of detachment; it runs thus:

T1. $q/CNpq \times C$T3–T4
T4. *CCCNpqrCpr*
T4. $q/p, r/p \times C$T2–T5
T5. *Cpp*.

The first line is called the derivational line. It consists of two parts separated from each other by the sign ×. The first part, T1. $q/CNpq$, means that in T1 *CNpq* has to be substituted for *q*. The thesis produced by this substitution is omitted in order to save space. It would be of the following form:

(I) *CCpCNpqCCCNpqrCpr*.

The second part, *C*T3–T4, shows how this omitted thesis is constructed, making it obvious that the rule of detachment may be applied to it. Thesis (I) begins with *C*, and then there follow axiom T3 as antecedent and thesis T4 as consequent. We can therefore detach T4 as a new thesis. The derivational line before T5 has a similar explanation. The stroke (/) is the sign of substitution and the short rule (–) the sign of detachment. Almost all subsequent deductions are performed in the same manner.

One must be very expert in performing such proofs if one wants to deduce from the axioms T1–T3 the law of commutation *CCpCqrCqCpr* or even the law of simplification *CpCqp*. I shall therefore explain an easy method of verifying expressions of our system without deducing them from the axioms. This method, invented by the American logician Charles S. Peirce about 1885, is based on the so-called principle of bivalence, which states that every proposition is either true or false, i.e. that it has one and only one of two possible truth-values: truth and falsity. This principle must not be mixed up with the law of the excluded middle, according to which of two contradictory propositions one must be true. It was stated as the basis of logic by the Stoics, in particular by Chrysippus.[1]

All functions of the theory of deduction are truth-functions, i.e. their truth and falsity depend only upon the truth and falsity of their arguments. Let us denote a constant false proposition by *o*, and a constant true proposition by *1*. We may define negation in the following way:

$$No = 1 \quad \text{and} \quad N1 = o.$$

This means: the negation of a false proposition means the same as a true proposition (or, shortly, is true) and the negation of a true proposition is false. For implication we have the following four definitions:

$$Coo = 1, \quad Co1 = 1, \quad C1o = o, \quad C11 = 1.$$

[1] Cicero, *Acad. pr.* ii. 95 'Fundamentum dialecticae est, quidquid enuntietur (id autem appellant ἀξίωμα) aut verum esse aut falsum'; *De fato* 21 'Itaque contendit omnes nervos Chrysippus ut persuadeat omne ἀξίωμα aut verum esse aut falsum.' In the Stoic terminology ἀξίωμα means 'proposition', not 'axiom'.

This means: an implication is false only when its antecedent is true and its consequent false; in all the other cases it is true. This is the oldest definition of implication, stated by Philon of Megara and adopted by the Stoics.[1] For conjunction we have the four evident equalities:

$$Koo = o, \quad Ko1 = o, \quad K1o = o, \quad K11 = 1.$$

A conjunction is true only when both its arguments are true; in all the other cases it is false.

Now if we want to verify a significant expression of the theory of deduction containing all or some of the functors *C*, *N*, and *K* we have to substitute for the variables occurring in the expression the symbols *o* and *1* in all possible permutations, and reduce the formulae thus obtained on the basis of the equalities given above. If after the reduction all the formulae give *1* as the final result, the expression is true or a thesis; if any one of them gives *o* as the final result, the expression is false. Let us take as an example of the first kind the law of transposition *CCpqCNqNp*; we get:

For $p/o,\ q/o$: $CCooCNoNo = C1C11 = C11 = 1,$
,, $p/o,\ q/1$: $CCo1CN1No = C1Co1 = C11 = 1,$
,, $p/1,\ q/o$: $CC1oCNoN1 = CoC1o = Coo = 1,$
,, $p/1,\ q/1$: $CC11CN1N1 = C1Coo = C11 = 1.$

As for all substitutions the final result is *1*, the law of transposition is a thesis of our system. Let us now take as an example of the second kind the expression *CKpNqq*. It suffices to try only one substitution:

$$p/1,\ q/o:\ CK1Noo = CK11o = C1o = o.$$

This substitution gives *o* as the final result, and therefore the expression *CKpNqq* is false. In the same way we may check all the theses of the theory of deduction employed as auxiliary premisses in Aristotle's syllogistic.

§ 24. *Quantifiers*

Aristotle had no clear idea of quantifiers and did not use them in his works; consequently we cannot introduce them into his syllogistic. But, as we have already seen, there are two points in his system which we can understand better if we explain them

[1] Sextus Empiricus, *Adv. math.* viii. 113 ὁ μὲν Φίλων ἔλεγεν ἀληθὲς γίνεσθαι τὸ συνημμένον, ὅταν μὴ ἄρχηται ἀπ' ἀληθοῦς καὶ λήγῃ ἐπὶ ψεῦδος, ὥστε τριχῶς μὲν γίνεσθαι κατ' αὐτὸν ἀληθὲς συνημμένον, καθ' ἕνα δὲ τρόπον ψεῦδος.

by employing quantifiers. Universal quantifiers are connected with the so-called 'syllogistic necessity', existential or particular quantifiers with the proofs by ecthesis. I shall now translate into symbols the proofs with existential quantifiers set down in section 19, and then the argument dependent on universal quantifiers mentioned in section 5.

I denote quantifiers by Greek capitals, the universal quantifier by *Π*, and the particular or existential quantifier by *Σ*. *Π* may be read 'for all', and *Σ* 'for some' or 'there exists'; e.g. *ΣcKAcbAca* means in words: 'There exists a *c* such that all *c* is *b* and all *c* is *a*', or more briefly: 'For some *c*, all *c* is *b* and all *c* is *a*.' Every quantified expression, for instance *ΣcKAcbAca*, consists of three parts: part one, in our example *Σ*, is always a quantifier; part two, here *c*, is always a variable bound by the preceding quantifier; part three, here *KAcbAca*, is always a propositional expression containing the variable just bound by the quantifier as a free variable. It is by putting *Σc* before *KAcbAca* that the free variable *c* in this last formula becomes bound. We may put it briefly: *Σ* (part one) binds *c* (part two) in *KAcbAca* (part three).

The rules of existential quantifiers have already been set out in section 19. In derivational lines I denote by *Σ1* the rule allowing us to put *Σ* before the antecedent, and by *Σ2* the rule allowing us to put it before the consequent of a true implication. The following deductions will be easily understood, as they are translations of the deductions given in words in section 19, the corresponding theses bearing the same running number and having corresponding small letters as variables instead of capitals.

Proof of conversion of the I-premiss

Theses assumed as true without proof:

(1) *CIabΣcKAcbAca*
(2) *CΣcKAcbAcaIab*

Theses (1) and (2) can be used as a definition of the *I*-premiss.

(3) *CKpqKqp* (commutative law of conjunction)

(3) *p/Acb, q/Aca* × (4)

(4) *CKAcbAcaKAcaAcb*

(4) *Σ2c* × (5)

(5) *CKAcbAcaΣcKAcaAcb*

(5) *Σ1c* × (6)

(6) *CΣcKAcbAcaΣcKAcaAcb*

T1. *CCpqCCqrCpr* (law of the hypothetical syllogism)

T1. *p/Iab, q/ΣcKAcbAca, r/ΣcKAcaAcb* × *C*(1)–*C*(6)–(7)

(7) *CIabΣcKAcaAcb*

(2) *b/a, a/b* × (8)

(8) *CΣcKAcaAcbIba*

T1. *p/Iab, q/ΣcKAcaAcb, r/Iba* × *C*(7)–*C*(8)–(9)

(9) *CIabIba*

The derivational lines show that (4) and (8) result from other theses by substitution only, and (7) and (9) by substitution and two detachments. Upon this pattern the reader himself may try to construct the proof of the mood Darapti, which is easy.

Proof of the mood Bocardo

(The variables *P*, *R*, and *S* used in section 19 must be re-lettered, as the corresponding small letters *p*, *r*, and *s* are reserved to denote propositional variables: write *d* for *P*, *a* for *R*, and *b* for *S*.)

Thesis assumed without proof:

(15) *CObdΣcKAcbEcd*

Two syllogisms taken as premisses:

(16) *CKAcbAbaAca* (Barbara)
(17) *CKAcaEcdOad* (Felapton)

T6. *CCKpqrCCKrstCKKpqst*

This is the 'synthetic theorem' ascribed to Aristotle.

T6. *p/Acb, q/Aba, r/Aca, s/Ecd, t/Oad* × *C*(16)–*C*(17)–(18)

(18) *CKKAcbAbaEcdOad*

T7. *CCKKpqrsCKprCqs* (auxiliary thesis)

T7. *p/Acb, q/Aba, r/Ecd, s/Oad* × *C*(18)–(19)

(19) *CKAcbEcdCAbaOad*

(19) *Σ1c* × (20)

(20) *CΣcKAcbEcdCAbaOad*

T1. *CCpqCCqrCpr*

T1. *p/Obd, q/ΣcKAcbEcd, r/CAbaOad* × *C*(15)–*C*(20)–(21)

(21) *CObdCAbaOad*

This is the implicational form of the mood Bocardo. If we wish to have the usual conjunctional form of this mood, we must apply to (21) the so-called law of importation:

T8. *CCpCqrCKpqr*.

We get:

T8. $p/Obd,\ q/Aba,\ r/Oad \times C(21)\text{–}(22)$

(22) *CKObdAbaOad* (Bocardo).

By the so-called law of exportation,

T9. *CCKpqrCpCqr*,

which is the converse of the law of importation, we can get the implicational form of the mood Bocardo back from its conjunctional form.

The rules of universal quantifiers are similar to the rules of particular quantifiers set out in section 19. The universal quantifier can be put before the antecedent of a true implication unconditionally, binding a free variable occurring in the antecedent, and before the consequent of a true implication only under the condition that the variable which is to be bound in the consequent does not occur in the antecedent as a free variable. I denote the first of these rules by *Π1*, the second by *Π2*.

Two derived rules result from the above primitive rules of universal quantifiers: first, it is permissible (by rule *Π2* and the law of simplification) to put universal quantifiers in front of a true expression binding free variables occurring in it; secondly, it is permissible (by rule *Π1* and the propositional law of identity) to drop universal quantifiers standing in front of a true expression. How these rules may be derived I shall explain by the example of the law of conversion of the *I*-premiss.

From the law of conversion

(9) *CIabIba*

there follows the quantified expression

(26) *ΠaΠbCIabIba*,

and from the quantified expression (26) there follows again the unquantified law of conversion (9).

First: from (9) follows (26).

T10. *CpCqp* (law of simplification)

T10. $p/CIabIba \times C(9)\text{–}(23)$

(23) *CqCIabIba*

To this thesis we apply rule *Π2* binding *b*, and then *a*, as neither *b* nor *a* occurs in the antecedent:

(23) $\Pi 2b \times (24)$

(24) *CqΠbCIabIba*

(24) $\Pi 2a \times (25)$

(25) *CqΠaΠbCIabIba*

(25) $q/CpCqp \times C\text{T}10\text{–}(26)$

(26) *ΠaΠbCIabIba*

Secondly: from (26) follows (9).

T5. *Cpp* (law of identity)

T5. $p/CIabIba \times (27)$

(27) *CCIabIbaCIabIba*

To this thesis we apply rule *Π1* binding *b*, and then *a*:

(27) $\Pi 1b \times (28)$

(28) *CΠbCIabIbaCIabIba*

(28) $\Pi 1a \times (29)$

(29) *CΠaΠbCIabIbaCIabIba*

(29) $\times C(26)\text{–}(9)$

(9) *CIabIba*

Aristotle asserts: 'If some *a* is *b*, it is necessary that some *b* should be *a*.' The expression 'it is necessary that' can have, in my opinion, only this meaning: it is impossible to find such values of the variables *a* and *b* as would verify the antecedent without verifying the consequent. That means, in other words: 'For all *a*, and for all *b*, if some *a* is *b*, then some *b* is *a*.' This is our quantified thesis (26). It has been proved that this thesis is equivalent to the unquantified law of conversion 'If some *a* is *b*, then some *b* is *a*', which does not contain the sign of necessity. Since the syllogistic necessity is equivalent to a universal quantifier it may be omitted, as a universal quantifier may be omitted at the head of a true formula.

§ 25. *Fundamentals of the syllogistic*

Every axiomatized deductive system is based on three fundamental elements: primitive terms, axioms, and rules of inference. I start from the fundamentals for asserted expressions, the fundamental elements for the rejected ones being given later.

As primitive terms I take the constants *A* and *I*, defining by them the two other constants, *E* and *O*:

Df 1. *Eab* = *NIab*
Df 2. *Oab* = *NAab*.

In order to abbreviate the proofs I shall employ instead of the above definitions the two following rules of inference:

Rule RE: *NI* may be everywhere replaced by *E* and conversely.

Rule RO: *NA* may be everywhere replaced by *O* and conversely.

The four theses of the system axiomatically asserted are the two laws of identity and the moods Barbara and Datisi:

1. *Aaa*
2. *Iaa*
3. *CKAbcAabAac* (Barbara)
4. *CKAbcIbaIac* (Datisi).

Besides the rules RE and RO I accept the two following rules of inference for the asserted expressions:

(*a*) Rule of substitution: If α is an asserted expression of the system, then any expression produced from α by a valid substitution is also an asserted expression. The only valid substitution is to put for term-variables *a*, *b*, *c* other term-variables, e.g. *b* for *a*.

(*b*) Rule of detachment: If $C\alpha\beta$ and α are asserted expressions of the system, then β is an asserted expression.

As an auxiliary theory I assume the *C–N*-system of the theory of deduction with *K* as a defined functor. For propositional variables propositional expressions of the syllogistic may be substituted, like *Aab*, *Iac*, *KEbcAab*, etc. In all subsequent proofs (and also for rejected expressions) I shall employ only the following fourteen theses denoted by roman numerals:

I. *CpCqp* (law of simplification)
II. *CCqrCCpqCpr* (law of hypothetical syllogism, 2nd form)
III. *CCpCqrCqCpr* (law of commutation)
IV. *CpCNpq* (law of Duns Scotus)
V. *CCNppp* (law of Clavius)
VI. *CCpqCNqNp* (law of transposition)
VII. *CCKpqrCpCqr* (law of exportation)
VIII. *CpCCKpqrCqr*
IX. *CCspCCKpqrCKsqr*
X. *CCKpqrCCsqCKpsr*
XI. *CCrsCCKpqrCKqps*
XII. *CCKpqrCKpNrNq*
XIII. *CCKpqrCKNrqNp*
XIV. *CCKpNqNrCKprq*

Thesis VIII is a form of the law of exportation, theses IX–XI are compound laws of hypothetical syllogism, and XII–XIV are compound laws of transposition. All of these can be easily verified by the *0–1* method explained in section 23. Theses IV and V give together with II and III the whole *C–N*-system, but IV and V are needed only in proofs for rejected expressions.

The system of axioms 1–4 is consistent, i.e. non-contradictory. The easiest proof of non-contradiction is effected by regarding term-variables as proposition-variables, and by defining the functions *A* and *I* as always true, i.e. by putting *Aab* = *Iab* = *KCaaCbb*. The axioms 1–4 are then true as theses of the theory of deduction, and as it is known that the theory of deduction is non-contradictory, the syllogistic is non-contradictory too.

All the axioms of our system are independent of each other. The proofs of this may be given by interpretation in the field of the theory of deduction. In the subsequent interpretations the term-variables are treated as propositional variables.

Independence of axiom 1: Take *K* for *A*, and *C* for *I*. Axiom 1 is not verified, for *Aaa* = *Kaa*, and *Kaa* gives *0* for *a*/*0*. The other axioms are verified, as can be seen by the *0–1* method.

Independence of axiom 2: Take *C* for *A*, and *K* for *I*. Axiom 2 is not verified, for *Iaa* = *Kaa*. The other axioms are verified.

Independence of axiom 4: Take *C* for *A* and *I*. Axiom 4 is not verified, for *CKAbcIbaIac* = *CKCbcCbaCac* gives *0* for *b*/*0*, *a*/*1*, *c*/*0*. The rest are verified.

Independence of axiom 3: it is impossible to prove the independence of this axiom on the ground of a theory of deduction with only two truth-values, *0* and *1*. We must introduce a third truth-value, let us say *2*, which may be regarded as another symbol for truth, i.e. for *1*. To the equivalences given for *C*, *N*, and *K* in section 23, we have to add the following formulae:

$$C02 = C12 = C21 = C22 = 1, \quad C20 = 0, \quad N2 = 0,$$
$$K02 = K20 = 0, \quad K12 = K21 = K22 = 1.$$

It can easily be shown that under these conditions all the theses of the *C–N*-system are verified. Let us now define *Iab* as a function always true, i.e. $Iab = 1$ for all values of *a* and *b*, and *Aab* as a function with the values

$Aa a = 1$, $A01 = A12 = 1$, and $A02 = 0$ (the rest is irrelevant).

Axioms 1, 2, and 4 are verified, but from 3 we get by the substitutions $b/1$, $c/2$, $a/0$: $CKA12A01A02 = CK110 = C10 = 0$.

It is also possible to give proofs of independence by interpretation in the field of natural numbers. If we want, for instance, to prove that axiom 3 is independent of the remaining axioms, we can define *Aab* as $a+1 \neq b$, and *Iab* as $a+b = b+a$. *Iab* is always true, and therefore axioms 2 and 4 are verified. Axiom 1 is also verified, for $a+1$ is always different from *a*. But axiom 3, i.e. 'If $b+1 \neq c$ and $a+1 \neq b$, then $a+1 \neq c$', is not verified. Take 3 for *a*, 2 for *b*, and 4 for *c*: the premisses will be true and the conclusion false.

It results from the above proofs of independence that there exists no single axiom or 'principle' of the syllogistic. The four axioms 1–4 may be mechanically conjoined by the word 'and' into one proposition, but they remain distinct in this inorganic conjunction without representing one single idea.

§ 26. *Deduction of syllogistic theses*

From axioms 1–4 we can derive all the theses of the Aristotelian logic by means of our rules of inference and by the help of the theory of deduction. I hope that the subsequent proofs will be quite intelligible after the explanations given in the foregoing sections. In all syllogistical moods the major term is denoted by *a*, the middle term by *b*, and the minor term by *c*.

The major premiss is stated first, so that it is easy to compare the formulae with the traditional names of the moods.[1]

A. The Laws of Conversion

VII. $p/Abc,\ q/Iba,\ r/Iac \times C4–5$

5. *CAbcCIbaIac*

5. $b/a,\ c/a,\ a/b \times C1–6$

6. *CIabIba* (law of conversion of the *I*-premiss)

III. $p/Abc,\ q/Iba,\ r/Iac \times C5–7$

7. *CIbaCAbcIac*

7. $b/a,\ c/b \times C2–8$

8. *CAabIab* (law of subordination for affirmative premisses)

II. $q/Iab,\ r/Iba \times C6–9$

9. *CCpIabCpIba*

9. $p/Aab \times C8–10$

10. *CAabIba* (law of conversion of the *A*-premiss)

6. $a/b,\ b/a \times 11$

11. *CIbaIab*

VI. $p/Iba,\ q/Iab \times C11–12$

12. *CNIabNIba*

12. RE × 13

13. *CEabEba* (law of conversion of the *E*-premiss)

VI. $p/Aab,\ q/Iab \times C8–14$

14. *CNIabNAab*

14. RE, RO × 15

15. *CEabOab* (law of subordination for negative premisses)

B. The Affirmative Moods

X. $p/Abc,\ q/Iba,\ r/Iac \times C4–16$

16. *CCsIbaCKAbcsIac*

16. $s/Iab \times C6–17$

17. *CKAbcIabIac* (Darii)

[1] In my Polish text-book, *Elements of Mathematical Logic*, published in 1929 (see p. 46, n. 3), I showed for the first time how the known theses of the syllogistic may be formally deduced from axioms 1–4 (pp. 180–90). The method expounded in the above text-book is accepted with some modifications by I. M. Bocheński, O.P., in his contribution: *On the Categorical Syllogism*, Dominican Studies, vol. i, Oxford (1948).

16. *s/Aab* × *C*10–18
18. *CKAbcAabIac* (Barbari)

8. *a/b*, *b/a* × 19
19. *CAbaIba*

16. *s/Aba* × *C*19–20
20. *CKAbcAbaIac* (Darapti)

XI. *r/Iba*, *s/Iab* × *C*11–21
21. *CCKpqIbaCKqpIab*

4. *c/a*, *a/c* × 22
22. *CKAbaIbcIca*

21. *p/Aba*, *q/Ibc*, *b/c* × *C*22–23
23. *CKIbcAbaIac* (Disamis)

17. *c/a*, *a/c* × 24
24. *CKAbaIcbIca*

21. *p/Aba*, *q/Icb*, *b/c* × *C*24–25
25. *CKIcbAbaIac* (Dimaris)

18. *c/a*, *a/c* × 26
26. *CKAbaAcbIca*

21. *p/Aba*, *q/Acb*, *b/c* × *C*26–27
27. *CKAcbAbaIac* (Bramantip)

C. The Negative Moods

XIII. *p/Ibc*, *q/Aba*, *r/Iac* × *C*23–28
28. *CKNIacAbaNIbc*

28. RE × 29
29. *CKEacAbaEbc*

29. *a/b*, *b/a* × 30
30. *CKEbcAabEac* (Celarent)

IX. *s/Eab*, *p/Eba* × *C*13–31
31. *CCKEbaqrCKEabqr*

31. *a/c*, *q/Aab*, *r/Eac* × *C*30–32
32. *CKEcbAabEac* (Cesare)

XI. *r/Eab*, *s/Eba* × *C*13–33
33. *CCKpqEabCKqpEba*

32. *c/a*, *a/c* × 34
34. *CKEabAcbEca*

33. *p/Eab*, *q/Acb*, *a/c*, *b/a* × *C*34–35
35. *CKAcbEabEac* (Camestres)

30. *c/a*, *a/c* × 36
36. *CKEbaAcbEca*

33. *p/Eba*, *q/Acb*, *a/c*, *b/a* × *C*36–37
37. *CKAcbEbaEac* (Camenes)

II. *q/Eab*, *r/Oab* × *C*15–38
38. *CCpEabCpOab*

38. *p/KEbcAab*, *b/c* × *C*30–39
39. *CKEbcAabOac* (Celaront)

38. *p/KEcbAab*, *b/c* × *C*32–40
40. *CKEcbAabOac* (Cesaro)

38. *p/KAcbEab*, *b/c* × *C*35–41
41. *CKAcbEabOac* (Camestrop)

38. *p/KAcbEba*, *b/c* × *C*37–42
42. *CKAcbEbaOac* (Camenop)

XIII. *p/Abc*, *q/Iba*, *r/Iac* × *C*4–43
43. *CKNIacIbaNAbc*

43. RE, RO × 44
44. *CKEacIbaObc*

44. *a/b*, *b/a* × 45
45. *CKEbcIabOac* (Ferio)

31. *a/c*, *q/Iab*, *r/Oac* × *C*45–46
46. *CKEcbIabOac* (Festino)

X. *p/Ebc*, *q/Iab*, *r/Oac* × *C*45–47
47. *CCsIabCKEbcsOac*

47. *s/Iba* × *C*11–48
48. *CKEbcIbaOac* (Ferison)

31. *a/c*, *q/Iba*, *r/Oac* × *C*48–49
49. *CKEcbIbaOac* (Fresison)

10. *a/b*, *b/a* × 50
50. *CAbaIab*

47. *s/Aba* × *C*50–51
51. *CKEbcAbaOac* (Felapton)

31. *a/c*, *q/Aba*, *r/Oac* × *C*51–52
52. *CKEcbAbaOac* (Fesapo)

As a result of all these deductions one remarkable fact deserves our attention: it was possible to deduce twenty syllogistic moods without employing axiom 3, the mood Barbara. Even Barbari could be proved without Barbara. Axiom 3 is the most important thesis of the syllogistic, for it is the only syllogism that yields a universal affirmative conclusion, but in the system of simple syllogisms it has an inferior rank, being necessary to prove only two syllogistic moods, Baroco and Bocardo. Here are these two proofs:

XII. p/Abc, q/Aab, $r/Aac \times C3$–53

53. *CKAbcNAacNAab*

53. RO × 54

54. *CKAbcOacOab*

54. b/c, $c/b \times 55$

55. *CKAcbOabOac* (Baroco)

XIII. p/Abc, q/Aab, $r/Aac \times C3$–56

56. *CKNAacAabNAbc*

56. RO × 57

57. *CKOacAabObc*

57. a/b, $b/a \times 58$

58. *CKObcAbaOac* (Bocardo)

§ 27. *Axioms and rules for rejected expressions*

Of two intellectual acts, to assert a proposition and to reject it,[1] only the first has been taken into account in modern formal logic. Gottlob Frege introduced into logic the idea of assertion, and the sign of assertion (⊢), accepted afterwards by the authors of *Principia Mathematica*. The idea of rejection, however, so far as I know, has been neglected up to the present day.

We assert true propositions and reject false ones. Only true propositions can be asserted, for it would be an error to assert a proposition that was not true. An analogous property cannot be asserted of rejection: it is not only false propositions that have to be rejected. It is true, of course, that every proposition is either true or false, but there exist propositional expressions that are neither true nor false. Of this kind are the so-called propositional functions, i.e. expressions containing free variables

[1] I owe this distinction to Franz Brentano, who describes the acts of believing as *anerkennen* and *verwerfen*.

and becoming true for some of their values, and false for others. Take, for instance, p, the propositional variable: it is neither true nor false, because for $p/1$ it becomes true, and for $p/0$ it becomes false. Now, of two contradictory propositions, α and $N\alpha$, one must be true and the other false, one therefore must be asserted and the other rejected. But neither of the two contradictory propositional functions, p and Np, can be asserted, because neither of them is true: they both have to be rejected.

The syllogistic forms rejected by Aristotle are not propositions but propositional functions. Let us take an example: Aristotle says that no syllogism arises in the first figure, when the first term belongs to all the middle, but to none of the last. The syllogistic form therefore:

(*i*) *CKAbcEabIac*

is not asserted by him as a valid syllogism, but rejected. Aristotle himself gives concrete terms disproving the above form: take for *b* 'man', for *c* 'animal', and for *a* 'stone'. But there are other values for which the formula (*i*) can be verified: by identifying the variables *a* and *c* we get a true implication *CKAbaEabIaa*, for its antecedent is false and its consequent true. The negation of the formula (*i*):

(*j*) *NCKAbcEabIac*

must therefore be rejected too, because for c/a it is false.

By introducing quantifiers into the system we could dispense with rejection. Instead of rejecting the form (*i*) we could assert the thesis:

(*k*) *ΣaΣbΣcNCKAbcEabIac.*

This means: there exist terms *a*, *b*, and *c* that verify the negation of (*i*). The form (*i*), therefore, is not true for all *a*, *b*, and *c*, and cannot be a valid syllogism. In the same way instead of rejecting the expression (*j*) we might assert the thesis:

(*l*) *ΣaΣbΣcCKAbcEabIac.*

But Aristotle knows nothing of quantifiers; instead of adding to his system new theses with quantifiers he uses rejection. As rejection seems to be a simpler idea than quantification, let us follow in Aristotle's steps.

Aristotle rejects most invalid syllogistic forms by exemplification through concrete terms. This is the only point where we cannot follow him, because we cannot introduce into logic such concrete terms as 'man' or 'animal'. Some forms must be rejected axiomatically. I have found[1] that if we reject axiomatically the two following forms of the second figure:

CKAcbAabIac
CKEcbEabIac,

all the other invalid syllogistic forms may be rejected by means of two rules of rejection:

(*c*) Rule of rejection by detachment: if the implication 'If α, then β' is asserted, but the consequent β is rejected, then the antecedent α must be rejected too.

(*d*) Rule of rejection by substitution: if β is a substitution of α, and β is rejected, then α must be rejected too.

Both rules are perfectly evident.

The number of syllogistic forms is $4 \times 4^3 = 256$; 24 forms are valid syllogisms, 2 forms are rejected axiomatically. It would be tedious to prove that the remaining 230 invalid forms may be rejected by means of our axioms and rules. I shall only show, by the example of the forms of the first figure with premisses *Abc* and *Eab*, how our rules of rejection work on the basis of the first axiom of rejection.

Rejected expressions I denote by an asterisk put before their serial number. Thus we have:

*59. *CKAcbAabIac* (Axiom)
*59*a*. *CKEcbEabIac*
I. *p*/*Iac*, *q*/*KAcbAab* × 60
60. *CIacCKAcbAabIac*
60 × *C**61–*59
*61. *Iac*.

Here for the first time is applied the rule of rejection by detachment. The asserted implication 60 has a rejected consequent, *59; therefore its antecedent, *61, must be rejected too. In this same way I get the rejected expressions *64, *67, *71, *74, and *77.

[1] See section 20.

V. *p*/*Iac* × 62
62. *CCNIacIacIac*
62. RE × 63
63. *CCEacIacIac*
63 × *C**64–*61
*64. *CEacIac*
1. *a*/*c* × 65
65. *Acc*
VIII. *p*/*Acc*, *q*/*Eac*, *r*/*Iac* × *C*65–66
66. *CCKAccEacIacCEacIac*
66 × *C**67–*64
*67. *CKAccEacIac*
*67 × *68. *b*/*c*
*68. *CKAbcEabIac*

Here the rule of rejection by substitution is applied. Expression *68 must be rejected, because by the substitution of *b* for *c* in *68 we get the rejected expression *67. The same rule is used to get* 75.

II. *q*/*Aab*, *r*/*Iab* × *C*8–69
69. *CCpAabCpIab*
69. *p*/*KAbcEab*, *b*/*c* × 70
70. *CCKAbcEabAacCKAbcEabIac*
70 × *C**71–*68
*71. *CKAbcEabAac*
XIV. *p*/*Acb*, *q*/*Iac*, *r*/*Aab* × 72
72. *CCKAcbNIacNAabCKAcbAabIac*
72. RE, RO × 73
73. *CCKAcbEacOabCKAcbAabIac*
73 × *C**74–*59
*74. *CKAcbEacOab*
*74 × *75. *b*/*c*, *c*/*b*
*75. *CKAbcEabOac*
38. *p*/*KAbcEab*, *b*/*c* × 76
76. *CCKAbcEabEacCKAbcEabOac*
76 × *C**77–*75
*77. *CKAbcEabEac*

The rejected expressions *68, *71, *75, and *77 are the four possible forms of the first figure having as premisses *Abc* and *Eab*. From these premisses no valid conclusion can be drawn in the first figure. We can prove in the same way on the basis of the two axiomatically rejected forms that all the other invalid syllogistic forms in all the four figures must be rejected too.

§ 28. *Insufficiency of our axioms and rules*

Although it is possible to prove all the known theses of the Aristotelian logic by means of our axioms and rules of assertion, and to disprove all the invalid syllogistic forms by means of our axioms and rules of rejection, the result is far from being satisfactory. The reason is that besides the syllogistic forms there exist many other significant expressions in the Aristotelian logic, indeed an infinity of them, so that we cannot be sure whether from our system of axioms and rules all the true expressions of the syllogistic can be deduced or not, and whether all the false expressions can be rejected or not. In fact, it is easy to find false expressions that cannot be rejected by means of our axioms and rules of rejection. Such, for instance, is the expression:

(F1) *CIabCNAabAba.*

It means: 'If some *a* is *b*, then if it is not true that all *a* is *b*, all *b* is *a*.' This expression is not true in the Aristotelian logic, and cannot be proved by the axioms of assertion, but it is consistent with them and added to the axioms does not entail any invalid syllogistic form. It is worth while to consider the system of the syllogistic as thus extended.

From the laws of the Aristotelian logic:

8. *CAabIab* and
50. *CAbaIab*

and the law of the theory of deduction:

(*m*) *CCprCCqrCCNpqr*

we can derive the following new thesis 78:

(*m*) *p/Aab*, *q/Aba*, *r/Iab* × *C*8–*C*50–78

78. *CCNAabAbaIab.*

This thesis is a converse implication with regard to (F1), and together with (F1) gives an equivalence. On the ground of this equivalence we may define the functor *I* by the functor *A*:

(F2) *Iab* = *CNAabAba.*

This definition reads: ' "Some *a* is *b*" means the same as "If it is not true that all *a* is *b*, then all *b* is *a*".' As the expression 'If not-*p*, then *q*' is equivalent to the alternation 'Either *p* or *q*', we can also say: ' "Some *a* is *b*" means the same as "Either all *a* is *b* or all *b* is *a*".' It is now easy to find an interpretation of this extended system in the so-called Eulerian circles. The terms *a*, *b*, *c* are represented by circles, as in the usual interpretation, but on the condition that no two circles shall intersect each other. Axioms 1–4 are verified, and the forms *59 *CKAcbAabIac* and *59*a* *CKEcbEabIac* are rejected, because it is possible to draw two circles lying outside each other and included in a third circle, which refutes the form *CKAcbAabIac*, and to draw three circles each excluding the two others, which refutes the form *CKEcbEabIac*. Consequently all the laws of the Aristotelian logic are verified, and all the invalid syllogistic forms are rejected. The system, however, is different from the Aristotelian syllogistic, because the formula (F1) is false, as we can see from the following example: it is true that 'Some even numbers are divisible by 3', but it is true neither that 'All even numbers are divisible by 3' nor that 'All numbers divisible by 3 are even'.

It results from this consideration that our system of axioms and rules is not categorical, i.e. not all interpretations of our system verify and falsify the same formulae or are isomorphic. The interpretation just expounded verifies the formula (F1) which is not verified by the Aristotelian logic. The system of our axioms and rules, therefore, is not sufficient to give a full and exact description of the Aristotelian syllogistic.

In order to remove this difficulty we could reject the expression (F1) axiomatically. But it is doubtful whether this remedy would be effective; there may be other formulae of the same kind as (F1), perhaps even an infinite number of such formulae. The problem is to find a system of axioms and rules for the Aristotelian syllogistic on which we could decide whether any given significant expression of this system has to be asserted or rejected. To this most important problem of decision the next chapter is devoted.

of some famous but fantastic philosophical speculations. Kant divided all propositions (he calls them 'judgements') into analytic and synthetic according to the relation of the predicate of a proposition to its subject. His *Critique of Pure Reason* is chiefly an attempt to explain the problem how true synthetic *a priori* propositions are possible. Now some Peripatetics, for instance Alexander, were apparently already aware that there exists a large class of propositions having no subject and no predicate, such as implications, disjunctions, conjunctions, and so on.[1] All these may be called functorial propositions, since in all of them there occurs a propositional functor, like 'if—then', 'or', 'and'. These functorial propositions are the main stock of every scientific theory, and to them neither Kant's distinction of analytic and synthetic judgements nor the usual criterion of truth is applicable, for propositions without a subject or predicate cannot be immediately compared with facts. Kant's problem loses its importance and must be replaced by a much more important problem: How are true functorial propositions possible? It seems to me that here lies the starting-point for a new philosophy as well as for a new logic.

[1] In connexion with Aristotle's definition of the πρότασις Alexander writes, 11. 17: εἰσὶ δὲ οὗτοι οἱ ὅροι προτάσεως οὐ πάσης ἀλλὰ τῆς ἁπλῆς τε καὶ καλουμένης κατηγορικῆς· τὸ γάρ τι κατά τινος ἔχειν καὶ τὸ καθόλου ἢ ἐν μέρει ἢ ἀδιόριστον ἴδια ταύτης· ἡ γὰρ ὑποθετικὴ οὐκ ἐν τῷ τι κατά τινος λέγεσθαι ἀλλ' ἐν ἀκολουθίᾳ ἢ μάχῃ τὸ ἀληθὲς ἢ τὸ ψεῦδος ἔχει.

CHAPTER VI
ARISTOTLE'S MODAL LOGIC OF PROPOSITIONS

§ 36. *Introduction*

THERE are two reasons why Aristotle's modal logic is so little known. The first is due to the author himself: in contrast to the assertoric syllogistic which is perfectly clear and nearly free of errors, Aristotle's modal syllogistic is almost incomprehensible because of its many faults and inconsistencies. He devoted to this subject some interesting chapters of *De Interpretatione*, but the system of his modal syllogistic is expounded in Book I, chapters 3 and 8–22 of the *Prior Analytics*. Gohlke[1] suggested that these chapters were probably later insertions, because chapter 23 was obviously an immediate continuation of chapter 7. If he is right, the modal syllogistic was Aristotle's last logical work and should be regarded as a first version not finally elaborated by the author. This would explain the faults of the system as well as the corrections of Theophrastus and Eudemus, made perhaps in the light of hints given by the master himself.

The second reason is that modern logicians have not as yet been able to construct a universally acceptable system of modal logic which would yield a solid basis for the interpretation and appreciation of Aristotle's work. I have tried to construct such a system, different from those hitherto known, and built up upon Aristotle's ideas.[2] The present monograph on Aristotle's modal logic is written from the standpoint of this system.

A modal logic of terms presupposes a modal logic of propositions. This was not clearly seen by Aristotle whose modal syllogistic is a logic of terms; nevertheless it is possible to speak of an Aristotelian modal logic of propositions, as some of his theorems are general enough to comprise all kinds of proposition, and some others are expressly formulated by him with propositional variables. I shall begin with Aristotle's modal logic of propositions,

[1] Paul Gohlke, *Die Entstehung der Aristotelischen Logik*, Berlin (1936), pp. 88–94.

[2] Jan Łukasiewicz, 'A System of Modal Logic', *The Journal of Computing Systems*, vol. i, St. Paul (1953), pp. 111–49. A summary of this paper appeared under the same title in the *Proceedings of the XIth International Congress of Philosophy*, vol. xiv, Brussels (1953), pp. 82–87. A short description of the system is given below in § 49.

which is logically and philosophically far more important than his modal syllogistic of terms.

§ 37. *Modal functions and their interrelations*

There are four modal terms used by Aristotle: ἀναγκαῖον—'necessary', ἀδύνατον—'impossible', δυνατόν—'possible', and ἐνδεχόμενον—'contingent'. This last term is ambiguous: in the *De Interpretatione* it means the same as δυνατόν, in the *Prior Analytics* it has besides a more complicated meaning which I shall discuss later.

According to Aristotle, only propositions are necessary, impossible, possible, or contingent. Instead of saying: 'The proposition "*p*" is necessary', where "*p*" is the name of the proposition *p*, I shall use the expression: 'It is necessary that *p*', where *p* is a proposition. So, for instance, instead of saying: 'The proposition "man is an animal" is necessary', I shall say: 'It is necessary that man should be an animal.' I shall express the other modalities in a similar way. Expressions like: 'It is necessary that *p*', denoted here by *Lp*, or 'It is possible that *p*', denoted by *Mp*, I call 'modal functions'; *L* and *M*, which respectively correspond to the words 'it is necessary that' and 'it is possible that', are 'modal functors', *p* is their 'argument'. As modal functions are propositions, I say that *L* and *M* are proposition-forming functors of one propositional argument. Propositions beginning with *L* or their equivalents are called 'apodeictic', those beginning with *M* or their equivalents 'problematic'. Non-modal propositions are called 'assertoric'. This modern terminology and symbolism will help us to give a clear exposition of Aristotle's propositional modal logic.

Two of the modal terms, 'necessary' and 'possible', and their interrelations, are of fundamental importance. In the *De Interpretatione* Aristotle mistakenly asserts that possibility implies non-necessity, i.e. in our terminology:

(*a*) *If it is possible that p, it is not necessary that p.*[1] He later sees that this cannot be right, because he accepts that necessity implies possibility, i.e.:

(*b*) *If it is necessary that p, it is possible that p*, and from (*b*) and (*a*) there would follow by the hypothetical syllogism that

[1] *De int.* 13, 22ᵃ15 τῷ μὲν γὰρ δυνατῷ εἶναι τὸ ἐνδέχεσθαι εἶναι (ἀκολουθεῖ), καὶ τοῦτο ἐκείνῳ ἀντιστρέφει, καὶ τὸ μὴ ἀδύνατον εἶναι καὶ τὸ μὴ ἀναγκαῖον εἶναι.

(*c*) *If it is necessary that p, it is not necessary that p*, which is absurd.[1] After a further examination of the problem Aristotle rightly states that

(*d*) *If it is possible that p, it is not necessary that not p*,[2] but does not correct his former mistake in the text of *De Interpretatione*. This correction is given in the *Prior Analytics* where the relation of possibility to necessity has the form of an equivalence:

(*e*) *It is possible that p—if and only if—it is not necessary that not p.*[3]

I gather from this that the other relation, that of necessity to possibility, which is stated in the *De Interpretatione* as an implication,[4] is also meant as an equivalence and should be given the form:

(*f*) *It is necessary that p—if and only if—it is not possible that not p.*

If we denote the functor 'if and only if' by *Q*,[5] putting it before its arguments, and 'not' by *N*, we can symbolically express the relations (*e*) and (*f*) thus:

1. *QMpNLNp*, i.e. *Mp—if and only if—NLNp*,
2. *QLpNMNp*, i.e. *Lp—if and only if—NMNp*.

The above formulae are fundamental to any system of modal logic.

§ 38. *Basic modal logic*

Two famous scholastic principles of modal logic: *Ab oportere ad esse valet consequentia*, and *Ab esse ad posse valet consequentia*, were known to Aristotle without being formulated by him explicitly. The first principle runs in our symbolic notation (*C* is the sign of the functor 'if–then'):

3. *CLpp*, i.e. *If it is necessary that p, then p.*

The second reads:

[1] Ibid. 22ᵇ11 τὸ μὲν γὰρ ἀναγκαῖον εἶναι δυνατὸν εἶναι . . . 14 ἀλλὰ μὴν τῷ γε δυνατὸν εἶναι τὸ οὐκ ἀδύνατον εἶναι ἀκολουθεῖ, τούτῳ δὲ τὸ μὴ ἀναγκαῖον εἶναι· ὥστε συμβαίνει τὸ ἀναγκαῖον εἶναι μὴ ἀναγκαῖον εἶναι, ὅπερ ἄτοπον.

[2] Ibid. 22ᵇ22 λείπεται τοίνυν τὸ οὐκ ἀναγκαῖον μὴ εἶναι ἀκολουθεῖν τῷ δυνατὸν εἶναι.

[3] *An. pr.* i. 13, 32ᵃ25 τὸ 'ἐνδέχεται ὑπάρχειν' καὶ 'οὐκ ἀδύνατον ὑπάρχειν' καὶ 'οὐκ ἀνάγκη μὴ ὑπάρχειν', ἤτοι ταὐτὰ ἔσται ἢ ἀκολουθοῦντα ἀλλήλοις.

[4] *De int.* 13, 22ᵃ20 τῷ δὲ μὴ δυνατῷ μὴ εἶναι καὶ μὴ ἐνδεχομένῳ μὴ εἶναι τὸ ἀναγκαῖον εἶναι καὶ τὸ ἀδύνατον μὴ εἶναι (ἀκολουθεῖ).

[5] I usually denote equivalence by *E*, but as this letter has already another meaning in the syllogistic, I have introduced (p. 108) the letter *Q* for equivalence.

4. *CpMp*, i.e. *If p, it is possible that p*.

According to a passage of the *Prior Analytics*[1] Aristotle knows that from the assertoric negative conclusion 'Not *p*', i.e. *Np*, there results the problematic consequence 'It is possible that not *p*', i.e. *MNp*. We have therefore *CNpMNp*. Alexander, commenting on this passage, states as a general rule that existence implies possibility, i.e. *CpMp*, but not conversely, i.e. *CMpp* should be rejected.[2] If we denote rejected expressions by an asterisk, we get the formula:[3]

*5. *CMpp*, i.e. *If it is possible that p, then p*—rejected.

The corresponding formulae for necessity are also stated by Alexander who says that necessity implies existence, i.e. *CLpp*, but not conversely, i.e. *CpLp* should be rejected.[4] We get thus another rejected expression:

*6. *CpLp*, i.e. *If p, it is necessary that p*—rejected.

Formulae 1–6 are accepted by the traditional logic, and so far as I know, by all the modern logicians. They are, however, insufficient to characterize *Mp* and *Lp* as modal functions, because all the above formulae are satisfied if we interpret *Mp* as always true, i.e. as '*verum* of *p*', and *Lp* as always false, i.e. as '*falsum* of *p*'. With this interpretation a system built up on the formulae 1–6 would cease to be a modal logic. We cannot therefore assert *Mp*, i.e. accept that all problematic propositions are true, or assert *NLp*, i.e. accept that all apodeictic propositions are false; both expressions should be rejected, for any expression which cannot be asserted should be rejected. We get thus two additional rejected formulae:

*7. *Mp*, i.e. *It is possible that p*—rejected, and
*8. *NLp*, i.e. *It is not necessary that p*—rejected.

Both formulae may be called Aristotelian, as they are consequences of the presumption admitted by Aristotle that there exist

[1] *An. pr.* i. 16, 36^{a}15 φανερὸν δ' ὅτι καὶ τοῦ ἐνδέχεσθαι μὴ ὑπάρχειν γίγνεται συλλογισμός, εἴπερ καὶ τοῦ μὴ ὑπάρχειν. — ἐνδέχεσθαι means here the 'possible', not the 'contingent'.

[2] Alexander 209. 2 τὸ μὲν γὰρ ὑπάρχον καὶ ἐνδεχόμενον ἀληθὲς εἰπεῖν, τὸ δ' ἐνδεχόμενον οὐ πάντως καὶ ὑπάρχον.

[3] Asserted expressions are marked throughout the Chapters VI–VIII by arabic numerals without asterisks.

[4] Alexander 152. 32 τὸ γὰρ ἀναγκαῖον καὶ ὑπάρχον, οὐκέτι δὲ τὸ ὑπάρχον ἀναγκαῖον.

asserted apodeictic propositions. For, if *Lα* is asserted, then *LNNα* must be asserted too, and from the principle of Duns Scotus *CpCNpq* we get by substitution and detachment the asserted formulae *CNLαp* and *CNLNNαp*. As *p* is rejected, *NLα* and *NLNNα* are rejected too, and consequently *NLp* and *NLNp*, i.e. *Mp*, must be rejected.

I call a system 'basic modal logic' if and only if it satisfies the formulae 1–8. I have shown that basic modal logic can be axiomatized on the basis of the classical calculus of propositions.[1] Of the two modal functors, *M* and *L*, one may be taken as the primitive term, and the other can be defined. Taking *M* as the primitive term and formula 2 as the definition of *L*, we get the following independent set of axioms of the basic modal logic:

4. *CpMp*   *5. *CMpp*   *7. *Mp*   9. *QMpMNNp*,

where 9 is deductively equivalent to formula 1 on the ground of the definition 2 and the calculus of propositions. Taking *L* as the primitive term and formula 1 as the definition of *M*, we get a corresponding set of axioms:

3. *CLpp*   *6. *CpLp*   *8. *NLp*   10. *QLpLNNp*,

where 10 is deductively equivalent to formula 2 on the ground of the definition 1 and the calculus of propositions. The derived formulae 9 and 10 are indispensable as axioms.

Basic modal logic is the foundation of any system of modal logic and must always be included in any such system. Formulae 1–8 agree with Aristotle's intuitions and are at the roots of our concepts of necessity and possibility; but they do not exhaust the whole stock of accepted modal laws. For instance, we believe that if a conjunction is possible, each of its factors should be possible, i.e. in symbols:

11. *CMKpqMp*   and   12. *CMKpqMq*,

and if a conjunction is necessary, each of its factors should be necessary, i.e. in symbols:

13. *CLKpqLp*   and   14. *CLKpqLq*.

None of these formulae can be deduced from the laws 1–8. Basic modal logic is an incomplete modal system and requires the addition of some new axioms. Let us see how it was supplemented by Aristotle himself.

[1] See pp. 114–17 of my paper on modal logic.

§ 39. *Laws of extensionality*

Aristotle's most important and—as I see it—most successful attempt to go beyond basic modal logic consisted in his accepting certain principles which may be called 'laws of extensionality for modal functors'. These principles are to be found in Book I, chapter 15 of the *Prior Analytics*, and are formulated in three passages. We read at the beginning of the chapter:

'First it has to be said that if (if α is, β must be), then (if α is possible, β must be possible too).'[1]

A few lines further Aristotle says referring to his syllogisms:

'If one should denote the premisses by α, and the conclusion by β, it would not only result that if α is necessary, then β is necessary, but also that if α is possible, then β is possible.'[2]

And at the end of the section he repeats:

'It has been proved that if (if α is, β is), then (if α is possible, then β is possible).'[3]

Let us first analyse these modal laws beginning with the second passage, which refers to syllogisms.

All Aristotelian syllogisms are implications of the form $C\alpha\beta$ where α is the conjunction of the two premisses and β the conclusion. Take as example the mood Barbara:

$$15.\quad C\underbrace{KAbaAcb}_{\alpha}\underbrace{Aca}_{\beta}.$$

According to the second passage we get two modal theorems, in the form of implications taking $C\alpha\beta$ as the antecedent and $CL\alpha L\beta$ or $CM\alpha M\beta$ as the consequent, in symbols:

16. $CC\alpha\beta CL\alpha L\beta$ and 17. $CC\alpha\beta CM\alpha M\beta$.

The letters α and β stand here for the premisses and the conclusion of an Aristotelian syllogism. As in the final passage there is

[1] *An. pr.* i. 15, 34ᵃ5 πρῶτον δὲ λεκτέον ὅτι εἰ τοῦ *A* ὄντος ἀνάγκη τὸ *B* εἶναι, καὶ δυνατοῦ ὄντος τοῦ *A* δυνατὸν ἔσται καὶ τὸ *B* ἐξ ἀνάγκης.

[2] Ibid. 34ᵃ22 εἴ τις θείη τὸ μὲν *A* τὰς προτάσεις, τὸ δὲ *B* τὸ συμπέρασμα, συμβαίνοι ἂν οὐ μόνον ἀναγκαίου τοῦ *A* ὄντος ἅμα καὶ τὸ *B* εἶναι ἀναγκαῖον, ἀλλὰ καὶ δυνατοῦ δυνατόν.

[3] Ibid. 34ᵃ29 δέδεικται ὅτι εἰ τοῦ *A* ὄντος τὸ *B* ἔστι, καὶ δυνατοῦ ὄντος τοῦ *A* ἔσται τὸ *B* δυνατόν.

no reference to syllogisms, we may treat these theorems as special cases of general principles which we get by replacing the Greek letters by propositional variables:

18. $CCpqCLpLq$ and 19. $CCpqCMpMq$.

Both formulae may be called in a wider sense 'laws of extensionality', the first for L, the second for M. The words 'in a wider sense' require an explanation.

The general law of extensionality, taken *sensu stricto*, is a formula of the classical calculus of propositions enlarged by the introduction of variable functors, and has the form:

20. $CQpqC\delta p\delta q$.

This means roughly speaking: If p is equivalent to q, then if δ of p, δ of q, where δ is any proposition-forming functor of one propositional argument, e.g. N. Accordingly, the strict laws of extensionality for L and M will have the form:

21. $CQpqCLpLq$ and 22. $CQpqCMpMq$.

These two formulae have stronger antecedents than formulae 18 and 19, and are easily deducible from them, 21 from 18, and 22 from 19, by means of the thesis $CQpqCpq$ and the principle of the hypothetical syllogism. It can be proved, however, on the ground of the calculus of propositions and the basic modal logic that conversely 18 is deducible from 21, and 19 from 22. I give here the full deduction of the L-formula:

The premisses:

23. $CCQpqrCpCCpqr$
24. $CCpqCCqrCpr$
25. $CCpCqCprCqCpr$
3. $CLpp$.

The deduction:

23. $r/CLpLq \times C21\text{–}26$
26. $CpCCpqCLpLq$
24. $p/Lp,\ q/p,\ r/CCpqCLpLq \times C3\text{–}C26\text{–}27$
27. $CLpCCpqCLpLq$
25. $p/Lp,\ q/Cpq,\ r/Lq \times C27\text{–}18$
18. $CCpqCLpLq$.

In a similar way 19 is deducible from 22 by means of the premisses *CCQpqrCNqCCpqr*, *CCpqCCqrCpr*, *CCNpCqCrpCqCrp*, and the transposition *CNMpNp* of the modal thesis *CpMp*.

We see from the above that, given the calculus of propositions and basic modal logic, formula 18 is deductively equivalent to the strict law of extensionality 21, and formula 19 to the strict law of extensionality 22. We are right, therefore, to call those formulae 'laws of extensionality in a wider sense'. Logically, of course, it makes no difference whether we complete the *L*-system of basic modal logic by the addition of *CCpqCLpLq* or by the addition of *CQpqCLpLq*; the same holds for the alternative additions to the *M*-system of *CCpqCMpMq* or *CQpqCMpMq*. Intuitively, however, the difference is great. Formulae 18 and 19 are not so evident as formulae 21 and 22. If *p* implies *q* but is not equivalent to it, it is not always true that if δ of *p*, δ of *q*; e.g. *CNpNq* does not follow from *Cpq*. But if *p* is equivalent to *q*, then always if δ of *p*, δ of *q*, i.e. if *p* is true, *q* is true, and if *p* is false, *q* is false; similarly if *p* is necessary, *q* is necessary, and if *p* is possible, *q* is possible. This seems to be perfectly evident, unless modal functions are regarded as intensional functions, i.e. as functions whose truth-values do *not* depend solely on the truth-values of their arguments. But what in this case the necessary and the possible would mean, is for me a mystery as yet.

§ 40. *Aristotle's proof of the M-law of extensionality*

In the last passage quoted above Aristotle says that he has proved the law of extensionality for possibility. He argues in substance thus: If α is possible and β impossible, then when α came to be, β would not come to be, and therefore α would be without β, which is against the premiss that if α is, β is.[1] It is difficult to recast this argument into a logical formula, as the term 'to come to be' has an ontological rather than a logical meaning. The comment, however, given on this argument by Alexander deserves a careful examination.

Aristotle defines the contingent as that which is not necessary and the supposed existence of which implies nothing impossible.[2]

[1] *An. pr.* i. 15, 34ᵃ8 *εἰ οὖν τὸ μὲν δυνατόν, ὅτε δυνατὸν εἶναι, γένοιτ' ἄν, τὸ δ' ἀδύνατον, ὅτ' ἀδύνατον, οὐκ ἂν γένοιτο, ἅμα δ' εἰ τὸ Α δυνατὸν καὶ τὸ Β ἀδύνατον, ἐνδέχοιτ' ἂν τὸ Α γενέσθαι ἄνευ τοῦ Β, εἰ δὲ γενέσθαι, καὶ εἶναι.* [2] See below, p. 154, n. 3.

Alexander assimilates this Aristotelian definition of contingency to that of possibility by omitting the words 'which is not necessary'. He says 'that a β which is impossible cannot follow from an α which is possible may also be proved from the definition of possibility: that is possible, the supposed existence of which implies nothing impossible'.[1] The words 'impossible' and 'nothing' here require a cautious interpretation. We cannot interpret 'impossible' as 'not possible', because the definition would be circular; we must either take 'impossible' as a primitive term or, taking 'necessary' as primitive, define the expression 'impossible that *p*' by 'necessary that not *p*'. I prefer the second way and shall discuss the new definition on the ground of the *L*-basic modal logic. The word 'nothing' should be rendered by a universal quantifier, as otherwise the definition would not be correct. We get thus the equivalence:

28. *QMpΠqCCpqNLNq*.

That means in words: 'It is possible that *p*—if and only if—for all *q*, if (if *p*, then *q*), it is not necessary that not *q*.' This equivalence has to be added to the *L*-basic modal logic as the definition of *Mp* instead of the equivalence 1 which must now be proved as a theorem.

The equivalence 28 consists of two implications:

29. *CMpΠqCCpqNLNq* and 30. *CΠqCCpqNLNqMp*.

From 29 we get by the theorem *CΠqCCpqNLNqCCpqNLNq* and the hypothetical syllogism the consequence:

31. *CMpCCpqNLNq*,

and from 31 there easily results by the substitution *q/p*, *Cpp*, commutation and detachment the implication *CMpNLNp*. The converse implication *CNLNpMp* which, when combined with the original implication, would give the equivalence 1, cannot be proved otherwise than by means of the law of extensionality for *L*: *CCpqCLpLq*. As this proof is rather complicated, I shall give it in full.

[1] Alexander 177. 11 *δεικνύοιτο δ' ἄν, ὅτι μὴ οἷόν τε δυνατῷ ὄντι τῷ Α ἀδύνατον ἕπεσθαι τὸ Β, καὶ ἐκ τοῦ ὁρισμοῦ τοῦ δυνατοῦ . . . δυνατόν ἐστιν, οὗ ὑποτεθέντος εἶναι οὐδὲν ἀδύνατον συμβαίνει διὰ τοῦτο.*

The premisses:

18. *CCpqCLpLq*
24. *CCpqCCqrCpr*
30. *CΠqCCpqNLNqMp*
32. *CCpqCNqNp*
33. *CCpCqrCqCpr*.

The deduction:

  18. *p/Nq, q/Np* × 34
34. *CCNqNpCLNqLNp*
  24. *p/Cpq, q/CNqNp, r/CLNqLNp* × *C*32–*C*34–35
35. *CCpqCLNqLNp*
  32. *p/LNq, q/LNp* × 36
36. *CCLNqLNpCNLNpNLNq*
  24. *p/Cpq, q/CLNqLNp, r/CNLNpNLNq* × *C*35–*C*36–37
37. *CCpqCNLNpNLNq*
  33. *p/Cpq, q/NLNp, r/NLNq* × *C*37–38
38. *CNLNpCCpqNLNq*
  38. *Π2q* × 39
39. *CNLNpΠqCCpqNLNq*
  24. *p/NLNp, q/ΠqCCpqNLNq, r/Mp* × *C*39–*C*30–40
40. *CNLNpMp*.

We can now prove the law of extensionality for *M*, which was the purpose of Alexander's argument. This law easily results from the equivalence 1 and thesis 37. We see besides that the proof by means of the definition with quantifiers is unnecessarily complicated. It suffices to retain definition 1 and to add to the *L*-system the *L*-law of extensionality in order to get the *M*-law of extensionality. In the same way we may get the *L*-law of extensionality, if we add the *M*-law of extensionality to the *M*-system and definition 2. The *L*-system is deductively equivalent to the *M*-system with the laws of extensionality as well as without them.

It is, of course, highly improbable that an ancient logician could have invented such an exact proof as that given above. But the fact that the proof is correct throws an interesting light on Aristotle's ideas of possibility. I suppose that he intuitively saw what may be shortly expressed thus: what is possible today, say a sea-fight, may become existent or actual tomorrow; but what is

impossible, can never become actual. This idea seems to lie at the bottom of Aristotle's proof and of Alexander's.

§ 41. *Necessary connexions of propositions*

The *L*-law of extensionality was formulated by Aristotle only once, together with the *M*-law, in the passage where he refers to syllogisms.[1]

According to Aristotle there exists a necessary connexion between the premisses α of a valid syllogism and its conclusion β. It would seem therefore that the laws of extensionality formulated above in the form:

16. *CCαβCLαLβ* and 17. *CCαβCMαMβ*,

should be expressed with necessary antecedents:

41. *CLCαβCLαLβ* and 42. *CLCαβCMαMβ*,

and the corresponding general laws of extensionality should run:

43. *CLCpqCLpLq* and 44. *CLCpqCMpMq*.

This is corroborated for the *M*-law by the first passage quoted above where we read: 'If (if α is, β must be), then (if α is possible, β is possible).'

Formulae 43 and 44 are weaker than the corresponding formulae with assertoric antecedents, 18 and 19, and can be got from them by the axiom *CLpp* and the hypothetical syllogism 24. It is not, however, possible to derive the stronger formulae conversely from the weaker. The problem is whether we should reject the stronger formulae 18 and 19, and replace them by the weaker formulae 43 and 44. To solve this problem we have to inquire into the Aristotelian concept of necessity.

Aristotle accepts that some necessary, i.e. apodeictic, propositions are true and should be asserted. Two kinds of asserted apodeictic proposition can be found in the *Analytics*: to the one kind there belong necessary connexions of propositions, to the other necessary connexions of terms. As example of the first kind any valid syllogism may be taken, for instance the mood Barbara:

(*g*) *If every b is an a, and every c is a b, then it is necessary that every c should be an a.*

Here the 'necessary' does not mean that the conclusion is an

[1] See p. 138, n. 2.

apodeictic proposition, but denotes a necessary connexion between the premisses of the syllogism and its assertoric conclusion. This is the so called 'syllogistic necessity'. Aristotle sees very well that there is a difference between syllogistic necessity and an apodeictic conclusion when he says, discussing a syllogism with an assertoric conclusion, that this conclusion is not 'simply' (ἁπλῶς) necessary, i.e. necessary in itself, but is necessary 'on condition', i.e. with respect to its premisses (τούτων ὄντων).[1] There are passages where he puts two marks of necessity into the conclusion saying, for instance, that from the premisses: 'It is necessary that every *b* should be an *a*, and some *c* is a *b*', there follows the conclusion: 'It is necessary that some *c* should be necessarily an *a*.'[2] The first 'necessary' refers to the syllogistic connexion, the second denotes that the conclusion is an apodeictic proposition.

By the way, a curious mistake of Aristotle should be noted: he says that nothing follows necessarily from a single premiss, but only from at least two, as in the syllogism.[3] In the *Posterior Analytics* he asserts that this has been proved,[4] but not even an attempt of proof is given anywhere. On the contrary, Aristotle himself states that 'If some *b* is an *a*, it is necessary that some *a* should be a *b*', drawing thus a necessary conclusion from only one premiss.[5]

I have shown that syllogistic necessity can be reduced to universal quantifiers.[6] When we say that in a valid syllogism the conclusion necessarily follows from the premisses, we want to state that the syllogism is valid for any matter, i.e. for all values of the variables occurring in it. This explanation, as I have found afterwards, is corroborated by Alexander who asserts that: 'syllogistic combinations are those from which something necessarily follows, and such are those in which for all matter the same comes to be'.[7] Syllogistic necessity reduced to universal quantifiers can

[1] *An. pr.* i. 10, 30^{b}32 τὸ συμπέρασμα οὐκ ἔστιν ἀναγκαῖον ἁπλῶς, ἀλλὰ τούτων ὄντων ἀναγκαῖον.

[2] Ibid. 9, 30^{a}37 τὸ μὲν Α παντὶ τῷ Β ὑπαρχέτω ἐξ ἀνάγκης, τὸ δὲ Β τινὶ τῷ Γ ὑπαρχέτω μόνον· ἀνάγκη δὴ τὸ Α τινὶ τῷ Γ ὑπάρχειν ἐξ ἀνάγκης.

[3] Ibid. 15, 34^{a}17 οὐ γὰρ ἔστιν οὐδὲν ἐξ ἀνάγκης ἑνός τινος ὄντος, ἀλλὰ δυοῖν ἐλαχίστοιν, οἷον ὅταν αἱ προτάσεις οὕτως ἔχωσιν ὡς ἐλέχθη κατὰ τὸν συλλογισμόν.

[4] *An. post.* i. 3, 73^{a}7 ἑνὸς μὲν οὖν κειμένου δέδεικται ὅτι οὐδέποτ' ἀνάγκη τι εἶναι ἕτερον (λέγω δ' ἑνός, ὅτι οὔτε ὅρου ἑνὸς οὔτε θέσεως μιᾶς τεθείσης), ἐκ δύο δὲ θέσεων πρώτων καὶ ἐλαχίστων ἐνδέχεται.

[5] *An. pr.* i. 2, 25^{a}20 εἰ γὰρ τὸ Α τινὶ τῷ Β, καὶ τὸ Β τινὶ τῷ Α ἀνάγκη ὑπάρχειν.

[6] See § 5.

[7] Alexander 208. 16 συλλογιστικαὶ δὲ αἱ συζυγίαι αὗται αἱ ἐξ ἀνάγκης τι συνάγουσαι· τοιαῦται δέ, ἐν αἷς ἐπὶ πάσης ὕλης γίνεται τὸ αὐτό.

be eliminated from syllogistic laws, as will appear from the following consideration.

The syllogism (*g*) correctly translated into symbols would have the form:

(*h*) *LCKAbaAcbAca*,

which means in words:

(*i*) *It is necessary that* (*if every b is an a*, *and every c is a b*, *then every c should be an a*).

The sign of necessity in front of the syllogism shows that not the conclusion, but the connexion between the premisses and the conclusion is necessary. Aristotle would have asserted (*h*). Formula

(*j*) *CKAbaAcbLAca*,

which literally corresponds to the verbal expression (*g*), is wrong. Aristotle would have rejected it, as he rejects a formula with stronger premisses, viz.

(*k*) *CKAbaLAcbLAca*,

i.e. '*If every b is an a and it is necessary that every c should be a b*, *it is necessary that every c should be an a*.'[1]

By the reduction of necessity to universal quantifiers formula (*h*) can be transformed into the expression:

(*l*) *ΠaΠbΠcCKAbaAcbAca*,

i.e. 'For all *a*, for all *b*, for all *c* (if every *b* is an *a* and every *c* is a *b*, then every *c* is an *a*).' This last expression is equivalent to the mood Barbara without quantifiers:

(*m*) *CKAbaAcbAca*,

since a universal quantifier may be omitted when it stands at the head of an asserted formula.

Formulae (*h*) and (*m*) are not equivalent. It is obvious that (*m*) can be deduced from (*h*) by the principle *CLpp*, but the converse deduction is not possible without the reduction of necessity to universal quantifiers. This, however, cannot be done at all, if the above formulae are applied to concrete terms. Put, for instance,

[1] *An. pr.* i. 9, 30^{a}23 εἰ δὲ τὸ μὲν ΑΒ μή ἐστιν ἀναγκαῖον, τὸ δὲ ΒΓ ἀναγκαῖον, οὐκ ἔσται τὸ συμπέρασμα ἀναγκαῖον.

in (*h*) 'bird' for *b*, 'crow' for *a*, and 'animal' for *c*; we get the apodeictic proposition:

(*n*) *It is necessary that* (*if every bird is a crow and every animal is a bird, then every animal should be a crow*).

From (*n*) results the syllogism (*o*):

(*o*) *If every bird is a crow and every animal is a bird, then every animal is a crow*,

but from (*o*) we cannot get (*n*) by the transformation of necessity into quantifiers, as (*n*) does not contain variables which could be quantified.

And here we meet the first difficulty. It is easy to understand the meaning of necessity when the functor *L* is attached to the front of an asserted proposition containing free variables. In this case we have a general law, and we may say: this law we regard as necessary, because it is true of all objects of a certain kind, and does not allow of exception. But how should we interpret necessity, when we have a necessary proposition without free variables, and in particular, when this proposition is an implication consisting of false antecedents and of a false consequent, as in our example (*n*)? I see only one reasonable answer: we could say that whoever accepts the premisses of this syllogism is necessarily compelled to accept its conclusion. But this would be a kind of psychological necessity which is quite alien from logic. Besides it is extremely doubtful that anybody would accept evidently false propositions as true.

I know no better remedy for removing this difficulty than to drop everywhere the *L*-functor standing in front of an asserted implication. This procedure was already adopted by Aristotle who sometimes omits the sign of necessity in valid syllogistical moods.[1]

§ 42. '*Material*' or '*strict*' *implication?*

According to Philo of Megara the implication 'If *p*, then *q*', i.e. *Cpq*, is true if and only if it does not begin with a true antecedent and end with a false consequent.[2] This is the so-called 'material' implication now universally accepted in the classical calculus of propositions. 'Strict' implication: 'It is necessary that

[1] See p. 10, n. 5.

[2] See p. 83, n. 1.

if *p*, then *q*', i.e. *LCpq*, is a necessary material implication and was introduced into symbolic logic by C. I. Lewis. By means of this terminology the problem we are discussing may be stated thus: Should we interpret the antecedent of the Aristotelian laws of extensionality as material, or as strict implication? In other words, should we accept the stronger formulae 18 and 19 (I call this the 'strong interpretation'), or should we reject them accepting the weaker formulae 43 and 44 (weak interpretation)?

Aristotle was certainly not aware of the difference between these two interpretations and of their importance for modal logic. He could not know Philo's definition of the material implication. But his commentator Alexander was very well acquainted with the logic of the Stoic–Megaric school and with the heated controversies about the meaning of the implication amidst the followers of this school. Let us then see his comments on our problem.

Commenting on the Aristotelian passage 'If (if α is, β must be), then (if α is possible, β must be possible)' Alexander emphasizes the necessary character of the premiss 'If α is, β must be'. It seems therefore that he would accept the weaker interpretation $CLC\alpha\beta CM\alpha M\beta$ and the weaker *M*-law of extensionality *CLCpqCMpMq*. But what he means by a necessary implication is different from strict implication in the sense of Lewis. He says that in a necessary implication the consequent should always, i.e. at any time, follow from the antecedent, so that the proposition 'If Alexander is, he is so and so many years old' is not a true implication, even if Alexander were in fact so many years old at the time when this proposition is uttered.[1] We may say that this proposition is not exactly expressed, and requires the addition of a temporal qualification in order to be always true. A true material implication must be, of course, always true, and if it contains variables, must be true for all values of the variables. Alexander's comment is not incompatible with the strong interpretation; it does not throw light on our problem.

Some more light is thrown on it, if we replace in Alexander's proof of the *M*-law of extensionality expounded in § 40 the

[1] Alexander 176. 2 ἔστι δὲ ἀναγκαία ἀκολουθία οὐχ ἡ πρόσκαιρος, ἀλλ' ἐν ᾗ ἀεὶ τὸ εἰλημμένον ἕπεσθαι ἔστι τῷ τὸ εἰλημμένον ὡς ἡγούμενον εἶναι. οὐ γαρ ἀληθὲς συνημμένον τὸ 'εἰ Ἀλέξανδρος ἔστιν, Ἀλέξανδρος διαλέγεται', ἢ 'εἰ Ἀλέξανδρος ἔστι, τοσῶνδε ἐτῶν ἔστι', καὶ <εἰ> εἴη, ὅτε λέγεται ἡ πρότασις, τοσούτων ἐτῶν.

material implication *Cpq* by the strict implication *LCpq*. Transforming thus the formula

31. *CMpCCpqNLNq*,

we get:

45. *CMpCLCpqNLNq*.

From 31 we can easily derive *CMpNLNp* by the substitution *q/p* getting *CMpCCppNLNp*, from which our proposition results by commutation and detachment, for *Cpp* is an asserted implication. The same procedure, however, cannot be applied to 45. We get *CMpCLCppNLNp*, but if we want to detach *CMpNLNp* we must assert the apodeictic implication *LCpp*. And here we encounter the same difficulty, as described in the foregoing section. What is the meaning of *LCpp*? This expression may be interpreted as a general law concerning all propositions, if we transform it into *ΠpCpp*; but such a transformation becomes impossible, if we apply *LCpp* to concrete terms, e.g. to the proposition 'Twice two is five'. The assertoric implication 'If twice two is five, then twice two is five' is comprehensible and true being a consequence of the law of identity *Cpp*; but what is the meaning of the apodeictic implication 'It is necessary that if twice two is five, then twice two should be five'? This queer expression is not a general law concerning all numbers; it may be at most a consequence of an apodeictic law, but it is not true that a consequence of an apodeictic proposition must be apodeictic too. *Cpp* is a consequence of *LCpp* according to *CLCppCpp*, a substitution of *CLpp*, but is not apodeictic.

It follows from the above that it is certainly simpler to interpret Alexander's proof by taking the word συμβαίνει of his text in the sense of material rather than strict implication. Nevertheless our problem is not yet definitively solved. Let us therefore turn to the other kind of asserted apodeictic proposition accepted by Aristotle, that is to necessary connexions of terms.

§ 43. *Analytic propositions*

Aristotle asserts the proposition: 'It is necessary that man should be an animal.'[1] He states here a necessary connexion between the subject 'man' and the predicate 'animal', i.e. a

[1] *An. pr.* i. 9, 30ᵃ30 ζῷον μὲν γὰρ ὁ ἄνθρωπος ἐξ ἀνάγκης ἐστί.

necessary connexion between terms. He apparently regards it as obvious that the proposition 'Man is an animal', or better 'Every man is an animal', must be an apodeictic one, because he defines 'man' as an 'animal', so that the predicate 'animal' is contained in the subject 'man'. Propositions in which the predicate is contained in the subject are called 'analytic', and we shall probably be right in supposing that Aristotle would have regarded all analytic propositions based on definitions as apodeictic, since he says in the *Posterior Analytics* that essential predicates belong to things necessarily,[1] and essential predicates result from definitions.

The most conspicuous examples of analytic propositions are those in which the subject is identical with the predicate. If it is necessary that every man should be an animal, it is, *a fortiori*, necessary that every man should be a man. The law of identity 'Every *a* is an *a*' is an analytic proposition, and consequently an apodeictic one. We get thus the formula:

(*p*) *LAaa*, i.e. *It is necessary that every a should be an a.*

Aristotle does not state the law of identity *Aaa* as a principle of his assertoric syllogistic; there is only one passage, found by Ivo Thomas, where in passing he uses this law in a demonstration.[2] We cannot expect, therefore, that he has known the modal thesis *LAaa*.

The Aristotelian law of identity *Aaa*, where *A* means 'every–is' and *a* is a variable universal term, is different from the principle of identity *Jxx*, where *J* means 'is identical with' and *x* is a variable individual term. The latter principle belongs to the theory of identity which can be established on the following axioms:

(*q*) *Jxx*, i.e. *x is identical with x*,

(*r*) *CJxyCφxφy*, i.e. *If x is identical with y, then if x satisfies φ, y satisfies φ*,

where *φ* is a variable proposition-forming functor of one individual argument. Now, if all analytic propositions are necessary, so also is (*q*), and we get the apodeictic principle:

(*s*) *LJxx*, i.e. *It is necessary that x should be identical with x.*

[1] *An. post.* i. 6, 74ᵇ6 τὰ δὲ καθ'αὑτὰ ὑπάρχοντα ἀναγκαῖα τοῖς πράγμασιν.

[2] Ivo Thomas, O.P., 'Farrago Logica', *Dominican Studies*, vol. iv (1951), p. 71. The passage reads (*An. pr.* ii. 22, 68ᵃ19) κατηγορεῖται δὲ τὸ Β καὶ αὐτὸ αὑτοῦ.

It has been observed by W. V. Quine that the principle (*s*), if asserted, leads to awkward consequences.[1] For if *LJxx* is asserted, we can derive (*t*) from (*r*) by the substitution *ϕ*/*LJx*'—*LJx* works here like a proposition-forming functor of one argument:

(*t*) *CJxyCLJxxLJxy*,

and by commutation

(*u*) *CLJxxCJxyLJxy*,

from which there follows the proposition:

(*v*) *CJxyLJxy*.

That means, any two individuals are necessarily identical, if they are identical at all.

The relation of equality is usually treated by mathematicians as identity and is based on the same axioms (*q*) and (*r*). We may therefore interpret *J* as equality, *x* and *y* as individual numbers and say that equality holds necessarily if it holds at all.

Formula (*v*) is obviously false. Quine gives an example to show its falsity. Let *x* denote the number of planets, and *y* the number 9. It is a factual truth that the number of (major) planets is equal to 9, but it is not necessary that it should be equal to 9. Quine tries to meet this difficulty by raising objections to the substitution of such singular terms for the variables. In my opinion, however, his objections are without foundation.

There is another awkward consequence of the formula (*v*) not mentioned by Quine. From (*v*) we get by the definition of *L* and the law of transposition the consequence:

(*w*) *CMNJxyNJxy*.

That means: 'If it is possible that *x* is not equal to *y*, then *x* is (actually) not equal to *y*.' The falsity of this consequence may be seen in the following example: Let us suppose that a number *x* has been thrown with a die. It is possible that the number *y* next thrown with the die will be different from *x*. But if it is possible that *x* will be different from *y*, i.e. not equal to *y*, then according to (*w*) *x* will actually be different from *y*. This consequence is obviously wrong, as it is possible to throw the same number twice.

[1] W. V. Quine, 'Three Grades of Modal Involvement', *Proceedings of the XIth International Congress of Philosophy*, vol. xiv, Brussels (1953). For the following argumentation I am alone responsible.

There is, in my opinion, only one way to solve the above difficulties: we must not allow that formula *LJxx* should be asserted, i.e. that the principle of identity *Jxx* is necessary. As *Jxx* is a typical analytic proposition, and as there is no reason to treat this principle in a different way from other analytic propositions, we are compelled to assume that no analytic proposition is necessary.

Before dealing with this important topic let us bring to an end our investigation of Aristotle's concepts of modalities.

§ 44. *An Aristotelian paradox*

There is a principle of necessity set forth by Aristotle which is highly controversial. He says in the *De Interpretatione* that 'anything existent is necessary when it exists, and anything non-existent is impossible when it does not exist'. This does not mean, he adds, that whatever exists is necessary, and whatever does not exist is impossible: for it is not the same to say that anything existent is necessary when it does exist, and to say that it is simply necessary.[1] It should be noted that the temporal 'when' (ὅταν) is used in this passage instead of the conditional 'if'. A similar thesis is set forth by Theophrastus. He says, when defining the kinds of things that are necessary, that the third kind (we do not know what the first two are) is 'the existent, for when it exists, then it is impossible that it should not exist'.[2] Here again we find the temporal particles 'when' (ὅτε) and 'then' (τότε). No doubt an analogous principle occurs in medieval logic and scholars could find it there. There is a formulation quoted by Leibniz in his *Theodicee* running thus: *Unumquodque, quando est, oportet esse.*[3] Note again in this sentence the temporal *quando*.

What does this principle mean? It is, in my opinion, ambiguous. Its first meaning seems to be akin to syllogistic necessity, which is a necessary connexion not of terms, but of propositions. Alexander commenting on the Aristotelian distinction between simple and conditional necessity,[4] says that Aristotle was himself

[1] *De int.* 9. 19^a23 τὸ μὲν οὖν εἶναι τὸ ὄν, ὅταν ᾖ, καὶ τὸ μὴ ὂν μὴ εἶναι, ὅταν μὴ ᾖ, ἀνάγκη· οὐ μὴν οὔτε τὸ ὂν ἅπαν ἀνάγκη εἶναι οὔτε τὸ μὴ ὂν μὴ εἶναι. Οὐ γὰρ ταὐτόν ἐστι τὸ ὂν ἅπαν εἶναι ἐξ ἀνάγκης ὅτε ἔστι, καὶ τὸ ἁπλῶς εἶναι ἐξ ἀνάγκης.

[2] Alexander 156. 29 ὁ γοῦν Θεόφραστος ἐν τῷ πρώτῳ τῶν Προτέρων ἀναλυτικῶν λέγων περὶ τῶν ὑπὸ τοῦ ἀναγκαίου σημαινομένων οὕτως γράφει· 'τρίτον τὸ ὑπάρχον· ὅτε γὰρ ὑπάρχει, τότε οὐχ οἷόν τε μὴ ὑπάρχειν.'

[3] *Philosophische Schriften*, ed. Gerhardt, vol. vi, p. 131.

[4] See p. 144, n. 1.

aware of this distinction, which was explicitly made by his friends (that is, by Theophrastus and Eudemus), and quotes as a further argument the passage of the *De Interpretatione* above referred to. He is aware that this passage is formulated by Aristotle in connexion with singular propositions about future events, and calls the necessity involved 'hypothetical necessity' (ἀναγκαῖον ἐξ ὑποθέσεως).[1]

This hypothetical necessity does not differ from conditional necessity, except that it is applied not to syllogisms, but to singular propositions about events. Such propositions always contain a temporal qualification. But if we include this qualification in the content of the proposition, we can replace the temporal particle by the conditional. So, for instance, instead of saying indefinitely: 'It is necessary that a sea-fight should be, when it is', we may say: 'It is necessary that a sea-fight should be tomorrow, if it will be tomorrow.' Keeping in mind that hypothetical necessity is a necessary connexion of propositions, we may interpret this latter implication as equivalent to the proposition: 'It is necessary that if a sea-fight will be tomorrow, it should be tomorrow' which is a substitution of the formula *LCpp*.

The principle of necessity we are discussing would lead to no controversy, if it had only the meaning explained above. But it may have still another meaning: we may interpret the necessity involved in it as a necessary connexion not of propositions, but of terms. This other meaning seems to be what Aristotle himself has in mind, when he expounds the determinist argument that all future events are necessary. In this connexion a general statement given by him deserves our attention. We read in the *De Interpretatione*: 'If it is true to say that something is white or not white, it is necessary that it should be white or not white.'[2] It seems that here a necessary connexion is stated between a 'thing' as subject and 'white' as predicate. Using a propositional variable instead of the sentence 'Something is white' we get the formula: 'If it is

[1] Alexander 141. 1 ἅμα δὲ καὶ τὴν τοῦ ἀναγκαίου διαίρεσιν ὅτι καὶ αὐτὸς οἶδεν, ἣν οἱ ἑταῖροι αὐτοῦ πεποίηνται, δεδήλωκε διὰ τῆς προσθήκης (*scil.* 'τούτων' ὄντων'), ἣν φθάσας ἤδη καὶ ἐν τῷ Περὶ ἑρμηνείας δέδειχεν, ἐν οἷς περὶ τῆς εἰς τὸν μέλλοντα χρόνον λεγομένης ἀντιφάσεως περὶ τῶν καθ' ἕκαστον εἰρημένων λέγει· 'τὸ μὲν οὖν εἶναι τὸ ὄν, ὅταν ᾖ, καὶ τὸ μὴ ὂν μὴ εἶναι, ὅταν μὴ ᾖ, ἀνάγκη'. τὸ γὰρ ἐξ ὑποθέσεως ἀναγκαῖον τοιοῦτόν ἐστι.

[2] *De int.* 9, 18ᵃ39 εἰ γὰρ ἀληθὲς εἰπεῖν ὅτι λευκὸν ἢ ὅτι οὐ λευκόν ἐστιν, ἀνάγκη εἶναι λευκὸν ἢ οὐ λευκόν.

true that *p*, it is necessary that *p*'. I do not know whether Aristotle would have accepted this formula or not, but in any case it is interesting to draw some consequences from it.

In two-valued logic any proposition is either true or false. Hence the expression 'It is true that *p*' is equivalent to '*p*'. Applying this equivalence to our case we see that the formula 'If it is true that *p*, it is necessary that *p*' would be equivalent to this simpler expression: 'If *p*, it is necessary that *p*' which reads in symbols: *CpLp*. We know, however, that this formula has been rejected by Alexander, and certainly by Aristotle himself. It must be rejected, for propositional modal logic would collapse, if it were asserted. Any assertoric proposition *p* would be equivalent to its apodeictic correspondent *Lp*, as both formulae, *CLpp* and *CpLp*, would be valid, and it could be proved that any assertoric proposition *p* was equivalent also to its problematic correspondent *Mp*. Under these conditions it would be useless to construct a propositional modal logic.

But it is possible to express in symbolic form the idea implied by the formula 'If it is true that *p*, it is necessary that *p*': we need only replace the words 'It is true that *p*' by the expression 'α is asserted'. These two expressions do not mean the same. We can put forward for consideration not only true, but also false propositions without being in error. But it would be an error to assert a proposition which was not true. It is therefore not sufficient to say '*p* is true', if we want to impart the idea that *p* is really true; *p* may be false, and '*p* is true' is false with it. We must say 'α is asserted' changing '*p*' into 'α', as '*p*' being a substitution-variable cannot be asserted, whereas 'α' may be interpreted as a true proposition. We can now state, not indeed a theorem, but a rule:

$(x)\quad \alpha \rightarrow L\alpha.$

In words: 'α, therefore it is necessary that α'. The arrow means 'therefore', and the formula (x) is a rule of inference valid only when α is asserted. Such a rule restricted to 'tautologous' propositions is accepted by some modern logicians.[1]

From rule (x) and the asserted principle of identity *Jxx* there follows the asserted apodeictic formula *LJxx* which leads, as we have seen, to awkward consequences. The rule seems to be doubtful, even if restricted to logical theorems or to analytic proposi-

[1] See, e.g. G. H. von Wright, *An Essay in Modal Logic*, Amsterdam (1951), pp. 14–15.

tions. Without this restriction rule (*x*) would yield, as appears from the example given by Aristotle, apodeictic assertions of merely factual truths, a result contrary to intuition. For this reason this Aristotelian principle fully deserves the name of a paradox.

§ 45. *Contingency in Aristotle*

I have already mentioned that the Aristotelian term ἐνδεχόμενον is ambiguous. In the *De Interpretatione*, and sometimes in the *Prior Analytics*, it means the same as δυνατόν, but sometimes it has another more complicated meaning which following Sir David Ross I shall translate by 'contingent'.[1] The merit of having pointed out this ambiguity is due to A. Becker.[2]

Aristotle's definition of contingency runs thus: 'By "contingent" I mean that which is not necessary and the supposed existence of which implies nothing impossible.'[3] We can see at once that Alexander's definition of possibility results from Aristotle's definition of contingency by omission of the words 'which is not necessary'. If we add, therefore, the symbols of these words to our formula 28 and denote the new functor by '*T*', we get the following definition:

46. $QTpKNLp\Pi qCCpqNLNq$.

This definition can be abbreviated, as $\Pi qCCpqNLNq$ is equivalent to $NLNp$. The implication:

39. $CNLNp\Pi qCCpqNLNq$

has been already proved; the converse implication

47. $C\Pi qCCpqNLNqNLNp$

easily results from the thesis $C\Pi qCCpqNLNqCCpqNLNq$ by the substitution q/p, commutation, Cpp, and detachment. By putting in 46 the simpler expression $NLNp$ for $\Pi qCCpqNLNq$ we get:

48. $QTpKNLpNLNp$.

This means in words: 'It is contingent that *p*—if and only if—it

[1] W. D. Ross, loc. cit., p. 296.

[2] See A. Becker, *Die Aristotelische Theorie der Möglichkeitsschlüsse*, Berlin (1933). I agree with Sir David Ross (loc. cit., Preface) that Becker's book is 'very acute', but I do not agree with Becker's conclusions.

[3] *An. pr.* i. 13, 32^{a}18 λέγω δ' ἐνδέχεσθαι καὶ τὸ ἐνδεχόμενον, οὗ μὴ ὄντος ἀναγκαίου, τεθέντος δ' ὑπάρχειν, οὐδὲν ἔσται διὰ τοῦτ' ἀδύνατον.

is not necessary that *p* and it is not necessary that not *p*.' As the phrase 'not necessary that not *p*' means the same as 'not impossible that *p*', we may say roughly speaking: 'Something is contingent if and only if it is not necessary and not impossible.' Alexander shortly says: 'The contingent is neither necessary nor impossible.'[1]

We get another definition of Tp, if we transform $NLNp$ according to our definition 1 into Mp, and NLp into MNp:

49. $QTpKMNpMp$ or 50. $QTpKMpMNp$.

Formula 50 reads: 'It is contingent that *p*—if and only if—it is possible that *p* and it is possible that not *p*.' This defines contingency as 'ambivalent possibility', i.e. as a possibility which can indeed be the case, but can also not be the case. We shall see that the consequences of this definition, together with other of Aristotle's assertions about contingency, raise a new major difficulty.

In a famous discussion about future contingent events Aristotle tries to defend the indeterministic point of view. He assumes that things which are not always in act have likewise the possibility of being or not being. For instance, this gown may be cut into pieces, and likewise it may not be cut.[2] Similarly a sea-fight may happen tomorrow, and equally it may not happen. He says that 'Of two contradictory propositions about such things one must be true and the other false, but not this one or that one, only whichever may chance (to be fulfilled), one of them may be more true than the other, but neither of them is as yet true, or as yet false.'[3]

These arguments, though not quite clearly expressed or fully thought out, contain an important and most fruitful idea. Let us take the example of the sea-fight, and suppose that nothing is decided today about this fight. I mean that there is nothing that is real today and that would cause there to be a sea-fight tomorrow, nor yet anything that would cause there not to be one. Hence, if

[1] Alexander 158. 20 οὔτε γὰρ ἀναγκαῖον οὔτε ἀδύνατον τὸ ἐνδεχόμενον.

[2] *De int.* 9, 19^{a}9 ἔστιν ἐν τοῖς μὴ ἀεὶ ἐνεργοῦσι τὸ δυνατὸν εἶναι καὶ μὴ ὁμοίως . . . 12 οἷον ὅτι τουτὶ τὸ ἱμάτιον δυνατόν ἐστι διατμηθῆναι, . . . ὁμοίως δὲ καὶ τὸ μὴ διατμηθῆναι δυνατόν.

[3] Ibid. 19^{a}36 τούτων γὰρ (i.e. ἐπὶ τοῖς μὴ ἀεὶ οὖσιν ἢ μὴ ἀεὶ μὴ οὖσιν) ἀνάγκη μὲν θάτερον μόριον τῆς ἀντιφάσεως ἀληθὲς εἶναι ἢ ψεῦδος, οὐ μέντοι τόδε ἢ τόδε ἀλλ' ὁπότερ' ἔτυχε, καὶ μᾶλλον μὲν ἀληθῆ τὴν ἑτέραν, οὐ μέντοι ἤδη ἀληθῆ ἢ ψευδῆ.

truth rests on conformity of thought with reality, the proposition 'The sea-fight will happen tomorrow' is today neither true nor false. It is in this sense that I understand the words 'not yet true or false' in Aristotle. But this would lead to the conclusion that it is today neither necessary nor impossible that there will be a sea-fight tomorrow; in other words, that the propositions 'It is possible that there will be a sea-fight tomorrow' and 'It is possible that there will not be a sea-fight tomorrow' are today both true, and this future event is contingent.

It follows from the above that according to Aristotle there exist true contingent propositions, i.e. that the formula *Tp* and its equivalent *KMpMNp* are true for some value of *p*, say α. For example, if α means 'There will be a sea-fight tomorrow', both $M\alpha$ and $MN\alpha$ would be accepted by Aristotle as true, so that he would have asserted the conjunction:

(A) $KM\alpha MN\alpha$.

There exists, however, in the classical calculus of propositions enlarged by the variable functor δ, the following thesis due to Leśniewski's protothetic:

51. $C\delta pC\delta Np\delta q$.

In words: 'If δ of *p*, then if δ of not *p*, δ of *q*', or roughly speaking: 'If something is true of the proposition *p*, and also true of the negation of *p*, it is true of an arbitrary proposition *q*.' Thesis 51 is equivalent to

52. $CK\delta p\delta Np\delta q$

on the ground of the laws of importation and exportation *CCpCqrCKpqr* and *CCKpqrCpCqr*. From (A) and 52 we get the consequence:

52. $\delta/M,\ p/\alpha,\ q/p \times C$(A)–(B)

(B) *Mp*.

Thus, if there is any contingent proposition that we accept as true, we are bound to admit of any proposition whatever that it is possible. But this would cause a collapse of modal logic; *Mp* must be rejected, and consequently $KM\alpha MN\alpha$ cannot be asserted.

We are at the end of our analysis of Aristotle's propositional

modal logic. This analysis has led us to two major difficulties: the first difficulty is connected with Aristotle's acceptance of true apodeictic propositions, the second with his acceptance of true contingent propositions. Both difficulties will reappear in Aristotle's modal syllogistic, the first in his theory of syllogisms with one assertoric and one apodeictic premiss, the second in his theory of contingent syllogisms. If we want to meet these difficulties and to explain as well as to appreciate his modal syllogistic, we must first establish a secure and consequent system of modal logic.