HISTORY AND PHILOSOPHY OF LOGIC, 16 (1995), 39–64

On the Completeness of Non-Philonian Stoic Logic

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Received 16 March 1993
Revised 15 April 1994

The majority of formal accounts attribute to Stoic logicians the classical truth-functional understanding of the material conditional and exclusive disjunction. These interpretations were disputed, some Stoic logicians favouring modal and/or temporal analyses; moreover, what comes down to us of Stoic logic fails to secure the classical interpretations on purely formal grounds. It is therefore of some interest to see how the non-classical interpretations fare. I argue that the strongest logic we have good grounds to attribute to Stoic logicians is not complete with respect to the non-classical interpretations of disjunction and the conditional.

> The Stoics maintained that their system of propositional logic was complete in the sense that every valid argument could be reduced to a series of arguments of five basic types. Even the method of reduction was not left vague, but was exactly characterized by four meta-rules, of which we possess two, and possibly three. Whether or not the Stoic system was actually complete could be decided only with the help of the missing rules.[1]

Thus Benson Mates. His conclusion is certainly correct. However, despite the missing rules, by adding to the basic types and known rules some further rules which there may be some grounds to suppose at least some Stoic logicians would have accepted, Stoic logic has been argued to be complete, notably by Oskar Becker and by Urs Egli.[2] A more modest claim, made by William and Martha Kneale and by Ian Mueller, is that Stoic logic is complete with respect to arguments employing statements containing only the connectives for negation and conjunction.[3] The two styles of completeness claim are related, for Becker and Egli both appeal to the representability of the material conditional and disjunction using only negation and conjunction. Now, any such representation takes it for granted that the conditional is to be understood as the material conditional, as Philo maintained, and that disjunction is to be understood in the usual classical manner, with the proviso that it be read as exclusive disjunction. Both these interpretations were subject to dispute by Stoic logicians. Consequently, any claim in favour of the completeness of Stoic logic that presupposes them is controversial, the more so as the rules that have come down to us are much too weak to secure the classical interpretations. On the other hand, under the rival non-classical interpretations one should expect certain purely classical

1 Benson Mates, *Stoic Logic* (second edition), University of California Press, Berkeley and Los Angeles, 1961, p. 4.
2 Oskar Becker, *Zwei Untersuchungen zur antiken Logik*, Otto Harrossowitz, Wiesbaden, 1957; Urs Egli, 'The Stoic Theory of Arguments', in Rainer Bäuerle *et al.* (eds.), *Meaning, Use, and Interpretation of Language*, Walter de Gruyter, Berlin, 1983, pp. 79–96.
3 William and Martha Kneale, *The Development of Logic*, Oxford University Press, Oxford, 1962, p. 174; Ian Mueller, 'The Completeness of Stoic Logic', *Notre Dame Journal of Formal Logic*, **20**, 1979, 201–215, p. 214.

principles to fail and hence to be absent from Stoic logic. What I wish to do in this article is to investigate the completeness claim with respect to non-classical Stoic interpretations of disjunction and the conditional. Stoic logicians who opposed the truth-functional readings of disjunction and the conditional proposed modal interpretations, whether temporally based (stemming from Diodorus Cronus) or employing some stringent notion of compatibility (Chrysippus). Ultimately we shall find that the Stoic claim to completeness appears unwarranted.

In Section I I present Basic Stoic Logic, as I shall call it. This is a system of logic that we know Stoics would have accepted but it is badly incomplete under both the classical and non-Philonian interpretations of the connectives. In Section II I augment this system to produce the strongest system of propositional logic that we have good grounds to attribute to Stoic logicians. Here I appeal to recent work by Katerina Ierodiakonou concerning 'three-component arguments'.[4] I call the system there obtained Extended Stoic Logic.

As already observed, the interpretation of the conditional exercised Stoic logicians. In Section III I argue, contrary to a number of authors, that it is unlikely that Stoic logicians employed anything like the modern rule of conditional proof. Finally, in Section IV I turn to the modal readings of the conditional and disjunction favoured by some Stoics. In this section investigation of Extended Stoic Logic is pursued and in the process a question posed by Ierodiakonou is answered. Extended Stoic Logic is found to be incomplete with respect to the modal readings of disjunction and the conditional.

I

The following argument-forms comprise the five Stoic indemonstrables:

(i) $p \rightarrow q,\ p \vdash q$;
(ii) $p \rightarrow q,\ \neg q \vdash \neg p$;
(iii) $\neg(p\ \&\ q),\ p \vdash \neg q$;
(iv) $p \blacktriangledown q,\ p \vdash \neg q$;
(v) $p \blacktriangledown q,\ \neg q \vdash p$;

where '$\blacktriangledown$' symbolises exclusive disjunction. In adopting this formulation of the fifth indemonstrable I follow Mates, the Kneales and Michael Frede.[5] Mueller prefers $p \blacktriangledown q,\ \neg p \vdash q$, and, as he shows, which is taken as basic can make a difference. Becker and I. M. Bocheński too favoured this form for the fifth indemonstrable.[6] While it is widely agreed that there were just five indemonstrables, and Chrysippus insisted on this,[7] were we to follow Diogenes Laertius's informal account there would be eight, (iii)–(v) having the parallel forms:

4 Katerina Ierodiakonou, 'Rediscovering Some Stoic Arguments', in Pantelis Nicolacopoulos (ed.), *Greek Studies in the Philosophy and History of Science*, Kluwer, Dordrecht, 1990, pp. 137–148.
5 See Michael Frede, *Die Stoische Logik*, Vandenhoeck & Ruprecht, Göttingen, 1974, p. 131.
6 Mueller, pp. 203, 211–212; Becker, p. 29; I. M. Bocheński, *Ancient Formal Logic*, North-Holland, Amsterdam, 1957, p. 98. Becker's formulation employs his contradiction operator (on which more below) rather than negation. Mates, aware of the rival formulation, gives his reasons for his choice, p. 73.
7 See Mates, p. 69. On candidates for further indemonstrables see Mates, *loc. cit.*, and Frede, pp. 157–167.

(iii′) $\neg(p\ \&\ q),\ q \vdash \neg p$;
(iv′) $p \blacktriangledown q,\ q \vdash \neg p$;
(v′) $p \blacktriangledown q,\ \neg p \vdash q$.[8]

The indemonstrables yield argument-forms all of whose instances are valid; they are not rules of inference.[9] In each case the minor premise and the conclusion are subformulae, or negations of subformulae, of the major premise. Each indemonstrable has two premises neither of which alone entails the conclusion. More remarkably they all enjoy a certain symmetry—the conclusion and minor premise classically entail the major premise. (The symmetry is not perfect for in cases (i)–(iii) the conclusion alone does this.)

Arguments are valid if they can be reduced to instances of the indemonstrables. More loosely, we shall say that argument-forms are valid if they can be reduced to the indemonstrables. The modern logician reads this as meaning that valid argument-forms are generated from the indemonstrables by means of transformation rules as in a sequent calculus, i.e., by means of rules that take one from valid argument-forms to the form of a valid argument, certain valid argument-forms being taken as basic. In what follows I shall analyze valid argument-forms in this way. Perhaps a better approximation to Stoic intentions would be achieved by inverting the proof-trees so that one works downwards from the complex forms towards the basic argument-forms.[10]

There are many ways in which we could set out a formalization of Stoic logic but a sequent calculus approach seems more faithful to Stoic intentions and practice.[11] As our transformation rules we take the first thema and the theorema that is mentioned by Sextus Empiricus. These too are not rules of inference; they are conditional assertions about validity (or, rather, generalizations of such). They are used to transform arguments into arguments. That is, although we present them in the form of rules for the transformation of argument-forms, their proper application is in the transformation of concrete arguments.[12] By '$\Sigma \vdash p$' we symbolize ambiguously *both* the assertion that the argument-form with premises Σ and conclusion p is valid where Σ is a finite, non-empty, set of formulae, p a single formula *and* the argument-form asserted to be valid. Context should make clear whether it is the assertion of validity or simply the argument-form that is intended.[13]

8 DL, *De Vitae*, vii, 76–81. See Mueller, p. 205.
9 On this important distinction see John Corcoran, 'Remarks on Stoic Deduction', in J. Corcoran (ed.), *Ancient Logic and Its Modern Interpretations*, Reidel, Dordrecht, pp. 169–81. As Corcoran observes, p. 180, if we accept that the aim and purpose of Stoic logic was the analysis or reduction of complex arguments in terms of simple arguments then there can be little doubt that the indemonstrables represent argument-forms, not rules of inference.
10 Corcoran makes this point, pp. 180–181. See also the proofs in Frede, pp. 186–190, which work down and across from left to right in the direction of simpler arguments.
11 Some authors take a different tack. Mario Mignucci, *Il significato della logica stoica*, Casa editrice Prof. Ricardo Pàtron, Bologna, 1967, p. 145, and Bocheński, pp. 93–98, for example, replace the indemonstrables by axiom-schemata and state versions of the first and third themata also as axiom-schemata.
12 See Frede, pp. 167–169. Frede observes that a number of modern commentators have erroneously taken Stoic logic to comprise rules of proof (the indemonstrables) and means to generate further rules from them (the themata). Becker, for example, declares, p. 27, 'die stoische Logik war eine Regellogik'. He refers to the themata as 'Metaregeln'.
13 This mildly reprehensible practice is copied from E. J. Lemmon's *Beginning Logic* (second edition), Chapman and Hall, London, 1987, pp. 11–12, 48–49. (For mild criticism see Stephen Read, *Relevant Logic*, Basil Blackwell, Oxford, 1988, p. 54.) The reader is referred to Lemmon, *op. cit.*, for metalogical notions.

Dropping the set braces for small sets, the first thema asserts:[14]

$$\text{if } p,\ q \vdash r \text{ then } p,\ \neg r \vdash \neg q \text{ and } q,\ \neg r \vdash \neg p,$$

that is, if the first argument-form is valid then so are the other two. I should perhaps say that the first thema asserts at least this much. This is the form in which it is presented by the Kneales and which Becker and Mueller attribute to Mates.[15] Further discussion is deferred until Section II.

In the light of this thema the second indemonstrable is reducible to the first, which raises questions as to the exact status of the indemonstrables and to the extent to which the themata were accepted by Stoic logicians. The exact significance of the term 'indemonstrable' is in any case unclear.[16]

Where Σ is a non-empty set of formulae and where Σ, q is the set obtained by adding q to Σ, the third thema asserts:[17]

$$\text{If } p,\ q \vdash r \text{ and } \Sigma \vdash p \text{ then } \Sigma,\ q \vdash r.$$

Sextus's theorema is not unambiguous but it appears to be a generalization of the third thema:[18]

$$\text{If } \Sigma \vdash p \text{ and } \Sigma,\ p \vdash q \text{ then } \Sigma \vdash q,$$

where Σ is a non-empty set of formulae. Henceforth I shall take the qualification 'non-empty' as read.

To what extent Sextus's theorema encompasses the missing second and fourth themata is not known. Anthony Long and David Sedley suggest it covers these and, in addition, the third thema. Some support for this comes from Galen's claim that Antipater simplified matters in showing that not all Chrysippus's themata are necessary.[19]

Now, I have already used the notation '$\Sigma \vdash p$' to stand for 'the argument-form with premises Σ and conclusion p' where Σ is understood to be a set of formulae. If we take this as the basic 'unit' of our sequent system we are committed to the view that the order and multiplicity of occurrence of premises is irrelevant to the validity of the argument in question. Sequent calculi, which come from Gentzen, are most often systematized using *ordered* sequences as antecedents, thus '$p \to q,\ p \vdash q$' is a different sequent from '$p,\ p \to q \vdash q$'. Structural rules permitting permutation of formulae and deletion of repetitions in the antecedent are usually added so that validity is effectively a property of (unordered) *sets* of premises and conclusions. For sure, a certain economy is achieved if, *ab initio*, we take the antecedents of sequents to comprise sets rather than ordered sequents but in doing so will we overlook anything important in the Stoic analysis of arguments?

14 Apuleius, *De Interpretatione*, 191, 5–10.
15 The Kneales, p. 169; Becker, p. 30, n.5; Mueller, p. 203.
16 See Mates, pp. 63–64.
17 Alexander, *In An. Pr. Comm.*, 278, 11–14.
18 SE, *Adv. Math.*, viii, 231. See Mates, pp. 77–78.
19 A. A. Long and D. N. Sedley, *The Hellenistic Philosophers*, Volume I, Cambridge University Press, Cambridge, 1987, pp. 219–220. Long and Sedley offer the following reconstructions of the three themata:
(II) if $\Sigma \vdash p$ and $p,\ q \vdash r$, where $q \in \Sigma$, then $\Sigma \vdash r$;
(III) if $p,\ q \vdash r$ and $\Sigma \vdash p$, where $p \notin \Sigma$ and $q \notin \Sigma$, then $\Sigma,\ q \vdash r$;
(IV) if $p,\ q \vdash r$, $\Sigma \vdash p$ and $\Gamma \vdash q$, where $p \notin \Sigma$, $q \notin \Sigma$, $p \notin \Gamma$, $q \notin \Gamma$, and $\Sigma \cap \Gamma = \varnothing$, then $\Sigma \cup \Gamma \vdash r$.

Stoic logicians appear to have taken exception to arguments with fewer than two premises.[20] The evidence on this is far from conclusive. Mueller observes that 'the only one-premised arguments to which we know Chrysippus objected were non-truth-functional ones such as 'You breathe; therefore you are alive'.'[21] Sextus Empiricus makes much of Antipater's assertion that there can be one-premised arguments.[22] Apuleius says that Antipater was wrong to go against the universal consensus, citing as an example of Antipater's single-premised arguments an enthymeme of exactly the sort Mueller mentions.[23] As Frede says, it is unclear on what grounds either side advocated its position.[24]

Following Mueller, the formalization of Stoic logic that employs sequents with set antecedents allows this derivation:[25]

$$\frac{\neg p \to p,\ \neg p \vdash p}{\neg p \vdash \neg(\neg p \to p)},$$

where the first sequent is an instance of the first indemonstrable and the second is obtained by an application of the first thema. Another example is obtained from the fifth indemonstrable:

$$\frac{p \blacktriangledown p,\ \neg p \vdash p}{\neg p \vdash \neg(p \blacktriangledown p)}.$$

Notice that it is immaterial which of the rival readings of the fifth indemonstrable is employed. Notice also that on any conception of exclusive disjunction the conclusion to the second sequent is a logical truth and the single premise is logically otiose. The important point here, however, is that our chosen formalization of Stoic logic allows the derivation of one-premised arguments from the indemonstrables. A sequent calculus formalization that uses sequence antecedents rather than set antecedents can avoid this consequence, for it differentiates between

$$\frac{\neg p \to p,\ \neg p \vdash p}{\neg p \vdash \neg(\neg p \to p)}$$

and

$$\frac{\neg p \to p,\ \neg p \vdash p}{\neg p,\ \neg p \vdash \neg(\neg p \to p)},$$

only the second of which it would count as derivable by a single application of the first thema to an instance of the first indemonstrable.[26] Thus there might well seem to be grounds for preferring this second type of sequent calculus presentation. In employing such a calculus we have to be careful about repetitions. Perhaps the Stoics were.[27] That said, the most natural readings of the third thema and Sextus's theorema appeal to sets of premises. Of course, they can be reformulated, at the cost of some complexity, in terms of sequents with sequence antecedents.

20 Mates, pp. 65–66. *Cf.* the Kneales, pp. 169, 174.
21 Mueller, pp. 203–204.
22 SE, viii, 443. On Antipater see Mates, p. 65.
23 Apuleius, *De Int.*, 184, 16–23.
24 Frede, p. 118.
25 See Mueller, p. 203.
26 *Cf.* the fourth row of the analysis presented at Frede, p. 186.
27 Egli suggests, p. 93, on the basis of a reading of Alexander, *In An. Pr. Comm.*, 17.11–18.7, that a rule permitted the introduction and deletion of repetitions of statements. For further discussion of this topic see Frede, pp. 181–190.

Irrespective of the form of sequents adopted, a further problem looms large. The sequent $p,\ p \rightarrow q \vdash q$ is valid, i.e., it is a valid argument-form. On almost any reasonable understanding of validity the sequent $p \,\&\, (p \rightarrow q) \vdash q$ would also be valid. But it is not, if arguments must have two premises. Becker suggests that failure to differentiate between $p,\ q \vdash r$ and $p \,\&\, q \vdash r$ *on the part of later commentators* is the root problem here.[28] This may indeed by the best explanation, for it requires some skill in distinguishing object-language and meta-language conjunctions to appreciate the difference, a skill which, Becker suggests, the formalistically-minded Stoics would have possessed but which a Peripatetic like Alexander would not. There is an alternate explanation, at once more charitable to Alexander and founded on Stoic concerns. Given the dialectical origins of Stoic logic it would be natural for Stoic logicians to think of the two sequents as inessential variants and to count both as having (at least) two premises, for having established that one's opponent accepts p and accepts q it is hardly an important dialectical move to conjoin them and have him concede $p \,\&\, q$.[29] (The Stoic criterion of validity, discussed below in Section III, also encourages assimilation of the two forms.)

As our aim is to arrive at the strongest logic that we can reasonably attribute to Stoic logicians we shall accept the sequent formulation with set antecedents and allow one-premise arguments. One reason for this is that the Stoic criterion for validity gives no reason to rule them out. Another is that, even if a one-premise argument is counted as invalid in virtue of its lack of premises, in a sequent formulation with sequence antecedents repetition produces a surrogate: $\neg p \vdash \neg(\neg p \rightarrow p)$ may not be valid but $\neg p,\ \neg p \vdash \neg(\neg p \rightarrow p)$ is. Finally, nothing of great moment will depend on this decision.

Corresponding to the first thema we have the rule:

$$\frac{\Sigma,\ p \vdash q}{\Sigma,\ \neg q \vdash \neg p}.$$

Its use in derivations will be signalled by (I).

In an orthodox sequent system Sextus's theorema is equivalent to the familiar *Cut* rule which we shall adopt as part of Stoic logic:[30]

$$\frac{\Sigma,\ p \vdash q \qquad \Gamma \vdash p}{\Sigma \cup \Gamma \vdash q}.$$

Its use in derivations will be signalled by (III).

In addition to the indemonstrables, which serve as basic sequents, one further type of basic sequent is admitted:

$$\Sigma \vdash p, \text{ where } \Sigma \text{ contains } p.$$

Justification for this rests on the Stoics' approval of what Mates calls tautologous inferences, i.e., inferences in which the conclusion is identical with one of the premises.[31] Stoics, in opposition to Aristotelians, counted inferences of the form

28 Becker, pp. 41–43.
29 On the dialectical origin of Stoic logic see Long and Sedley, p. 218.
30 *Cf.* Mueller, p. 203.
31 Mates, p. 66; in translation of Alexander, *In. Top.*, 10, 5ff, he has, p. 125 'unanalyzed inferences'. Long and Sedley, p. 219, have 'tautologically valid' in translation of Galen, *De plac. Hipp. et Plat.*, 2.3.18–19, but note, p. 220, that literally the expression used is 'indifferently valid'.

$p,\ q \vdash p$ as obviously valid. Mates gives no indication as to whether these inferences were thought reducible to the indemonstrables. Two passages, one from Alexander, the other from Galen, state that they did and that the missing second thema was used for this purpose.[32] No indication is given, however, of the second thema and its application in this context. For this reason I choose to adopt sequents of the form $\Sigma \vdash p$, where Σ contains p, as basic.[33]

This completes the formulation of what I shall henceforth call Basic Stoic Logic (BSL).

As a first meta-theorem we show that the addition of tautologous inferences to the stock of basic sequents extends what would otherwise be BSL. Call a sequent *complex* if any formula in its antecedent contains $\neg$, $\&$, $\blacktriangledown$, or $\rightarrow$. The indemonstrables are all complex in this sense. Complexity is preserved under the first thema, (I), and cut, (III), hence any sequent obtained from the indemonstrables by means of these transformation rules is complex. On the other hand, where p and q contain no logical constants, $p,\ q \vdash p$ is a basic sequent but is not complex.

The indemonstrable and the forms of tautologous arguments have in common the property that the antecedent is never empty. This property is preserved by the sequent rules (I) and (III), hence if $\Sigma \vdash p$ is derivable in BSL then $\Sigma \neq \varnothing$.

We next show:

Lemma 1 Gentzen's structural thinning rule preserves validity in BSL:

$$\frac{\Sigma \vdash p}{\Sigma \cup \Gamma \vdash p},$$

where Γ is some set, possibly empty, of formulae.

Proof $\Gamma,\ p \vdash p$ is a basic sequent. A single application of *Cut* gives us:

$$\frac{\Gamma,\ p \vdash p \qquad \Sigma \vdash p}{\Sigma \cup \Gamma \vdash p} \text{ (III)}.$$

As the added basic sequent is valid thinning (augmentation of premises) preserves validity. (End of proof)

32 Alexander, *In An. Pr. Comm.*, 164, 29–31. Galen, *loc. cit.*.
33 Mueller, p. 204, has reservations about Becker's adoption of $p \vdash p$ but in view of the approval of tautologous inferences there seems to be little reason for such hesitancy. It is hard to believe that $p,\ q \vdash p$, an instance of which is explicitly given as a tautologous inference, would be accepted whilst $p \vdash p$ were not. Only dogmatic adherence to the notion that every argument must have two or more premises could explain such a difference, for in holding $p,\ q \vdash p$ valid, only q's presence is required, its form is immaterial. That being the case it is difficult to see how one might argue that $p,\ q \vdash p$ is valid without tacitly appealing to the validity of $p \vdash p$.

In *La sillogistica stoica*, an unpublished typescript, Paolo Crivelli observes that if the two rules

$$\frac{\Sigma \vdash p \,\&\, q}{\Sigma \vdash p} \quad \text{and} \quad \frac{\Sigma \vdash p \,\&\, q}{\Sigma \vdash q}$$

are adopted in Stoic logic then tautologous inferences are derivable with the aid of version (Ic) of the first thema (see §II below) and the third indemonstrable. Crivelli cites the passages from Galen and Alexander in support of the view that these rules might correspond to the missing second thema. Egli, p. 89, based on the same sources, suggests instead the rule (his word), 'if the same is posited and denied with or without other things, then whatsoever is posited', and the schema, 'the first, not the first, therefore the second'. I take this to amount, in the present setting, to taking $p,\ \neg p \vdash q$ as a basic sequent. All tautologous inferences then become derivable with the aid of (Ic). Crivelli's rules have the additional virtue that they obviate the need for the rule (IV) introduced in §II below but, being derivable in Extended Stoic Logic, they do not add to the strength of Stoic logic as here formalized.

In view of Lemma 1 we shall henceforth use the thinning rule, signalling its use by *Aug*.

Despite Lemma 1, thinning calls for comment. Sextus Empiricus maintained that the Stoics counted as invalid any inference with logically redundant premises.[34] If that were so, thinning could not possibly preserve validity. To avoid outright contradiction with other sources one would have to maintain, firstly, that valid arguments must have at least two premises and, secondly, that the only valid tautologous arguments contain exactly two premises. Also, not all instances of our reconstruction of the third thema would preserve validity.[35] But worse than that, invalidity by redundancy robs Sextus's theorema of any interesting application, for if $\Sigma \vdash q$ is valid and $p \notin \Sigma$ then $\Sigma, p \vdash q$ is invalid by redundancy; hence the only circumstance in which the conditions for application of the theorema, namely $\Sigma \vdash p$ and $\Sigma, p \vdash q$, are satisfied occurs when $p \in \Sigma$ and nothing is gained. Moreover, Sextus's assertion is difficult to reconcile with the Stoic criterion of validity, as we shall see. For these reasons I shall not seek to make the formalizations of Stoic logic here presented compatible with the idea of invalidity by redundancy.

As a further meta-theoretic observation we note:

Lemma 2 Let Q be a property of sequents (argument-forms) and suppose that the sequents corresponding to tautologous inferences possess Q and that the cut rule, i.e., the third thema, preserves Q. Then the sequent rule

$$\frac{\Sigma \vdash p \qquad \Sigma \vdash q}{\Sigma \vdash r}$$

preserves Q if, and only if, the sequent $p, q \vdash r$ possesses Q.

Proof. Suppose first that the rule preserves Q. Then

$$\frac{p, q \vdash p \qquad p, q \vdash q}{p, q \vdash r}$$

is an instance of the rule in which the premises, being the forms of tautologous inferences, possess Q. As the rule preserves Q the conclusion also possesses Q.

Next suppose that $p, q \vdash r$ possesses Q. Then

$$\dfrac{\dfrac{\dfrac{p, q \vdash r}{\Sigma, p, q \vdash r}\,\text{Aug} \qquad \Sigma \vdash p}{\Sigma, q \vdash r}\,\text{(III)} \qquad \Sigma \vdash q}{\Sigma \vdash r}\,\text{(III)}$$

employs only the Q-preserving (III) and, implicitly, the forms of tautologous inferences in the instance of thinning, so $\Sigma \vdash r$ possesses Q if $\Sigma \vdash p$ and $\Sigma \vdash q$ do. (End of proof.)

To some extent Lemma 2 allows us to ignore the difference between taking Stoic logic to comprise a set of inference rules and seeing it as a set of argument-forms together with transformation rules. Consequently, in what follows I shall freely make use of the validity-preserving sequent rules associated with valid argument-forms. For example, the form $p, p \to q \vdash q$ is associated with the rule

34 See, e.g., SE, viii, 431.

35 I owe this observation to Paolo Crivelli. His example: suppose that $p_1, p_2 \vdash q$, then by *Cut*, since $p_1, p_3 \vdash p_1$, we have $p_1, p_2, p_3 \vdash q$ although p_3 is logically redundant, contrary to Sextus's stipulation.

$$\frac{\Sigma \vdash p \qquad \Sigma \vdash p \to q}{\Sigma \vdash q}.$$

BSL is hugely incomplete. One way to appreciate this is to observe how little it does to fix the interpretations of &, $\blacktriangledown$ and $\to$. Reading negation classically, take $p \,\&\, q$ to be uniformly true, $p \blacktriangledown q$ and $p \to q$ uniformly false for all p and q. The indemonstrables are all trivially correct under this reading, for each then contains a necessarily false formula in its antecedent. Moreover, as negation is classical the first thema preserves correctness under these readings. (Just as, strictly, it is arguments that are valid, not argument-forms, so properly, it is sentences and/or propositions, not formulae, that are true or false; however, we shall follow common logical practice. A formula is necessarily false if all its instances are false.)

Under these interpretations the sequent $\neg p, \neg q \vdash \neg(p \,\&\, q)$ ought not to be derivable (and is not), for if p and q are both false the antecedent comprises true formulae but the conclusion is necessarily false. However, the sequent is certainly valid under the standard truth-functional reading of conjunction adopted by all Stoic logicians: there are explicit statements of this reading of conjunction.[36] Consequently, BSL is incomplete.

The unorthodox readings of the connectives also suffice to show that the Philonian, truth-functional readings of $\blacktriangledown$ and $\to$ also fail to be forthcoming in BSL. We can also easily show that $\Sigma, p \vdash \neg\neg p$ is not derivable in BSL. Consider $p, q \vdash \neg\neg q$ where p and q contain no connectives. This is not itself a basic sequent. No basic sequent contains atomic antecedents and a complex consequent. No application of (I) or (III) can generate such a sequent from sequents that are not themselves of this form. So $p, q \vdash \neg\neg q$ is not derivable in BSL.

II

There is some evidence that Stoic negation is classical. Mates quotes Alexander attributing acceptance of the equivalence of p and $\neg\neg p$ to Stoics: 'For "Not: not: it is day" differs from "It is day" only in manner of speech'.[37] Probably the best way to accommodate this observation in a formal logic would be to adopt the substitution rule:[38]

> any occurrence of a subformula $\neg\neg p$ within a formula in a sequent may be replaced by p; any occurrence of a subformula p within a formula in a sequent may be replaced by $\neg\neg p$.

However, Frede finds Alexander's testimony that a sentence and its double-negation differ only in expression 'more than questionable'.[39] On the other hand he acknowledges that we know from other sources that the double negation is supposed to cancel itself (*sich aufheben*).[40] Becker too says that 'the rule of

36 Mates, p. 54; Frede, p. 77. On anti-Stoic criticisms of the truth-functional understanding see Jacques Brunschwig, 'Le modèle conjonctif', in *Les Stoïciens et leur logique*, J. Vrin, Paris, 1978, pp. 58–86.

37 Alexander, *In An. Pr. Comm.*, 18, 6–7.

38 On double negation rules as substitution rules see, e.g., Irving Copi, *Symbolic Logic* (fifth edition), Macmillan, New York, 1979, pp. 39–40.

39 Frede, p. 72.

40 *Ibid.*

self-cancellation of a double negative was known in Stoic logic'.[41] The source most commonly indicated for an account of double negation is Diogenes Laertius, *Lives of Eminent Philosophers*, vii, 69. Both Bocheński and the Kneales cite this passage, the former in asserting that Stoics 'stated the principle of double negation', the latter when claiming that Stoics 'formulated the appropriate rules' for double negation.[42] Its brevity alone ensures that the passage in question is hardly conclusive.[43]

Perhaps we are worrying about nothing here. Egli informs us that 'Stoics avoided double negation by using the concept of the contrary'.[44] Now, 'avoided' is a strong term. It can hardly be true, given their discussion of the doubly negated sentence (*hyperapophatikon*), that they avoided double negation. The most that can be said is that they avoided explicit rules for it in their logic.[45] Accounts of how they managed this appeal to their use of the notion of the 'opposite' (*das Gegenteil*, Becker) or 'contrary' (Egli) of a sentence. Mueller finds obscurity in Becker's account, offering two readings in accord with Becker's stipulations.[46] I find incoherence in Egli's, unless, as he says, Stoic logicians did avoid double negations altogether. His formal account merely repeats what he says informally:[47]

> [T]he contrary of an unnegated sentence is its negation, the contrary of the negation of a sentence is this sentence, the contrary of the contrary of a sentence is therefore this very sentence.

What is the contrary of the contrary of a doubly negated sentence? By Egli's stipulations the contrary of the contrary of $\neg\neg p$ is the contrary of $\neg p$, which, in turn, is just p. But p and $\neg\neg p$ are not the same formula.

Dismissing Egli's unjustifiable conclusion, his definition of the contrary corresponds to one of Mueller's proposals. Mueller's other suggestion is that the contrary of p stands ambiguously for q and $\neg\neg q$ when p is $\neg q$, and is $\neg p$ otherwise.[48] The important consequence, on either reading, is that a stronger first thema results. Apuleius says:[49]

41 Becker, p. 47.
42 Bocheński, p. 89. The Kneales, p. 169. As Becker, p. 29, and Mates, p. 31, both observe, Diogenes's putative example of a doubly negated sentence does not meet his own definition. See Frede, pp. 71–72, on amendment to the text.
43 Long and Sedley, p. 205, omit it from their translation of DL, vii, 69–70. In translations of this passage concerning the relation of the double-negation to the unnegated original Hicks has 'presupposes'; Mates, Mueller and Egli put 'posits', although Mates also has 'asserts'; Egli asserts that 'posit' is the contrary of 'deny'; in commentary rather than translation the Kneales have 'is equivalent to'. See Diogenes Laertius, *Lives of Eminent Philosophers*, revised translation in two volumes by R. D. Hicks, Heinemann, London, 1950, p. 179; Mates, pp. 31, 113; Mueller, p. 207; Egli, p. 89; the Kneales, p. 147.
44 Egli, p. 81.
45 Even this is implicitly denied by Egli, pp. 92–93, who infers a substitution rule for double negations, or, as he has it, a 'convention of replacement of statements of the same force' to the effect that 'double negation signs may be omitted or introduced', from the cited passage by Alexander and urges comparison with DL, vii, 69.
46 Mueller, pp. 204–205. It is strange that Becker should be so lax in defining the operation he calls *Kontradiktion*, for he informs us, p. 28, that 'logically the difference between *antikeimenon* [contrary] and *apophatikon* [negative] is [. . .] of quite fundamental significance'.
47 Egli, p. 81.
48 Mueller, p. 204. Read, pp. 178–182, uses the same formulation in an analysis of the operator N in Wittgenstein's account of negation in the *Tractatus*.
49 Apuleius, *De Int.*, 191, 5–10. Mueller, p. 201, has 'opposite' for 'contradictory'. Frede, p. 167, has '*das kontradiktorische Gegenteil*', Mates, p. 31, has 'denial'; neither expands on its interpretation although Frede, p. 82, does describe the term '*antikeimenon*' as characteristically Stoic.

> If from two propositions a third is deduced, then from either one of them together with the contradictory of the conclusion the contradictory of the other is deduced.

This represents a strengthening of what we have taken to be the first thema in BSL. Becker observes that in this form $p, q \vdash r$ and $p, \neg r \vdash \neg q$ are equivalent, whereas, under (I) all we have is that the latter follows from the former.[50]

This opens up a number of formal possibilities. We could follow Becker and Egli in introducing a formal notion of contradictory. However, since any such notion is defined in terms of negations and double negations, exactly the same logical force is attained by introducing appropriate rules for negations.[51] We might contemplate adding two more basic sequents:

$$p \vdash \neg\neg p; \qquad \neg\neg p \vdash p.$$

However, as Stoic logicians appear not to have explicitly formulated argument-forms employing double negations, perhaps a better tactic, and the one to be adopted here, is to use these variants of the first thema:

$$(a)\ \frac{\Sigma,\ p \vdash \neg q}{\Sigma,\ q \vdash \neg p}, \quad (b)\ \frac{\Sigma,\ \neg p \vdash q}{\Sigma,\ \neg q \vdash p}, \quad (c)\ \frac{\Sigma,\ \neg p \vdash \neg q}{\Sigma,\ q \vdash p}.$$

In the sequel I shall count all three as instances of the first thema and signal which is employed by the appropriate letter.

In an example of his own devising Mates goes from $p \rightarrow \neg q$, $p \vdash \neg q$ to p, $q \vdash \neg(p \rightarrow \neg q)$, an instance of (I*a*), and cites the first thema in justification.[52] Frede too uses a step of exactly this form in the analysis of a Stoic argument-form, citing it as an application of the first thema. Other of Frede's examples use instances of (I*b*).[53] In view of Mates's use of (I*a*) and of the fact that he gives no formal version of the first thema I find it less than evident that, as Becker and Mueller claim, he sanctioned only (I).[54]

In the presence of the first thema and other rules of BSL (I*a*), (I*b*) and (I*c*) are interderivable with the explicit double negation sequents

$$\Sigma,\ p \vdash \neg\neg p; \qquad \Sigma,\ \neg\neg p \vdash p.$$

50 Becker, p. 30, n. 5. While this is clear in the case in which none of p, q and r are negated formulae it is much less evident how Becker established it in the general case.
51 Becker, p. 41, asserts that the equivalence of p and $\neg\neg p$ is derivable from his system. Mueller, p. 207, shows that this is so but comments, *loc. cit.*, that 'Becker's treatment of the principle [of double negation elimination] seems to amount to little more than adding T1 [the sequent $\neg\neg p \vdash p$] to the initial sequents of his system'.
52 Mates, p. 78, n. 37.
53 Frede, pp. 186–190.
54 Corcoran, p. 178, whose formulation of Stoic logic follows Mates's exposition, gives these two forms for the first thema:

$$\frac{p,\ q \vdash r}{p,\ \neg r \vdash \neg q} \qquad \frac{p,\ q \vdash r}{\neg r,\ q \vdash \neg p},$$

then goes on to say that 'the Stoics could have been using the term 'the denial' ambiguously to indicate either the result of adding a negation to a sentence or the result of deleting a negation from a sentence (which stands with a negation). If this is so, one would get sixteen subrules (when r is a negation, when p is a negation and when q is a negation)'. Notice that in order to get sixteen subrules Corcoran has to employ sequents with sequence antecedents and some might be counted special cases of others as, e.g., when '$\neg$' is placed in front of 'p in the left-hand sequent rule above. When compared to the four in the text—(I), (I*a*), (I*b*) and (I*c*)—this furnishes a good illustration of the economy achieved in opting for set-antecedents.

That the latter are derivable from the tautologous argument-form $\Sigma, \neg p \vdash \neg p$ by single applications of (I*a*) and (I*b*) respectively should be obvious. The converse derivations are perhaps less obvious. Here is a derivation of (I*c*):

$$\dfrac{\dfrac{\dfrac{\Sigma, \neg p \vdash \neg q}{\Sigma, \neg\neg q \vdash \neg\neg p}\,(\mathrm{I}) \qquad \Sigma, q \vdash \neg\neg q}{\Sigma, q \vdash \neg\neg p}\,(\mathrm{III}) \qquad \Sigma, \neg\neg p \vdash p}{\Sigma, q \vdash p}\,(\mathrm{III})$$

It is readily shown, as the reader may care to discover, that if any two of (I*a*), (I*b*) and (I*c*) are added to BSL then the third is a derived rule. Let us call the system that results when any two are added to Basic Stoic Logic BSL+. BSL+ incorporates Frede's and Mates's working Stoic logics, i.e., their derivations can be reproduced in BSL+. Thanks to Frede, we therefore know that the following Stoic argument-forms are derivable in BSL+:[55]

$$p \to q,\ p \to \neg q \vdash \neg p; \qquad p \to q,\ r \to q,\ p \blacktriangledown r \vdash q.$$

We know already that BSL+ yields classical rules for double negation. As $p, \neg q \vdash p$ is a basic sequent we have, by (I*a*), that $p, \neg p \vdash q$ is derivable and hence, by Lemma 2, that the rule of *ex falso quodlibet* (EFQ),

$$\dfrac{\Sigma \vdash p \qquad \Sigma \vdash \neg p}{\Sigma \vdash q},$$

preserves derivability, i.e., it is a derived rule in BSL+. We can also show that the rule of *reductio ad absurdum* (RAA) is a derived rule in BSL+. As $q, \neg q \vdash \neg r$ is derivable in BSL we know, by Lemma 2, that the rule

$$\dfrac{\Sigma \vdash q \qquad \Sigma \vdash \neg q}{\Sigma \vdash \neg r},$$

preserves derivability. Suppose, then, that $\Sigma, p \vdash q$ and $\Sigma, p \vdash \neg q$, where $\Sigma \neq \varnothing$. Choose $r \in \Sigma$. Then, by EFQ, $\Sigma, p \vdash \neg r$. By (I*a*), $\Sigma, r \vdash \neg p$. But $r \in \Sigma$, so $\Sigma \vdash \neg p$. Consequently,

$$\dfrac{\Sigma, p \vdash q \qquad \Sigma, p \vdash \neg q}{\Sigma \vdash \neg p} \text{ where } \Sigma \neq \varnothing,$$

is a derived rule of BSL+. Given EFQ, RAA, and the rules for double negation the rule of dilemma,

$$\dfrac{\Sigma, p \vdash q \qquad \Sigma, \neg p \vdash q}{\Sigma \vdash q} \text{ where } \Sigma \neq \varnothing,$$

is easily shown to preserve derivability.

BSL+ appears to give us a full classical account of negation, for it gives us all the rules pertaining to negation that are standardly used in natural deduction systems for classical propositional logic. The system of Lemmon's *Beginning Logic*, for example, contains *modus tollendo tollens*, which is the rule corresponding to the second indemonstrable, the rules corresponding to the two double-negation sequents above, and *reductio ad absurdum*, which, as we now

55 Frede, pp. 186, 187. Frede also discusses, p. 188, this special case of the second, $p \to p$, $\neg p \to p$, $p \blacktriangledown \neg p \vdash p$, pointing out that the third premise is redundant.

know, preserves derivability in BSL+. In this case, as we shall see below, appearances are deceptive.

Also in BSL+, using the third indemonstrable and (I*c*), we can demonstrate the validity of &-introduction: $p, q \vdash p \,\&\, q$. Perhaps more surprisingly, the rule known as disjunction-elimination in natural deduction systems and as proof-by-cases to mathematicians is forthcoming:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Sigma, q \vdash r}{\Sigma, \neg r \vdash \neg q}\,(\mathrm{I}) \qquad \dfrac{\Sigma \vdash p \blacktriangledown q}{\Sigma, \neg r \vdash p \blacktriangledown q}\,\mathrm{Aug}}{\Sigma, \neg r \vdash p}\,(\mathrm{v})}{\Sigma, \neg p \vdash r}\,(\mathrm{I}b) \qquad \Sigma, p \vdash r}{\Sigma \vdash r}\,\mathrm{Dilemma}.$$

Here (v) marks a use of the validity-preserving rule associated with the fifth indemonstrable.

Before we leave the subject of negation we must attend to an observation made by Mates and by the Kneales.[56] This is that

$$p \blacktriangledown \neg p,\ p \vdash \neg\neg p \text{ and } p \blacktriangledown \neg p,\ \neg\neg p \vdash p$$

are instances of the fourth and fifth indemonstrables, respectively, and, consequently, if a logical truth such as $p \blacktriangledown \neg p$ may be assumed as a premise without explicit mention in any argument there is no need of explicit rules for double negation. $p \blacktriangledown \neg p$ is a logical truth on any of the competing Stoic views concerning disjunction but there is no record that Stoic logicians made any such use of logical truths.[57] As we shall see in Section IV, $p \blacktriangledown \neg p$ is not a theorem of BSL+, nor indeed of what we shall call below Extended Stoic Logic, unlike $\neg(p \blacktriangledown p)$. Of course, BSL+ has *no* theorems proper, for every derivable sequent has a non-empty antecedent, but we may call a formula p a theorem of BSL+ if $\Sigma \vdash p$ is derivable in BSL+ for all sets of formulae Σ.[58]

As an instance of the fourth indemonstrable we get $p \blacktriangledown p, p \vdash \neg p$, the fifth delivers $p \blacktriangledown p, \neg p \vdash p$. Using (I*a*) and (I) respectively we obtain $p \vdash \neg(p \blacktriangledown p)$ and $\neg p \vdash \neg(p \blacktriangledown p)$; whence, employing (I*a*) and (I*c*) respectively, we get $p \blacktriangledown p \vdash \neg p$ and $p \blacktriangledown p \vdash p$. By thinning, $\Sigma, p \blacktriangledown p \vdash \neg p$ and $\Sigma, p \blacktriangledown p \vdash p$, for any non-empty set Σ, and consequently, by RAA, $\Sigma \vdash \neg(p \blacktriangledown p)$.

Looking solely at the forms of the indemonstrable we might expect to find indemonstrables with these major premises:

$$(a)\ \neg(p \to q); \qquad (b)\ p \,\&\, q; \qquad (c)\ \neg(p \blacktriangledown q).$$

That we do not may tell us something about Stoic logic. For example, the absence of (*a*) and (*b*) might be explained by the fact that each alone determines the truth-values of its subformulae, p and q; at least that is the case if we adopt the reading of $\to$ as the material conditional. We would then, under

56 Mates, pp. 31–32; the Kneales, pp. 167–169.
57 For an argument that the law of excluded middle obtains see SE, vii, 466ff.
58 Cf. Mueller, p. 209. The fact that a reconstruction of Stoic logic has no theorems seems to me no drawback except if one wants to claim it complete in the modern sense with respect to arguments with no premises, a class of argument Stoic logicians would not have recognized. Completeness with respect to Stoic conceptions of validity should be the most that one asks of a reconstruction.

that supposition, obtain the single-premise argument-forms:

$$\neg(p \rightarrow q) \vdash p; \quad \neg(p \rightarrow q) \vdash \neg q; \quad p \,\&\, q \vdash p; \quad p \,\&\, q \vdash q.$$

We have noted already that single-premise argument-forms were the subject of debate, if not dispute, among Stoic logicians.

As suggested in Section I, the absence of (*b*) might also be accounted for by indifference to the distinction between an antecedent containing the conjunction $p \,\&\, q$ and one containing the two conjuncts separately. If we follow Becker's hypothesis of not indifference but blindness to the distinction we would find that the commentators on Stoic logic, but perhaps not Stoic logicians themselves, failed to distinguish:

between $p, q \vdash p$ and $p \,\&\, q \vdash p$;
between $p, q \vdash q$ and $p \,\&\, q \vdash q$;
and between $p \,\&\, q \vdash p \,\&\, q$ and $p, q \vdash p \,\&\, q$.

In each case the first member of the pair has the form of a tautologous inference. Either way, indifference or blindness, Stoics would have counted the second member valid because the first is valid. (The fact that in the third case we have a single-premise argument-form is not important, for 'spurious' antecedents may be added.) A full formalization must make this evident. There are a number of ways to achieve this. One such is Becker's proposal for the missing fourth thema:

$$\frac{\Sigma, p, q \vdash r}{\Sigma, p \,\&\, q \vdash r}.$$

Adopting this sequent rule may well be, as Mueller says in commentary on Becker, 'a conjecture where little more than conjecture is possible'.[59] Nevertheless, the suggestion about the counting of premises gives it some small measure of plausibility when taken in conjunction with the insistence, or, as Apuleius has it, the universal consensus, on there being two or more premises in any valid inference. Following Becker, and like him with no pretence of historical accuracy, I shall call the transformation rule the *fourth thema*.[60] Its use in derivations will be signalled by (IV).

With the addition of (IV) we find that $\neg(p \,\&\, \neg p)$ is a theorem:

$$\dfrac{\dfrac{\Sigma, p, \neg p \vdash p}{\Sigma, p \,\&\, \neg p \vdash p}\,\text{(IV)} \qquad \dfrac{\Sigma, p, \neg p \vdash \neg p}{\Sigma, p \,\&\, \neg p \vdash \neg p}\,\text{(IV)}}{\Sigma \vdash \neg(p \,\&\, \neg p)}\,\text{RAA.}$$

Case (*c*), the absence of indemonstrables with major premises of the form $\neg(p \blacktriangledown q)$, is very intriguing, for the following two argument-forms are sound under the truth-functional reading of exclusive disjunction and possess the same irredundancy and symmetry properties as (iv) and (v):

$$\neg(p \blacktriangledown q), p \vdash q; \qquad \neg(p \blacktriangledown q), \neg q \vdash \neg p.$$

Their absence from the list of indemonstrables can be explained in two ways.

59 Mueller, p. 206.
60 Becker frankly acknowledges that his aim is to show that Stoic propositional logic is complete with respect to the standard two-valued semantics and chooses his candidate for fourth thema accordingly.

Firstly, it may be taken as evidence that (i) Stoic logicians adopted the truth-functional reading of $\blacktriangledown$ and (ii) that they were familiar with the logical equivalence under that reading of $\neg(p \blacktriangledown q)$ and $p \blacktriangledown \neg q$. As far as I am aware, however, there is no independent evidence for part (ii) of this suggestion.[61]

A second explanation, no less compatible with their absence, is that these argument-forms are invalid on the alternative, modal readings of $\blacktriangledown$, just as are

$$\neg(p \rightarrow q) \vdash p \text{ and } \neg(p \rightarrow q) \vdash \neg q$$

on the modal readings of the conditional. On the non-Philonian interpretations of $\blacktriangledown$ and $\rightarrow$ we should not expect there to be indemonstrables with major premises of the forms (*a*) and (*c*).[62]

As a final addition to BSL we take the 'three-component arguments', recently argued by Katerina Ierodiakonou to be of Stoic origin.[63] These are of the form: $p \rightarrow q,\ q \rightarrow r \vdash p \rightarrow r$. That they do extend BSL will be shown in Section IV. One comment is necessary. The sources Ierodiakonou cites, namely Blemmidas and the unknown author of the *Logica et Quadrivium*, nominate this form along with the indemonstrables. Chrysippus discussed such arguments but evidently did not count them among the indemonstrables. Did he, then, possess a means by which he reduced them to the indemonstrables? The suspicion that he did might give some clues as to the identity of the missing themata. The most obvious candidate would be a principle of conditional proof of the form

$$\frac{\Sigma, p \vdash q}{\Sigma \vdash p \rightarrow q},$$

which I shall denote by CP, for then we would obtain

$$\dfrac{\dfrac{p \rightarrow q, p \vdash q \qquad q \rightarrow r, q \vdash r}{p \rightarrow q, q \rightarrow r, p \vdash r}\,\text{(III)}}{p \rightarrow q, q \rightarrow r \vdash p \rightarrow r}\,\text{CP},$$

thereby deriving three-component arguments from the first indemonstrable; but, as I am about to argue, although obvious, CP is probably not one of the missing themata.

Let Extended Stoic Logic (ESL) be the system that results when (IV) is added as a sequent-rule to BSL+ and, in addition to those of BSL+, sequents of the form $p \rightarrow q,\ q \rightarrow r \vdash p \rightarrow r$ are taken as basic. It is, I submit, the strongest logic we have good grounds to attribute to Stoic logicians.

61 The nearest we get to it is Galen's observation that $p \blacktriangledown q$ is equivalent to $\neg p \equiv q$; Galen, *Inst. Log.*, 9. See Bocheński p. 92; Mates, p. 56. Cf. the Kneales, p. 162. See also the discussion in Frede, pp. 102–105. Provided Galen held that $\neg(p \equiv q)$ is equivalent to $p \equiv \neg q$ it would have been but a short step from $\neg(p \blacktriangledown q)$ to $p \blacktriangledown \neg q$. Despite a problem with the particular passage cited by Bocheński, by Mates and by the Kneales, a problem acknowledged by them, there can be little doubt that Galen well understood the interpretation of exclusive disjunction, for in another passage, *Inst. Log.*, 32, we find him emphasising that proper disjunctions require complete incompatibility, that is, that the disjuncts cannot both hold or fail to hold. Unfortunately, neither passage puts it beyond doubt whether a truth-functional or a modal reading of disjunction is intended.
62 The parallel explanation in the case of (*b*), granted correctness of the third indemonstrable, would require that $p \,\&\, q$ is true just in case p and q are compossible. No Stoic logician proposed this reading of conjunction.
63 Ierodiakonou, pp. 137–148.

III

Did Stoic logicians advance a rule like conditional proof as one of the missing themata? The suggestion has certainly tempted a number of modern commentators. Let us first note that CP is not a derived rule of ESL. This is easily established if we go back to the unorthodox interpretation of $\rightarrow$ which declares that $p \rightarrow q$ is uniformly false for all p and q. On this reading three-component arguments are valid—there can be no instance with true premises and false conclusion, for both premises and conclusion are necessarily false. On this reading CP does not preserve validity, for while both $p, q \vdash q$ and $\neg p, p \vdash q$ are derivable in ESL neither $q \vdash p \rightarrow q$ nor $\neg p \vdash p \rightarrow q$ is. The fact that neither of these sequents is derivable in ESL shows that there is no other direct route to the Philonian material reading of the conditional in ESL.

The material reading of the conditional, attributed to Philo, holds that $p \rightarrow q$ is false only when p is true and q is false, and is otherwise true. Hence it identifies $p \rightarrow q$ with $\neg(p \& \neg q)$. Now, although CP is not a derived rule of ESL, this surrogate is:

$$\frac{\Sigma, p \vdash q}{\Sigma \vdash \neg(p \& \neg q)}.$$

There are a number of ways to show this. Perhaps the shortest is based on Becker's own derivation:[64]

Suppose that Σ, $p \vdash q$ and that, as $\Sigma \neq \varnothing$, $r \in \Sigma$. Then

$$\cfrac{\cfrac{\cfrac{\Sigma, r, p \vdash q}{\Sigma, p, \neg q \vdash \neg r}\,(\mathrm{I})}{\Sigma, p \& \neg q \vdash \neg r}\,(\mathrm{IV})}{\Sigma, r \vdash \neg(p \& \neg q)}\,(\mathrm{I}a).$$

As $r \in \Sigma$ we have the desired sequent: $\Sigma \vdash \neg(p \& \neg q)$.[65] Consequently, *if the Philonian understanding of the conditional were already adopted* there would be every reason to treat CP as validity-preserving and use it in the analysis of arguments. On the other hand, the first and second indemonstrables would both be superfluous, for as an instance of the third indemonstrable we have

64 See Becker, p. 49. Egli, p. 91, presents a partly semantic version of Becker's proof:

$$\begin{array}{ll}
\Sigma, r, p \vdash q & \\ \hline
\Sigma, p, \neg q \vdash \neg r & (\mathrm{I}) \\ \hline
\Sigma, p \text{ is true, } q \text{ is false} \vdash \neg r & \\ \hline
\Sigma, p \rightarrow q \text{ is false} \vdash \neg r & \\ \hline
\Sigma, \neg(p \rightarrow q) \vdash \neg r & \\ \hline
\Sigma, r \vdash p \rightarrow q & (\mathrm{I}c).
\end{array}$$

To what extent such semantic ascent and descent formed a part of the Stoic analysis of arguments is unclear. Egli (*loc. cit.*) suggests such procedures—subsyllogistic reasoning as he terms it—formed a central part of the enterprise yet his argument in favour of this conclusion (*op. cit.*, §4.E) is almost entirely speculative. Moreover, while happy to allow the semantic ascent and descent in the proof above, which employs 'well known Stoic type equivalences' (p. 91), Egli, p. 92, is more hesitant in ascribing knowledge of the equivalence between $p \rightarrow q$ and $\neg(p \& \neg q)$, despite the fact that it is almost inconceivable that Stoic logicians would accept the equivalence of '$p \rightarrow q$ is false' and 'p is true, q is false' without accepting the latter. Given bivalence, stoutly defended by Chrysippus (Cicero, *De fato*, 20–21, 38; cf. Bocheński, p. 91, and the references to 'bivalence' in Long and Sedley), the two equivalences are, effectively, one.

$\neg(p \& \neg q)$, $p \vdash \neg\neg q$, which, by a single application if (Ia), yields $\neg(p \& \neg q)$, $\neg q \vdash \neg p$, whence, by an application of (Ic), $\neg(p \& \neg q)$, $p \vdash q$. The first of these mirrors the second indemonstrable, the second mirrors the first, given the Philonian interpretation of the conditional. Of course, we know already that there is redundancy among the indemonstrables so it is hard to say what weight ought to be given to this observation.

We have seen that on the Philonian interpretation of the conditional there would be good reason to adopt CP. As we shall now see, the converse is also true, i.e., adoption of CP as one of the missing themata compels acceptance of the Philonian interpretation of the conditional. This is shown by these three derivations:

$$\cfrac{\cfrac{q, p \vdash q}{q \vdash p \rightarrow q}\,\mathrm{CP}}{\neg(p \rightarrow q) \vdash \neg q}\,(\mathrm{I}); \qquad \cfrac{\cfrac{\cfrac{p, \neg q \vdash p}{p, \neg p \vdash q}\,(\mathrm{I}b)}{\neg p \vdash p \rightarrow q}\,\mathrm{CP}}{\neg(p \rightarrow q) \vdash p}\,(\mathrm{I}b);$$

and

$$\frac{p, p \rightarrow q \vdash q}{p, \neg q \vdash \neg(p \rightarrow q)}\,(\mathrm{I}).$$

In each case the sequent at the head of the derivation is a basic sequent. Consequently, the final sequent in each is valid, i.e., represents an argument-form all of whose instances are valid with respect to the orthodox classical two-valued semantics. That being so, a semantic reading of the final sequents informs us that $p \rightarrow q$ is false when, *and only when*, p is true and q is false, which is exactly what the material construal of the conditional demands. Therefore, if conditional proof is indeed one of the missing themata, the dispute

65 This derivation uses (IV). Becker, pp. 43–44, played with the idea of using CP as his fourth thema, in place of (IV). He claims that the two are interderivable and purports to prove it. Both derivations depend on the identification of $p \rightarrow q$ with $\neg(p \& \neg q)$. His derivation of CP in a system with (IV) has, in essence, just been given. The derivation of (IV) when CP is given is as follows (Becker, p. 49):

$$\begin{array}{ll}
p, q \vdash r & \\ \hline
p, \neg r \vdash \neg q & (\mathrm{I}) \\ \hline
\neg r \vdash p \rightarrow \neg q & \mathrm{CP} \\ \hline
\neg r \vdash \neg(p \& \neg\neg q) & \mathrm{Def} \\ \hline
\neg r \vdash \neg(p \& q) & (?) \\ \hline
p \& q \vdash r & (\mathrm{I}c).
\end{array}$$

The annotations are mine. *Def* is the point where Becker appeals to the material interpretation of the conditional; in context there are no grounds for complaint there. What of (?)? Only appeal to some substitution-rule version of double negation elimination allows *immediate* passage from $\neg r \vdash \neg(p \& \neg\neg q)$ to $\neg r \vdash \neg(p \& q)$ but Becker nowhere explains his entitlement to its use. His preferred reconstruction of Stoic logic can be shown to be complete with respect to all classically valid inferences containing no connectives other than $\neg$ and $\&$ and so, in that system, $\neg r \vdash \neg(p \& q)$ is derivable from $\neg r \vdash \neg(p \& \neg\neg q)$. But in this instance he cannot appeal to that fact, for he has temporarily rejected (IV) in favour of CP. What guarantee do we have that a substitution-rule formulation of the negation rules can be shown to preserve derivability in this context? Arguably none, for in the terminology of §IV below, while (I), (Ic), CP and Def all preserve validity$_2$ the sequent $\neg(p \& \neg\neg q) \vdash \neg(p \& q)$ is not valid$_2$. Certainly, Becker's proof cannot be carried out in BSL+ augmented by CP and the substitution rule Def: any occurrence of a subformula $p \rightarrow q$ within a formula in a sequent may be replaced by $\neg(p \& \neg q)$; any occurrence of a subformula $\neg(p \& \neg q)$ within a formula in a sequent may be replaced by $p \rightarrow q$.

amongst Stoic logicians over the conditional was an egregious mistake—the Philonian interpretation was demonstrably correct. Notice too that there is nothing recondite about the derivations; in addition to CP they make very limited use of the resources of BSL+

Worse is to come. We know that given the first indemonstrable and the first thema the second indemonstrable is redundant, i.e., derivable. Given conditional proof and the second indemonstrable the first thema, (I), is redundant! There are a number of ways one could approach proof of this fact. The following is pleasingly short:

$$\dfrac{\dfrac{\dfrac{\Sigma,\ p \vdash q}{\Sigma \vdash p \rightarrow q}\ \text{CP}}{\Sigma,\ \neg q \vdash p \rightarrow q}\ \text{Aug} \qquad \Sigma,\ \neg q \vdash \neg q}{\Sigma,\ \neg q \vdash \neg p}\ \text{(ii)},$$

where (ii) is the rule associated with the second indemonstrable. As the only added sequent, Σ, $\neg q \vdash \neg q$, is a basic sequent (I) is a derived rule. We have been given no reason to think that all, or indeed any, of (I*a*), (I*b*) and (I*c*) are redundant. Nevertheless it is disconcerting to find that (I) is unnecessary, especially as, again, the derivation is by no means difficult.

What are we to make of these results? They seem to present us with a stark choice: either we must accept that Chrysippus and his associates were poor logicians or we must deny that CP is one of the missing themata. Unsurprisingly, I shall adopt the second alternative.

The Kneales prefer a different version of conditional proof, namely

$$\dfrac{p_1,\ p_2,\ \ldots,\ p_n \vdash p}{\vdash (p_1\ \&\ p_2\ \&\ \ldots\ \&\ p_n) \rightarrow p},$$

which Mueller tentatively approves using in a formalization of Stoic logic.[66] Let us call this rule CP_K. Mueller says that CP_K is close to a rule Stoics did use and refers us to Mates's discussion of what he, Mates, calls the principle of conditionalization, but Mates informs us that there is no indication that Stoics used the principle as a rule of inference.[67] There is every reason to think this correct.

Stoic logicians looked at conditionals of the form $(p_1\ \&\ p_2\ \&\ \ldots\ \&\ p_n) \rightarrow p$ in the context of determining the validity of inferences. In providing a *criterion* for validity Stoic logicians almost certainly did not say that $p_1, p_2, \ldots, p_n \vdash p$ is valid only if $\vdash (p_1\ \&\ p_2\ \&\ \ldots\ \&\ p_n) \rightarrow p$ is valid. There are two reasons for this. Firstly, this succeeds only in reducing the validity of all inferences to the validity of a certain type of inference, hence supplies no general criterion of validity. Secondly, the type to which all are reduced is that of inferences with no premises. Given the dialectical origins of Stoic logic it is hard to envisage that they would even have contemplated the possibility of arguments with no premises, let alone that they should have accepted any as valid. It is improbable in the extreme that they should provide a criterion of validity which depends on giving pride of place to such arguments.

It is, of course, true that if $p_1, p_2, \ldots, p_n \vdash p$ is valid in the modern sense, i.e., with respect to two-valued classical semantics, then so is

66 The Kneales, p. 170; Mueller, p. 214.
67 Mates, p. 74.

$\vdash (p_1\ \&\ p_2\ \&\ \ldots\ \&\ p_n) \rightarrow p$, and hence CP_K preserves validity in the modern sense. Moreover, if Stoic logicians considered arguments of the form $\vdash (p_1\ \&\ p_2\ \&\ \ldots\ \&\ p_n) \rightarrow p$ valid it would almost certainly have been in virtue of the validity of the related argument-form $p_1, p_2, \ldots, p_n \vdash p$. But what evidence is there that they ever did consider such arguments valid? And notice that CP_K inverts what little is established of Stoic practice: it takes us from the argument-form $p_1, p_2, \ldots, p_n \vdash p$ to the form $\vdash (p_1\ \&\ p_2\ \&\ \ldots\ \&\ p_n) \rightarrow p$ whereas in determining validity we move in the reverse direction.

Mates is quite clear as to what Stoic logicians were about: the *inference* from $p_1, p_2, \ldots, p_n$ to p is valid if, and only if, the *sentence* $(p_1\ \&\ p_2\ \&\ \ldots\ \&\ p_n) \rightarrow p$ is 'necessarily' or 'logically' true.[68] There are, as Mates points out, two Stoic ways to express this conception of validity. One, due to Sextus Empiricus, asserts that the inference from $p_1, p_2, \ldots, p_n$ to p is valid if the sentence $(p_1\ \&\ p_2\ \&\ \ldots\ \&\ p_n) \rightarrow p$ is *true*.[69] As Mates makes clear, it is the Diodorean interpretation of the conditional that is intended, hence his own interpolations of 'necessarily' and 'logically'. The other approach may be extrapolated from but is not quite explicit in Diogenes Laertius's writings. He says that an argument is invalid if the conjunction of the premises is compatible with the contradictory of the conclusion.[70] But this, as Diogenes himself writes, is the truth-condition of the conditional whose antecedent is the conjunction of the premises and whose consequent is the conclusion; at least this is so on the Chrysippean understanding of the conditional.[71] The criteria proposed by Diogenes and Sextus coincide up to a point: they diverge on the interpretation of the modality, Diodorean or Chrysippean, implicit in the conditional.[72]

That Sextus must have had in mind a modal conception of the conditional is shown by his argument against the first indemonstrable on grounds of redundancy.[73] The first indemonstrable indicates that the inference to q from p and $p \rightarrow q$ is valid. But, says Sextus, the premise $p \rightarrow q$ asserts that q follows from p, and if true means that one can infer q from p directly, without appeal to any extra premise such as $p \rightarrow q$. Hence, if the second premise is true the inference is invalid in virtue of a redundant premise. On the other hand, if $p \rightarrow q$ is false then q does not follow from p and, says Sextus, will not follow on the addition of the false claim that it does. The last clause appears to be itself redundant, for presumably it would be sufficient for invalidity through redundancy that whenever the premises are true one is superfluous in obtaining the conclusion. Whatever one makes of Sextus's argument it would make less sense, if any, on the Philonian understanding of the conditional. It scarcely needs to be added that the concept of invalidity by redundancy squares ill with either reading of the criterion of validity.

68 Mates, pp. 74–77.
69 E.g., SE, viii, 304, 415, 417.
70 DL, vii, 77.
71 DL vii. 73. Cf. Mates, p. 60.
72 Frede, p. 120, identifies Diogenes's formulation as based on the Chrysippean notion, Sextus's in the Diodorean.
73 SE, viii, 440. Here I may be in disagreement with Mueller who implies that it is the truth-conditional interpretation that underpins attacks against Stoic logic including those by Sextus. See pp. 22–31 of Mueller, 'An Introduction to Stoic Logic', in John Rist (ed.), *The Stoics*, University of California Press, Berkeley, 1978, pp. 1–26. Long and Sedley, p. 219, suggest that Sextus' argument may have prompted Antipater to introduce single-premise arguments.

Egli suggests that the Philonian interpretation was used in syllogistic theory while the Diodorean reading was employed in the criterion of validity.[74] If that is so there would seem to be no straightforward way to read off a principle of conditionalization such as CP_K from the criterion of validity, for they would involve different conditionals. From Frede's commentary it is clear that a modal interpretation of the conditional was fundamental in the Stoic criterion of validity.[75] If we suppose, contrary to Egli's hypothesis, that the same conditional is involved in arguments and in the criterion, a principle such as CP_K is valid but CP is not, for it does not follow from the fact that p and q entail p that p entails 'Necessarily: whenever q then p'.

There is, it seems, little justification for adding CP_K as a sequent rule to ESL. If one persists in doing so two facts should be noted. Firstly, (IV) is redundant. We have this derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{p, q \vdash p}{\vdash (p \,\&\, q) \to p}\,CP_K}{\Sigma, p \,\&\, q \vdash (p \,\&\, q) \to p}\,\text{Aug} \qquad \Sigma, p \,\&\, q \vdash p \,\&\, q}{\Sigma, p \,\&\, q \vdash p}\,\text{(i)} \qquad \Sigma, p, q \vdash r}{\Sigma, p \,\&\, q, q \vdash r}\,\text{(III)}$$

(where (i) denotes the sequent rule associated with the first indemonstrable) and further steps of a similar kind can be used to eliminate the 'q' in the antecedent of the last sequent in order to obtain $\Sigma, p \,\&\, q \vdash r$.

Secondly, under the interpretations that reads $p \to q$ as $\Box(p \supset q)$, where '$\Box$' is read either as 'necessarily' or as 'whenever' and $\supset$ is material implication, the first and second indemonstrables are valid argument-forms and CP_K preserves validity. On the other hand, as just noted CP does not preserve validity. Also, $\neg p \vdash p \to q$ and $q \vdash p \to q$ are not valid argument-forms. Consequently, neither of these sequents is derivable in ESL augmented by CP_K, and CP is not a derived rule. In view of this the system comprising ESL together with CP_K is not complete with respect to the Philonian interpretation of the conditional.[76]

I have argued that CP_K ought not to be added to ESL. That it is not a derived rule in ESL is clear because every sequent derivable in ESL has a non-empty antecedent. That no surrogate of the form

$$\frac{p_1, p_2, \ldots, p_n \vdash p}{\Sigma \vdash (p_1 \,\&\, p_2 \,\&\, \ldots \,\&\, p_n) \to p},$$

where Σ is any non-empty set of formulae, is derivable is shown by another appeal to the uniform-falsity reading of $\to$. While $p, q \vdash p$ is basic sequent in ESL, $\Sigma \vdash (p \,\&\, q) \to p$ has an arbitrary, hence possibly true, antecedent and a necessarily false consequent.

IV

ESL has been obtained by adding (Ia), (Ib), (Ic) and (IV) as sequent rules and the three-component argument-form as a basic sequent to BSL. Let us abbreviate by saying that we have added a negation-, a conjunction- and an

74 Egli, p. 92.
75 Frede, p. 119–121.
76 *Cf.* Mueller, 'On the Completeness...', p. 214.

implication-condition to BSL. Each of these conditions is independent of BSL augmented by the others, as we shall now show.

We consider first the negation condition and in particular (Ib). Let us take $p \blacktriangledown q$ to be an abbreviation for $(p \to \neg q) \,\&\, (\neg q \to p)$. Under that reading all basic sequents of ESL, including three-component arguments, represent intuitionistically valid argument-forms and (I), (III) and (IV) preserve intuitionistic validity. The basic sequent $p \to p, \neg p \vdash \neg p$ is intuitionistically valid but the sequent $p \to p, \neg\neg p \vdash p$, obtained by a single application of (Ib), is not, for famously intuitionists do not permit the inference from $\neg\neg p$ to p. Therefore (Ib) is independent of the other additions to BSL that go to make up ESL. Notice that, since (Ia) preserves intuitionistic validity, adding (Ia) to BSL is not sufficient to obtain (Ib) and/or (Ic).

Adding (Ia) on its own to BSL suffices to show that nothing is lost by ignoring the extra content in Diogenes's informal exposition of the third and fourth indemonstrables, for the argument-forms numbered (iii') and (iv') above follow immediately from the indemonstrables (iii) and (iv), respectively, by a single application of (Ia). On the other hand, Mueller's form of the fifth indemonstrable, (v'), is not derivable in BSL together with (Ia). To see this, again read $p \blacktriangledown q$ as $(p \to \neg q) \,\&\, (\neg q \to p)$ and note that $(p \to \neg q) \,\&\, (\neg q \to p), \neg q \vdash p$ is intuitionistically valid, although $(p \to \neg q) \,\&\, (\neg q \to p), \neg p \vdash q$ is not: letting p be $\neg q$ we obtain $(\neg q \to \neg q) \,\&\, (\neg q \to \neg q), \neg q \vdash \neg q$ for the first, which is intuitionistically valid and a basic sequent in BSL, and, for the second, $(\neg q \to \neg q) \,\&\, (\neg q \to \neg q), \neg\neg q \vdash q$, which is neither.

Given (Ib) the two forms of the fifth indemonstrable, (v) and (v'), are immediately interderivable. Consequently, it makes no difference to the deductive power of ESL which form is chosen as basic.

In order to show independence of the conjunction- and implication-conditions, and for further results below, we introduce a variety of many-valued rules of semantic evaluation. In effect we have already used non-standard two-valued rules when considering the uniform-falsity interpretations of $\to$ and $\blacktriangledown$. Below we employ three- and four-valued semantic rules. These have, admittedly, little in the way of intuitive interpretation.

We shall call the following the 'given matrices' for the connectives:

p	$\neg p$	$p \,\&\, q$ (q=0)	q=1	q=2	$p \blacktriangledown q$ (q=0)	q=1	q=2	$p \to q$ (q=0)	q=1	q=2
0	2	0	0	0	0	0	2	2	2	2
1	2	0	0	0	0	0	2	2	2	2
2	0	0	0	2	2	2	0	0	0	2.

If we consider valuations defined in accordance with these matrices and take 2 as the designated value we can readily see that these are just expanded classical valuations with an extra 'falsity-value' thrown in for good measure, for, writing 'u' for undesignated and 'd' for desiganted, they generate these familiar classical patterns:

p	$\neg p$	$p \,\&\, q$ (q=u)	q=d	$p \blacktriangledown q$ (q=u)	q=d	$p \to q$ (q=u)	q=d
u	d	u	u	d	d	d	d
d	u	u	d	u	d	u	d.

Recall that all basic sequents of ESL are classically valid and all sequent rules of ESL preserve classical validity. We therefore know that the basic sequents of ESL are valid$_1$ and that its sequent rules preserve validity$_1$ where a sequent $\Sigma \vdash p$ is defined to be valid$_1$ if every valuation v that assigns 2 to all the members of Σ also assigns 2 to p.

Having just introduced the given matrices we now replace the given matrix for & with this matrix:

$$\begin{array}{cc|ccc} & & & q & \\ \multicolumn{2}{c|}{p \,\&\, q} & 0 & 1 & 2 \\ \hline & 0 & 0 & 0 & 0 \\ p & 1 & 0 & 1 & 1 \\ & 2 & 0 & 2 & 2. \end{array}$$

Let v be any valuation defined in accordance with this matrix and the given matrices for the other connectives. A sequent is said to be valid$_2$ if every such valuation that assigns the value 2 to all the members of the antecedent assigns the same value to the consequent. Clearly basic sequents not essentially involving '&' are valid$_2$, since valid$_1$, and the sequent rules other than (IV) preserve validity$_2$, since they preserve validity$_1$. If $v(p) = v(\neg(p \,\&\, q)) = 2$ then $v(p) = 2$ and $v(p \,\&\, q) < 2$, and so $v(q) = 0$. Hence $v(\neg q) = 2$. Consequently, the third indemonstrable preserves validity$_2$. However, when $v(p) = 2$ and $v(q) = 1$, $v(p \,\&\, q) = 2$, so $p \,\&\, q \vdash q$ is not valid$_2$. As $p \,\&\, q \vdash q$ is derivable in ESL by an application of (IV) to the basic sequent $p, q \vdash q$ we see that (IV) is independent of the other additions to BSL that comprise ESL, for they preserve validity$_2$.[77]

Reinstating the given matrix for & and replacing the given matrix for $\rightarrow$ with

$$\begin{array}{cc|ccc} & & & q & \\ \multicolumn{2}{c|}{p \rightarrow q} & 0 & 1 & 2 \\ \hline & 0 & 2 & 2 & 2 \\ p & 1 & 2 & 1 & 2 \\ & 2 & 0 & 1 & 2, \end{array}$$

we may define valuations in accordance with this matrix and the given matrices for the other connectives. A sequent is valid$_3$ if any such valuation that assigns 2 to all members of its antecedent also assigns that value to its consequent. If $v(p \rightarrow q) = v(p) = 2$ then $v(q) = 2$, so the first indemonstrable preserves validity$_3$. The third, fourth and fifth indemonstrables are valid$_3$ since valid$_1$. We do not need to check the second indemonstrable as it is derivable from the first by (I) in ESL and (I), (I*a*), (I*b*), (I*c*), (III) and (IV) all preserve validity$_3$, since they preserve validity$_1$.

We can now use validity$_3$ to show that three-component arguments are independent of the other additions to BSL. Setting $v(p) = v(r) = 1$ and $v(q) = 0$ we find that $v(p \rightarrow q) = v(q \rightarrow r) = 2$ but $v(p \rightarrow r) = 1$, so $p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$ is not valid$_3$. We can thus provide an answer to Ierodiakonou's question as to 'whether this type of argument [three-component arguments] augements the logical power of the Stoic calculus'.[78] In so far as ESL is an accurate formalization of the Stoic calculus (or is an extension of it) the answer is yes, the addition of three-component arguments does augment Stoic logic. The qualification is, of course, all-important.

77 It is not enough to show that (IV) does not preserve validity$_2$, for we have not claimed completeness with respect to validity$_2$ for the system without (IV); we must also show that (IV) applied to a sequent derivable in ESL without (IV) leads to a sequent that is not valid$_2$.

How strong is ESL? Unsurprisingly in the light of similar results concerning their formulations of Stoic logic obtained by Becker, the Kneales, and Mueller, ESL is complete with respect to all classically valid inferences involving only conjunction and negation.[79] A demonstration of this fact is easily obtained from Becker's and Mueller's proofs.

ESL is far from being complete with respect to the classical, truth-functional interpretations of the conditional and disjunction. It is quite clear why this is the case. The first and second indemonstrables function as elimination rules for the conditional. Likewise, the fourth and fifth function as elimination rules for exclusive disjunction. One passes from premises that contain these connectives to simpler conclusions. The sequent-rules that are part of ESL and the basic sequents added in addition to the indemonstrables do not suffice to provide introduction rules for these connectives, i.e. rules that permit inferences to formulas of the forms $p \rightarrow q$ and $p \blacktriangledown q$, either directly from premises containing simpler propositions, e.g., p, q and their negations, or indirectly on the basis of sequents characterized in terms of these simpler propositions.[80]

There are two modal interpretations of the conditional and disjunction that I shall consider: the Diodorean and the Chrysippean. With regard to the conditional the first holds that $p \rightarrow q$ is true if, and only if, it was not, is not now, and never will be the case that p is true and q is false, the second that $p \rightarrow q$ is true if, and only if, p is incompatible with the contradictory of q. With regard to disjunction the second holds that $p \blacktriangledown q$ is true if, and only if, p and q are completely incompatible, i.e., p and q are incompatible and so are their contradictories.[81] As far as we now know, no explicit Diodorean account of disjunction was proposed by any Stoic logician but one can readily be constructed: $p \blacktriangledown q$ is true if, and only if, at no time, past, present or future, are p and q either both true or both false. I shall argue that ESL is complete with respect to neither the Diodorean nor the Chrysippean readings of these connectives. The arguments against the two readings run in parallel and are quite straightforward. The idea is that, given the modal readings, there are obvious inferential links between the connectives that turn out not to be derivable. In what follows, negation is always to be understood classically.

If $p \rightarrow q$ and $\neg p \rightarrow \neg q$ are both true in the Diodorean sense then it was not, is not now, and never will be the case that p and $\neg q$ are both true, nor was it, is it, or will it be the case that p and $\neg q$ are both false. Hence $p \blacktriangledown \neg q$ is true on its Diodorean interpretation (and also on the classical truth-functional

78 Ierodiakonou, p. 145.

79 See Becker, pp. 46–48; the Kneales, p. 174; Mueller, pp. 208–209, 213, 214. The Kneales refer to Becker's proof.

80 If that characterization seems obscure contrast the direct rule of &-introduction, which amounts in the present context to derivability of the sequent $p, q \vdash p \,\&\, q$, to the indirect rule CP, the rule of conditional proof, commonly known as $\rightarrow$-introduction.
On the absence of the requisite introduction rules *cf.* Mueller, pp. 210–211, 214.

81 See, for example, Frede, pp. 80–96; Bocheński, pp. 89–91; Mates, pp. 42–51; Mueller, 'An Introduction . . .', pp. 15–20; Long and Sedley, pp. 210–201.

reading). Likewise, if $p \rightarrow q$ and $\neg p \rightarrow \neg q$ are both true in the Chrysippean sense then p is incompatible with the contradictory of q and the contradictory of p is incompatible with the contradictory of the contradictory of q. Hence $p \blacktriangledown \neg q$ is true on its Chrysippean interpretation.

On either reading the sequent $p \rightarrow q,\ \neg p \rightarrow \neg q \vdash p \blacktriangledown \neg q$ is valid. However, it is not derivable in ESL. To show this we return to our many-valued matrices, defining validity$_4$ in the now standard way, with 2 as designated value, relative to the given matrices for $\neg$, &, and $\rightarrow$ and to this matrix for $\blacktriangledown$:

$p \blacktriangledown q$	q: 0	1	2
p 0	0	0	2
1	0	0	1
2	2	2	0.

When $v(p \blacktriangledown q) = v(p) = 2$ then $v(q) \in \{0, 1\}$, so $v(\neg q) = 2$ and the fourth indemonstrable is valid$_4$. When $v(p \blacktriangledown q) = v(\neg q) = 2$ then $v(q) \in \{0, 1\}$ and $v(p) = 2$: the fifth indemonstrable is valid$_4$. However, $p \rightarrow q,\ \neg p \rightarrow \neg q \vdash p \blacktriangledown \neg q$ is not valid$_4$: set $v(p) = 1$ and $v(q) = 0$, so that $v(p \rightarrow q) = v(\neg p \rightarrow \neg q) = 2$ and $v(p \blacktriangledown \neg q) = 1$.

The system ESL is, therefore, not complete relative to the modal interpretation of disjunction and the conditional (nor with respect to a Diodorean reading of the conditional and a classical reading of disjunction). We have used an intuitively valid argument-form to demonstrate this incompleteness. Validity$_4$ permits demonstration of perhaps even more blatant shortcomings of ESL. For example, notice that when $v(p) = 1$, $v(p \blacktriangledown \neg p) = 1$, so $p \blacktriangledown \neg p$ is not a theorem of ESL. On the other hand, $\neg p \blacktriangledown p$ always takes the value 2. As $p \rightarrow p,\ \neg p \blacktriangledown p \vdash \neg p \blacktriangledown p$ is a basic sequent of ESL and valid$_4$ but the sequent $p \rightarrow p,\ \neg p \blacktriangledown p \vdash p \blacktriangledown \neg p$ is not valid$_4$—take $v(p) = 1$ again—the commutativity of disjunction, i.e., the logical equivalence of $p \blacktriangledown q$ and $q \blacktriangledown p$, which is certainly intuitively valid with respect to both modal readings of $\blacktriangledown$, is not derivable in ESL. Finally, notice that $\neg\neg p \blacktriangledown \neg p$ always takes the value 2 and that $p \rightarrow p,\ \neg\neg p \blacktriangledown \neg p \vdash \neg\neg p \blacktriangledown \neg p$ is a basic sequent of ESL, therefore valid$_4$, but—$v(p) = 1$ again—the sequent $p \rightarrow p,\ \neg\neg p \blacktriangledown \neg p \vdash p \blacktriangledown \neg p$ is not valid$_4$. What this shows is that the substitution rule for double negations, that is the rule which decrees:

> any occurrence of a subformula $\neg\neg p$ within a formula in a sequent may be replaced by p; any occurrence of a subformula p within a formula in a sequent may be replaced by $\neg\neg p$,

does not preserve derivability in ESL.

These are, it has to be said, rather gross deficiencies. I have said that ESL is the strongest logic we have good grounds for attributing to Stoic logicians. One might be tempted by the thought that the very grossness of the deficiencies undermines this claim. Let us then consider ESL+, the system obtained by adding to ESL all instances of $\Sigma \vdash p \blacktriangledown \neg p$ as basic sequents, the substitution rule for double negations, and this substitution rule for the commutativity of disjunction:

> any occurrence of a subformula $p \blacktriangledown q$ within a formula in a sequent may be replaced by $q \blacktriangledown p$.

Perhaps only the negation rule can plausibly be attributed to Stoic logicians but Diogenes's informal account of the fourth and fifth indemonstrables, which speaks of a disjunctive proposition and its disjuncts without reference to order, might give some support to a Stoic provenance for the commutativity rule.

ESL+ removes what were characterized as the gross deficiencies of ESL.[82] However, ESL+ does not greatly advance the cause of non-Philonian Stoic logic, for our sequent $p \rightarrow q,\ \neg p \rightarrow \neg q \vdash p \blacktriangledown \neg q$ remains underivable. To demonstrate this fact we use these matrices for the connectives:

p	$\neg p$
0	3
1	2
2	1
3	0

$p\ \&\ q$	q: 0	1	2	3
p 0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

$p \blacktriangledown q$	q: 0	1	2	3
p 0	0	0	2	3
1	0	0	3	1
2	2	3	0	0
3	3	1	0	0

$p \rightarrow q$	q: 0	1	2	3
p 0	3	3	3	3
1	2	2	2	3
2	1	1	2	3
3	0	1	2	3.

We take 2 and 3 as the designated values, so that the matrices for $\neg$, &, and $\rightarrow$ are just expansions of the classical case, as is seen from the fact that, writing 'u' for undesignated and 'd' for designated, they again generate the familiar classical patterns for these connectives:

p	$\neg p$
u	d
d	u

$p\ \&\ q$	q: u	d
p u	u	u
d	u	d

$p \rightarrow q$	q: u	d
p u	d	d
d	u	d.

A sequent $\Sigma \vdash p$ is defined to be valid$_5$ if every valuation v that assigns 2 or 3 to all the members of Σ also assigns 2 or 3 to p. In view of the classical nature of $\neg$, & and $\rightarrow$, the only basic sequents of ESL+ that we need to check are the fourth and fifth indemonstrables and the law of excluded middle. Now, if $v(p \blacktriangledown q) \in \{2,3\}$ and $v(p) \in \{2,3\}$ then $v(q) \in \{0,1\}$, i.e., $v(\neg q) \in \{2,3\}$, so $p \blacktriangledown q,\ p \vdash \neg q$ is valid$_5$. And if $v(p \blacktriangledown q) \in \{2,3\}$ and $v(\neg q) \in \{2,3\}$, i.e., $v(q) \in \{0,1\}$, then $v(p) \in \{2,3\}$, so $p \blacktriangledown q,\ \neg q \vdash p$ is valid$_5$. Also, $v(p \blacktriangledown \neg p) = 3$ for all valuations v, so $\Sigma \vdash p \blacktriangledown \neg p$ is valid$_5$ for all sets Σ.

The only sequent rules we need look at are the substitution rules for double negations and disjunctions. Now, by the negation matrix $v(\neg\neg p) = v(p)$ for all valuations v, so the value which v assigns to any formula must be the same as that assigned to any formula obtained by introducing or eliminating double

82 I should point out that the three deficiencies, i.e., the failure of ESL to deliver the law of excluded middle, the double negation rule, and the commutativity of disjunction, can be shown to be independent failings; one can strengthen ESL so as to remedy any two while the third still fails.

negations—the substitution rule for double negations preserves validity$_5$. The disjunction matrix is symmetric—swapping rows for columns yields the same matrix—hence the value which v assigns to $p \blacktriangledown q$ is always the same as that assigned to $q \blacktriangledown p$ and so swapping over the disjuncts makes no difference to the value v assigns to the formula as a whole. The substitution rule for disjunctions preserves validity$_5$.[83]

Lastly, when $v(p) = 1$ and $v(q) = 0$ the sequent $p \to q, \neg p \to \neg q \vdash p \blacktriangledown \neg q$ is found, as predicted, not to be valid$_5$. ESL+ is not complete with respect to the non-Philonian interpretations of the conditional and disjunction.[84]

Acknowledgements

My thanks to Katerina Ierodiakonou for discussions of Stoic logic. My thanks to Paolo Crivelli for helpful comments on the penultimate draft of this article.

83 Although in the present context, i.e., given the modal interpretation of $\to$, it would make little sense to add CP to ESL+ the reader may like to know that CP preserves validity$_5$ and so its addition would not remove any incompleteness demonstrated using that formal notion.

84 An ancestor of the present article was read, under the title 'On the Completeness of Stoic Logic', at the Australasian Association of Logic/Australasian Association of Philosophy (NZ Division) Arthur Prior Memorial Conference, Christchurch, New Zealand, 1989. An abstract appeared in *The Journal of Symbolic Logic*, **56**, (1991), 378.

HISTORY AND PHILOSOPHY OF LOGIC, **16** (1995), 65–73

Prior and Rennie on Times and Tenses

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Received 2 December 1993, Revised 13 October 1994

One of Arthur Prior's constructions of the relational calculus for times within tense logic plus propositional quantifiers is considered using Malcolm Rennie's multimodal semantics and found wanting in two respects: Prior's use of an independent necessity operator was both incorrect and unnecessary, and his system imported an objectionable metaphysics namely temporal relationism. When these defects are repaired it is shown that temporal relationism is not entailed by the modified system. Finally, there is considered the possibility of describing an inconsistent temporal structure in which distinct, classically incompatible conceptions of time can be unified within an inconsistent framework.

1. Introduction

This paper is dedicated to the Australian logician Malcolm Rennie (1940–1980), whose brilliant logical career was cut short at an early age. Rennie was responsible for educating a generation of Australian logicians, and the present author considers it a privilege to have been his student. Rennie was a great admirer of Arthur Prior. Rennie, following Prior, took the view that logic is a vehicle in which conflicting metaphysical views can be expressed, so that disputes can be conducted precisely and rigorously and thus with some chance of resolution. I have subsequently come to think that Prior's tense-logical work is significant in a way Prior did not suspect: as the doom of the view of logic as the purveyor of necessary truth. It takes only a little study of Prior's work on tense logic to expand greatly one's conceptions of what is possible for time, and thus to erode confidence in the necessity of any thesis about time; the example generalizes rapidly. Neither Prior nor Rennie, necessitarians both, drew that consequence of course. In this paper, completeness results due to Rennie are applied in criticism of one of Prior's constructions of times from tenses. Further metaphysical implications of the construction, specifically temporal relationism, are then uncovered and repairs are suggested. Finally, it is shown that the inconsistency between the competing conceptions of time, temporal relationism and temporal absolutism, is not the worst kind of inconsistency since they can both be made to hold in an appropriate inconsistent but nontrivial framework.

2. Prior's system

Prior began with the classical propositional calculus expressed in Polish notation, and added to its language the usual tense logical unary propositional operators F, P, G, H, for 'It will be that', 'It has been that', 'It will always be that', and 'It has always been that' respectively. The relational calculus for times, or U-calculus, is an applied first order functional calculus with identity; and with the primitives Utt' and Ttp to be read respectively 't is earlier than t'' and 'p is true at t', where the t-variables range over times or temporal instants. An

