Chapter 2
Semantics: Making Sense of the Symbols

There are two different views to a given set of formulae T, namely the *syntactical* view and the *semantical* view.

From the syntactical point of view (presented in the previous chapter), we consider the set T just as a set of well-formed formulae—regardless of their intended sense or meaning—from which we can prove some formulae. The only thing that matters is the relationship between the objects, which is given by the axioms (*i.e.*, by the formulae of T), and not the objects themselves. So, from a formal point of view there is no need to assign real objects (what ever this means) to our strings of symbols.

In contrast to this very formal syntactical view, there is also the semantical point of view from which we consider the intended meaning of the formulae in T and then seeking for a *model* in which all formulae of T become true. For this, we have to explain some basic notions of Model Theory like *structure* and *interpretation*, which we will do in an natural, informal language. In this language, we will use words like "or", "and", or phrases like "if. . .then". These words and phrases have the usual meaning. Furthermore, we assume that in our normal world, which we describe with our informal language, the basic rules of *common logic* apply. For example, a statement φ is true or false, and if φ is true, then $\neg\varphi$ is false; and vice versa. Hence, the statement "φ or $\neg\varphi$" is always true, which means that we tacitly assume the LAW OF EXCLUDED MIDDLE, also known as TERTIUM NON DATUR, which corresponds to the logical axiom L_0. Furthermore, we assume DEMORGAN'S LAWS and we apply MODUS PONENS as inference rule.

Structures & Interpretations

In order to define structures and interpretations, we have to assume some notions of NAIVE SET THEORY like *subset*, *cartesian product*, or *relation*, which shall be defined properly in Part IV. On this occasion we also make use of the set theoretical symbol "$\in$", which stands for the binary *membership relation*.

Let $\mathscr{L}$ be an arbitrary but fixed language. An **$\mathscr{L}$-structure** $\mathbf{M}$ consists of a non-empty set A, called the **domain** of $\mathbf{M}$, together with a mapping which assigns to each constant symbol $c \in \mathscr{L}$ an element $c^{\mathbf{M}} \in A$, to each n-ary relation symbol $R \in \mathscr{L}$ a set of n-tuples $R^{\mathbf{M}}$ of elements of A, and to each n-ary function symbol $F \in \mathscr{L}$ a function $F^{\mathbf{M}}$ from n-tuples of A to A. In other word, the constant symbols become elements of A, n-ary relation symbols become subsets of A^n (*i.e.*, subsets of the n-fold cartesian product of A), and n-ary functions symbols become n-ary functions from A^n to A.

The interpretation of variables is given by a so-called assignment: An **assignment** in an $\mathscr{L}$-structure $\mathbf{M}$ is a mapping j which assigns to each variable an element of the domain A.

Finally, an **$\mathscr{L}$-interpretation** $\mathbf{I}$ is a pair $(\mathbf{M}, j)$ consisting of an $\mathscr{L}$-structure $\mathbf{M}$ and an assignment j in $\mathbf{M}$. For a variable ν, an element $a \in A$, and an assignment j in $\mathbf{M}$ we define the assignment $j\frac{a}{\nu}$ by stipulating

$$j\tfrac{a}{\nu}(\nu') = \begin{cases} a & \text{if } \nu' \equiv \nu, \\ j(\nu') & \text{otherwise.} \end{cases}$$

For an interpretation $\mathbf{I} = (\mathbf{M}, j)$ and an element $a \in A$, let

$$\mathbf{I}\tfrac{a}{\nu} := (\mathbf{M}, j\tfrac{a}{\nu})\,.$$

We associate with every interpretation $\mathbf{I} = (\mathbf{M}, j)$ and every $\mathscr{L}$-term τ an element $\mathbf{I}(t) \in A$ as follows:

- For a variable ν let $\mathbf{I}(\nu) := j(\nu)$.
- For a constant symbol $c \in \mathscr{L}$ let $\mathbf{I}(c) := c^{\mathbf{M}}$.
- For an n-ary function symbol $F \in \mathscr{L}$ and terms $\tau_1, \ldots, \tau_n$ let

$$\mathbf{I}\big(F(\tau_1, \ldots, \tau_n)\big) := F^{\mathbf{M}}\big(\mathbf{I}(\tau_1), \ldots, \mathbf{I}(\tau_n)\big).$$

Now, we are able to define precisely when a formula φ becomes *true* under an interpretation $\mathbf{I} = (\mathbf{M}, j)$; in which case we write $\mathbf{I} \vDash \varphi$ and say that φ is **true** in $\mathbf{I}$ (or that φ **holds** in $\mathbf{I}$). The definition is by induction on the complexity of the formula φ. By the rules (F0)–(F4), φ must be of the form $\tau_1 = \tau_2$, $R(\tau_1, \ldots, \tau_n)$, $\neg\psi$, $\psi_1 \wedge \psi_2$, $\psi_1 \vee \psi_2$, $\psi_1 \rightarrow \psi_2$, $\exists\nu\psi$, or $\forall\nu\psi$:

$$\mathbf{I} \vDash \tau_1 = \tau_2 \quad :\Longleftrightarrow \quad \mathbf{I}(\tau_1) \text{ IS THE SAME OBJECT AS } \mathbf{I}(\tau_2)$$

$$\mathbf{I} \vDash R(\tau_1, \ldots, \tau_n) \quad :\Longleftrightarrow \quad \big\langle \mathbf{I}(\tau_1), \ldots, \mathbf{I}(\tau_n) \big\rangle \text{ BELONGS TO } R^{\mathbf{M}}$$

$$\mathbf{I} \vDash \neg\psi \quad :\Longleftrightarrow \quad \text{NOT } \mathbf{I} \vDash \psi$$

$$\mathbf{I} \vDash \psi_1 \wedge \psi_2 \quad :\Longleftrightarrow \quad \mathbf{I} \vDash \psi_1 \text{ AND } \mathbf{I} \vDash \psi_2$$

$$\mathbf{I} \vDash \psi_1 \vee \psi_2 \quad :\Longleftrightarrow \quad \mathbf{I} \vDash \psi_1 \ \text{OR} \ \mathbf{I} \vDash \psi_2$$

$$\mathbf{I} \vDash \psi_1 \rightarrow \psi_2 \quad :\Longleftrightarrow \quad \text{IF} \ \mathbf{I} \vDash \psi_1 \ \text{THEN} \ \mathbf{I} \vDash \psi_2$$

$$\mathbf{I} \vDash \exists \nu \psi \quad :\Longleftrightarrow \quad \text{IT EXISTS } a \text{ IN } A: \ \mathbf{I}\tfrac{a}{\nu} \vDash \psi$$

$$\mathbf{I} \vDash \forall \nu \psi \quad :\Longleftrightarrow \quad \text{FOR ALL } a \text{ IN } A: \ \mathbf{I}\tfrac{a}{\nu} \vDash \psi$$

Notice that by the logical rules in our informal language, for *every* $\mathscr{L}$-formula φ we have either $\mathbf{I} \vDash \varphi$ or $\mathbf{I} \vDash \neg\varphi$. So, every $\mathscr{L}$-formula is either true or false in $\mathbf{I}$.

The following fact summarises a few immediate consequences of the definitions above:

FACT 2.1. (a) *If φ is a formula and $\nu \notin \text{free}(\varphi)$, then:*

$$\mathbf{I}\tfrac{a}{\nu} \vDash \varphi \quad \textit{if and only if} \quad \mathbf{I} \vDash \varphi$$

(b) *If $\varphi(\nu)$ is a formula and the substitution $\varphi(\nu/\tau)$ is admissible, then:*

$$\mathbf{I}\tfrac{\mathbf{I}(\tau)}{\nu} \vDash \varphi(\nu) \quad \textit{if and only if} \quad \mathbf{I} \vDash \varphi(\tau)$$

Models

Let T be an arbitrary set of $\mathscr{L}$-formulae. Then an $\mathscr{L}$-structure $\mathbf{M}$ is a **model of** T if for every assignment j and for each formula $\varphi \in \mathsf{T}$ we have $(\mathbf{M}, j) \vDash \varphi$, *i.e.*, φ is true in the $\mathscr{L}$-interpretation $\mathbf{I} = (\mathbf{M}, j)$. Instead of saying "$\mathbf{M}$ is a model of T" we just write $\mathbf{M} \vDash \mathsf{T}$. If φ fails in $\mathbf{M}$, then we write $\mathbf{M} \nvDash \varphi$, which is equivalent to $\mathbf{M} \vDash \neg\varphi$, because for any $\mathscr{L}$-formula φ we have *either* $\mathbf{M} \vDash \varphi$ *or* $\mathbf{M} \vDash \neg\varphi$.

Example 2.1. Beispiel

As an immediate consequence of the definition of models we get:

FACT 2.2. *If φ is an $\mathscr{L}$-formula, ν a variable, and $\mathbf{M}$ a model of some $\mathscr{L}$-theory, then $\mathbf{M} \vDash \varphi$ if and only if $\mathbf{M} \vDash \forall\nu\varphi$.*

This leads to the following definition: Let $\langle \nu_0, \dots, \nu_n \rangle$ be the sequence of variables which appear free in the $\mathscr{L}$-formula φ, where the variables appear in the sequence as they appear in φ if one reads φ from left to right. Then the **universal closure** of φ, denoted $\overline{\varphi}$, is defined by stipulating

$$\overline{\varphi} :\equiv \forall \nu_0 \cdots \forall \nu_n \, \varphi \, .$$

As a generalisation of FACT 2.2 we get:

FACT 2.3. *If φ is an $\mathscr{L}$-formula and $\mathbf{M}$ a model of some $\mathscr{L}$-theory, then:*

$$\mathbf{M} \vDash \varphi \iff \mathbf{M} \vDash \overline{\varphi}$$

Basic Notions of Model Theory

Let $\mathscr{L}$ be a signature, *i.e.*, a possibly empty set of constant symbols c, n-ary function symbols F, and n-ary relation symbols R. Two $\mathscr{L}$-structures $\mathbf{M}$ & $\mathbf{N}$ with domains A & B are **isomorphic**, denoted $\mathbf{M} \cong \mathbf{N}$, if there is a bijection $f : A \to B$ such that

$$f\big(c^{\mathbf{M}}\big) = c^{\mathbf{N}} \quad \text{(for all } c \in \mathscr{L}\text{)}$$

and for all $a_1, \ldots, a_n \in A$:

$$f\big(F^{\mathbf{M}}(a_1, \ldots, a_n)\big) = F^{\mathbf{N}}\big(f(a_1), \ldots, f(a_n)\big) \qquad \text{(for all } F \in \mathscr{L}\text{)}$$

$$\langle a_1, \ldots, a_n \rangle \in R^{\mathbf{M}} \Leftrightarrow \big\langle f(a_1), \ldots, f(a_n) \big\rangle \in R^{\mathbf{N}} \quad \text{(for all } R \in \mathscr{L}\text{)}$$

FACT 2.4. (a) *If $\mathbf{M}$ & $\mathbf{N}$ are isomorphic $\mathscr{L}$-structures and σ is an $\mathscr{L}$-sentence, then:*

$$\mathbf{M} \vDash \sigma \iff \mathbf{N} \vDash \sigma$$

(b) *If $\mathbf{M}$ & $\mathbf{N}$ are isomorphic models of some $\mathscr{L}$-theory and φ is an $\mathscr{L}$-formula, then:*

$$\mathbf{M} \vDash \varphi \iff \mathbf{N} \vDash \varphi$$

It may happen that although two $\mathscr{L}$-structures $\mathbf{M}$ & $\mathbf{N}$ are not isomorphic there is no $\mathscr{L}$-sentence that can distinguish between them. In this case we say that $\mathbf{M}$ & $\mathbf{N}$ are elementarily equivalent. More formally, we say that $\mathbf{M}$ is **elementarily equivalent** to $\mathbf{N}$, denoted $\mathbf{M} \equiv \mathbf{N}$, if each $\mathscr{L}$-sentence σ true in $\mathbf{M}$ is also true in $\mathbf{N}$. The following lemma shows that "$\equiv$" is symmetric:

LEMMA 2.5. *If $\mathbf{M}$ and $\mathbf{N}$ are $\mathscr{L}$-structures and $\mathbf{M} \equiv \mathbf{N}$, then for each $\mathscr{L}$-sentence σ we have:*

$$\mathbf{M} \vDash \sigma \iff \mathbf{N} \vDash \sigma$$

Proof. One direction is immediate from the definition. For the other direction, assume that σ is not true in $\mathbf{M}$, *i.e.*, $\mathbf{M} \nvDash \sigma$. Then $\mathbf{M} \vDash \neg\sigma$, which implies $\mathbf{N} \vDash \neg\sigma$, and hence, σ is not true in $\mathbf{N}$. ⊣