DIODORAN MODALITIES

1. DIODORAN MODALITIES AND CONTEMPORARY SYSTEMS

The Megaric logician Diodorus defined the possible as that which either is or at some time will be true, the impossible as that which neither is nor ever will be true, and the necessary as that which both is and always will be true. These definitions assume—as ancient and medieval logic generally assumes—that the same proposition may be true at one time and false at another ; Dr. Benson Mates has accordingly remarked, in his recent study of Stoic logic, that Diodoran ' propositions ' are not ' propositions ' in the modern sense, but something more like propositional functions, and he represents them as such in his symbolic treatment of the Diodoran definitions of the modal operators.[1]

I propose here to do something a little different, namely to employ the ordinary propositional variables ' p ', ' q ', ' r ', etc., for ' propositions ' in the Diodoran sense, and to use certain operators which take such propositions as arguments, and which form functions taking such propositions as values. I shall use

' Fp ' for ' It will be the case that p '
' Mp ' for ' It is (now) possible that p '
' Lp ' for ' It is (now) necessary that p '
' Np ' for ' It is not (now) the case that p '
' Cpq ' for ' If p then q '
' Apq ' for ' Either p or q '
' Kpq ' for ' Both p and q '
' Epq ' for ' If and only if p then q '.

' Np ' is understood as being true at any time at which ' p ' is false, and *vice versa* ; ' Kpq ' as true at any time at which ' p ' and ' q ' are both true (and only at such times) ; ' Apq ' as being false at any time at which ' p ' and ' q ' are both false (and only at such times) ; ' Cpq ' as false at any time at which ' p ' is true and ' q ' false (and only at such times) ; ' Epq ' as true at any time at which ' p ' and ' q ' have the same truth-value (and only at such times) ; and only two truth-values are admitted. When a formula is laid down as expressing a logical law, it is understood as yielding, for all substitutions of actual propositions for its variables, a proposition which is at all times true.

' Cpq ' as above used does not symbolise the locution ' If p then q ' in

[1]B. Mates, *Stoic Logic*, University of California Press, 1953, pp. 36-37. It has been pointed out to me by Mr. W. W. Sawyer that the Diodoran view that whatever is possible either is or will be true is very like the ergodic hypothesis in the kinetic theory of gases.

the sense given to 'If' by Diodorus, but rather in the sense given to it by his pupil and opponent Philo.[2] Diodoran implication may, however, easily be expressed in our symbolism as 'LCpq' (or more accurately by the equivalent 'LNKpNq'), and little would be gained by employing a distinct symbol for it. We shall find it useful, however, to introduce the following abbreviation :—

Df. G : Gp = NFNp.

That is, 'Gp' means 'It will not be the case that not p', or 'It will always be the case that p'. We may now express the Diodoran definitions of possibility and necessity as follows :—

Df. M : Mp = ApFp.

Df. L : Lp = KpGp.

The definition of 'Lp' as 'NMNp' would yield an equivalent form, for NMNp = NANpFNp = KNNpNFNp = KpNFNp = KpGp. This proof assumes that de Morgan's laws and the law of double negation hold under the present interpretation of 'proposition'; but it is clear enough that they do, and that in fact the whole of the classical assertoric propositional calculus will hold under the present interpretation. A question of more interest is that as to which, if any, of the classical modal calculi the Diodoran definitions will yield.

Von Wright[3] has shown that a system equivalent to Lewis's S4 may be obtained by adding the axiom

W3. CMMpMp

to the classical assertoric propositional calculus enriched by the axioms

W1. CpMp

W2. EMApqAMpMq,

the definition of 'Lp' as 'NMNp', and the rules

RL. If α is a thesis, so is $L\alpha$;

RE. If $E\alpha\beta$ is a thesis, so is $EM\alpha M\beta$;

and that a system equivalent to Lewis's S5 is obtained if W3 is replaced by

W4. CMNMpNMp.

We have already seen that the Diodoran definition of 'L' is equivalent to von Wright's, and we may now see how his other postulates fare.

The formula W1 asserts that whenever a proposition is true it is possible. If 'It is possible that p' means 'It either is or will be the case that p', this clearly follows from the simple 'p' by the substitution q/Fp in the law CpApq. In view of the type of proofs which will occur later, it is worth pointing out that CpFp, the 'F-analogue' of CpMp, does not express a law, for it is not always true that if it is the case that p then it will be the case that p. In this respect the logic of futurity is like the logic of moral permissibility, in which, although permissibility and possibility exhibit

[2] *Ibid.*, pp. 42-51.

[3] G. H. von Wright, *An Essay in Modal Logic*, Amsterdam, 1951, Appendix II. See also B. Sobocinski, 'Note on a Modal System of Feys-von Wright', *Journal of Computing Systems*, July 1953.

many analogies, we do not have as a law ' If p is done, then the doing of p is permissible '.[4]

The formula W2 asserts that if and only if it is possible that either p or q then either it is possible that p or it is possible that q. On the Diodoran interpretation, this means that if it is or will be the case that either-p-or-q, then either it is or will be the case that p or it is or will be the case that q ; and *vice versa.* If the truth of this is not intuitively obvious, we may prove it from the corresponding law directly relating to the primitive operator F,

A1. EFApqAFpFq,

' If and only if it will be the case that either p or q, then either it will be the case that p or it will be the case that q '. The proof (introducing laws of the assertoric calculus as we require them) is as follows :—

1. EAApqArsAAprAqs
2. CEqpCEArpsEArqs

   2 p/AFpFq, q/FApq, r/Apq, s/AApFpAqFq
   = CA1 —— C1 r/Fp, s/Fq —— 3.
3. EAApqFApqAApFpAqFq

   3 X Df. M = 4.
4. EMApqAMpMq.

The rule RL asserts, in Diodoran terms, that if a formula expresses something which is always true, then the formula asserting that that thing is necessary, i.e. that it is and always will be true, is always true. This again is obviously true, and is formally derivable if we lay down the simpler rule

RG. If α is a thesis, so is Gα ;

that is, if α is always true, then the assertion that α will always be true is always true. The rule RL follows from this, for if we are given a thesis α we may always deduce Lα as follows :—

1. α
2. CpCqp
3. CCpqCpKpq

   1 X RG = 4.
4. Gα

   2 p/Gα, q/α = C4 —— 5
5. CαGα

   3 p/α, q/Gα = C5 —— C1 —— 6.
6. KαGα

The rule RE asserts in Diodoran terms that if a pair of formulae α and β at all times have the same truth-value, then it will always be true that it either is or will be the case that α if and only if it either is or will be the case that β. This, again, is deducible from the corresponding rule for ' F ',

RF : If E$\alpha\beta$ is a thesis, so is EFαFβ.

For given RF, and a thesis E$\alpha\beta$, we may always prove EMαMβ as follows :—

1. E$\alpha\beta$

[4]Von Wright, *op. cit.*, Ch. V, p. 41.

2. CEpqCErsEAprAqs
    1 X RF = 3.
3. $EF\alpha F\beta$
    2 p/α, q/β, $r/F\alpha$, $s/F\beta$ = C1 —— C3 —— 4.
4. $EA\alpha F\alpha A\beta F\beta$
    4 X Df.M = 5.
5. $EM\alpha M\beta$.

The S4 formula W3 asserts that what is possibly possible is actually possible, i.e., in Diodoran terms, that if it either is or will be the case that it either is or will be the case that p, then it either is or will be the case that p. This again, if not obvious, is deducible from the corresponding law for F,

A2. CFFpFp,

'If it will be the case that it will be the case that p, then it will be the case that p', which is obvious enough. The deduction is as follows :—

A1. EFApqAFpFq
A2. CFFpFp
1. CCqpCApqp
2. CCpqCCqrCpr
3. CpAqp
4. CEpqCpq
    1 p/Fp, q/FFp = CA2 —— 5.
5. CAFpFFpFp
    2 p/AFpFFp, q/Fp, r/ApFp
    = C5 —— C3 p/Fp, q/p —— 6.
6. CAFpFFpApFp
    4 p/FApq, q/AFpFq = CA1 —— 7
7. CFApqAFpFq
    2 p/FApFp, q/AFpFFp, r/ApFp
    = C7 q/Fp —— C6 —— 8.
8. CFApFpApFp
    1 p/ApFp, q/FApFp = C8 —— 9
9. CAApFpFApFpApFp
    9 X Df. M = 10.
10. CMMpMp.

The stronger formula W4, characteristic of S5, asserts that whatever is possibly impossible is in fact impossible, i.e., in Diodoran terms, that whenever it either is or will be the case that it neither is nor will be the case that p, it neither is nor will be the case that p. This is plainly untrue, for if it is now the case that p, but will later on cease to be the case that p, and will never thereafter be the case that p, then (i) it will later be the case that it neither is nor will be the case that p (i.e. the antecedent is now true), but (ii) this is not the case now (i.e. the consequent is now false). And this rejection of the thesis CMNMpNMp is derivable from the rejection of the thesis CFNFpNFp, 'If it will be the case that it will not be the case that p, then it will not be the case that p', i.e. 'If p will ever cease to be the case, it has done so now', which is clearly no law. The rejection of W4 follows from that of the corresponding formula in F because if W4 were a thesis the other would follow from it, thus :—

W4. CMNMpNMp
A2. CFFpFp
1. CCApqpCqp
2. CCqpENApqNp
3. CEpqCErsCCprCqs
W4 X Df. M = 4.
4. CANApFpFNApFpNApFp
1 p/NApFp, q/FNApFp = C4 —— 5
5. CFNApFpNApFp
2 p/Fp, q/FFp = CA2 —— 6
6. ENAFpFFpNFp
6 X RF = 7.
7. EFNAFpFFpFNFp
3 p/FNAFpFFp, q/FNFp, r/NAFpFFp, s/NFp
= C7 —— C6 —— C5 p/Fp —— 8.
8. CFNFpNFp.

In sum, the Diodoran definitions of the modal operators yield a system more like the Lewis system S4 than any other. The system contains all theses of S4, and they arise from a more economical basis than von Wright's (axioms A1 and A2 in place of von Wright's W1, W2, and W3 ; on the side of rules, the honours are even). It also contains further theses (including the two axioms) which are not interpretable modally ; whether it contains theses over and above those of S4 in which 'F' occurs only as an implicit constituent of 'M', is a question which remains to be investigated. But it does not contain the characteristic theses of the main modal system known to be thus stronger than S4, namely S5. It may be observed, however, that if Diodorus had defined the possible not merely as that which is or will be the case, but as that which is, has been or will be the case, the S5 thesis W4 (CMNMpNMp) would have held. For if it is, has been or will be the case that it neither is, has been nor will be the case that p, then it neither is, has been nor will be the case that p. The proposal to extend the notion of the possible in this or any other direction has, however, a very formidable piece of adverse reasoning to face.

2. THE 'MASTER-ARGUMENT'

Against those who rejected his definition on the ground that some propositions are possible though they neither are nor ever will be true, Diodorus had an argument which came to be known as the 'Master Argument'.[5] Its main premisses were that

(*a*) Every true proposition about the past is necessary ;

(*b*) An impossible proposition never follows from a possible one.

[5]B. Mates, *op. cit.*, pp. 38-40.

Other ancient logicians rejected the Diodoran conclusion (that what neither is nor will be true is not possible), but agreed with him that they could only do so by rejecting at least one of the propositions (*a*) and (*b*) ; that is, they admitted that his reasoning was valid. Modern scholars have wondered why they did so ; but before attending to this problem, we may glance at a somewhat simpler one, namely that of the consistency of these propositions with the Diodoran definitions of modal terms.

The necessary, according to Diodorus, both is and always will be true, and it must be admitted that some true propositions about the past do not always remain so. Thus 'I had soup for tea last night' may be true now, but unless I have soup tonight also, it will be false tomorrow.[6] This makes (*a*) as it stands inconsistent with the Diodoran account of necessity. It is true, however, of all propositions of the form 'It has been the case that p' that once they are true they are true for ever, and we shall see that if we understand (*a*) as referring to propositions of this form, it will suffice for the purpose to which Diodorus puts it. (*b*) holds in the Diodoran system, as it holds in S4.

In dealing with the main problem, our starting-point must be a little different. We cannot assume the Diodoran definitions, but must take necessity and possibility as undefined, and use (*a*) and (*b*), which we might restate as

(*a*) When anything has been the case, it cannot not have been the case,

(*b*) If anything is impossible, then anything that necessarily implies it is impossible,

to prove the conclusion

(*z*) What neither is nor will be true, is not possible. Stated more precisely, our problem is to discover what broad assumptions about time, likely to have been taken for granted both by Diodorus and by his main opponents, would make (*z*) demonstrable from (*a*) and (*b*). In answer to this question, it can be shown that the following two will suffice :—

(*c*) When anything is the case, it has always been the case that it will be the case ;

(*d*) When anything neither is nor will be the case, it has been the case that it will not be the case.

The way the proof proceeds may be best understood by considering an example. One of the opponents of Diodorus is said to have contended that we may rightly say of a shell at the bottom of the sea that it can be seen there, even if in fact it is not being seen and never will be. But by (*d*), if the shell neither is nor will be seen, it has been the case that it will not be seen. Hence, by (*a*), it cannot (now) not have been the case that it has been the case that it will not be seen. That is, the proposition that it has not been the case that it will not be seen, i.e. that it has always been the case that it will be seen, is impossible. But by (*c*), the proposition that the shell

[6]Cf. Martha Kneale, 'Logical and Metaphysical Necessity', *Proceedings of the Aristotelian Society*, 1937-8.

is now being seen entails this impossible proposition that it has always been the case that the shell will be seen. Hence, by (*b*), the proposition that the shell is being seen is itself impossible.

We may generalise and formalise this reasoning by introducing the forms ' Pp ', for ' It has been the case that p ', and ' Hp ' for ' It has always been the case that p ', the latter being defined as an abbreviation for ' NPNp ', ' It has not been the case that not p '. Our four premisses then become

(*a*) CPpNMNPp.
(*b*) CNMqCLCpqNMp
(*c*) CpHFp
(*d*) CKNpNFpPNFp.

And the conclusion to be drawn from them is

(*z*) CKNpNFpNMp.

The proof (with two further premisses drawn from the ordinary propositional calculus) is as follows :—

1. CCpqCCqrCpr
2. CCpCqrCqCpr.
   (*a*) p/NFp X Df. H = 3.
3. CPNFpNMHFp
   1 p/KNpNFp, q/PNFp, r/NMHFp
   = C (*d*) —— C3 —— 4
4. CKNpNFpNMHFp
   1 p/KNpNFp, q/NMHFp, r/CLCpHFpNMp
   = C4 —— C (*b*) q/HFp —— 5
5. CKNpNFpCLCpHFpNMp
   (*c*) X RL = 6
6. LCpHFp
   2 p/KNpNFp, q/LCpHFp, r/NMp
   = C5 —— C6 —— (*z*).

(*z*) CKNpNFpNMp.

It may, I think, be reasonably assumed that the schools of Cleanthes and Chrysippus, who admitted the validity of the ' Master Argument ' while denying its conclusion, would have joined Diodorus in taking our (*c*) and (*d*) for granted, and it must have been in some such way that the argument was originally filled in, though whether it was completed in exactly the manner outlined can of course only be a matter for conjecture. But it may be noted that if we admit three truth-values, and assign the ' neuter ' value to propositions about undetermined future events,[7] we may accept (*a*) and (*b*) and still deny (*z*), for with such a logic we may well deny one of the conjectural additional premisses. The bearing of three-valued logic on this argument is not, however, quite as simple as it may at first appear to be.

The assignment of a 'neuter' truth-value to propositions about contingent

[7]Cf. A. N. Prior, ' Three-valued Logic and Future Contingents ', *Philosophical Quarterly*, October 1953.

future events would seem at first sight to cast suspicion upon our premiss (*c*), asserting that whatever is the case has always been going to be the case. It should be remembered, however, that in the system with which we have been working the form 'It has always been the case that p' is simply an abbreviation for 'It has not been the case that not p', and even on the three-valued hypothesis it must be admitted that when anything is the case it has not been the case that it will not be the case. If three truth-values are admitted it is no doubt misleading and unwise to use the form 'It has always been the case that p' as an abbreviation for 'It has not been the case that not p'; but the fact remains that it is only in this sense that the form occurs in our reconstruction of the Master Argument, and that in this sense our premiss (*c*) is unobjectionable even in a three-valued system.

On the other hand, a reconsideration of the Diodoran conclusion, that what neither is nor will be the case is not possible, might suggest that an exponent of three-valued logic would not really wish to deny this. Would he not rather say that once it is already a determinate fact that something is not and will not be the case, the possibility of its being the case has gone, just as Diodorus says it has? His qualms, one might think, would only arise when Diodorus passes from this to the positive assertion that what is possible either definitely is the case or definitely will be the case. But in fact the transposition by which the second of these two propositions is derived from the first—the use, that is, of the law CCNpNqCqp—is quite valid in three-valued logic, at least if we interpret three-valued implication as Lukasiewicz does. And if he uses 'If' in this sense, the three-valued logician is not in fact free to accept even the negative formula CKNpNFpNMp as expressing a logical law. This implication does work out as a true one when 'p' and 'Fp' are definitely true or definitely false; but when the antecedent is indeterminate, i.e. when it is not yet either true or false that it either is or will be the case that p, then the consequent, asserting p's impossibility, is false, and an implication with a neuter antecedent and a false consequent is itself not true but neuter ($C\frac{1}{2}O = \frac{1}{2}$). The point may be clarified by returning to the illustration of the shell. If it is already a determinate fact that the shell neither is nor will be seen, the three-valued logician will share the Diodoran view that it is now impossible that the shell should be seen. But suppose that whether the shell will or will not be seen is not yet determinate, and our enquiry is as to the possibility, not that the shell is being seen, but that the shell will be seen. That it *is not* the case that the shell will be seen is on this supposition indeterminate; that it *will not be* the case that it will be seen is also indeterminate; hence the whole assertion that it neither is nor will be the case that the shell will be seen is indeterminate. But the assertion that it is impossible that the shell will be seen is on this supposition definitely false. Hence the implication 'If it neither is nor will be the case that the shell will be seen, it is not possible that the shell will be seen' is not true but neuter.

An analysis of the same sort makes it clear that the real way of escape from the Diodoran conclusion which the admission of a third truth-value offers is through the rejection not of (*c*) but of (*d*). That is, we may deny that propositions of the form CKNpNFpPNFp are in all cases true. For if it is indeterminate whether p is or will be the case, the assertion that it has been the case that p will not be the case is false, so that once again we shall have an implication with a neuter antecedent and a false consequent, making the whole not true but neuter.

A. N. Prior

Canterbury University College, Christchurch, N.Z.